Module 46: Time Dependent Boundary Conditions

Consider the problem $\frac{\partial}{\partial t} w = \frac{\partial^2}{\partial x^2} w$ with

$$\frac{\partial}{\partial x} w(t, 0) = - f(t)$$

and

$$\frac{\partial}{\partial x} w(t, 1) = g(t).$$

To do this problem, create u and F so that

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u \; + \; F(t, x)$$

$$\frac{\partial}{\partial x} u\,(t, 0) = 0,$$

$$\frac{\partial}{\partial x} u\,(t, 1) = 0$$

and

u and w are related in a simple manner.

The w and u connection:

$$w(t, x) = u(t, x) - f(t)\,\left(x - \frac{x^2}{2}\right) + g(t)\,\frac{x^2}{2}$$

$$\frac{\partial}{\partial t} w = \frac{\partial}{\partial t} u - f\,'(t)\,\left(x - \frac{x^2}{2}\right) + g\,'(t)\,\frac{x^2}{2}$$

$$\frac{\partial^2}{\partial x^2} w = \frac{\partial^2}{\partial x^2} u + f(t) + g(t)$$

Thus, if the diffusion equation holds for w,

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u + f(t) + f\,'(t)\,(x - \frac{x^2}{2}) +$$

$$g(t) - g\,'(t)\,\frac{x^2}{2}\ .$$

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u \ + \ F(t,\ x).$$

What about the boundary conditions?

$$w(t, x) = u(t, x) - f(t)\left(x - \frac{x^2}{2}\right) + g(t)\frac{x^2}{2}$$

$$-f(t) = \frac{\partial}{\partial x} w(t, 0) = \frac{\partial}{\partial x} u(t, 0) - f(t)$$

$$g(t) = \frac{\partial}{\partial x} w(t, 1) = \frac{\partial}{\partial x} u(t, 1) + g(t)$$

Good News! We need to solve

$$\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u \;+\; F(t, x)$$

$$\frac{\partial}{\partial x} u\,(t, 0) = 0, \quad \frac{\partial}{\partial x} u\,(t, 1) = 0$$

where w(0, x) =

$$u(0, x) - f(0)\left(x - \frac{x^2}{2}\right) + g(0)\,\frac{x^2}{2}$$

BAD NEWS!

We need a worked out example.

$$\frac{\partial}{\partial t} w = \frac{\partial^2}{\partial x^2} w \quad \text{with}$$

$$\frac{\partial}{\partial x} w\,(t,\,0) = -\sin(2\,\pi\,t)$$

and

$$\frac{\partial}{\partial x} w\,(t,\,1) = \sin(2\,\pi\,t)$$

$$w(0,\,x\,) = 0.$$

Be aware of the physical interpretation.

To use the method of separation of variables, we suppose that

$$u(t, x) = \sum_n T_n(t) \cos(n \pi x)$$

Break up that F to

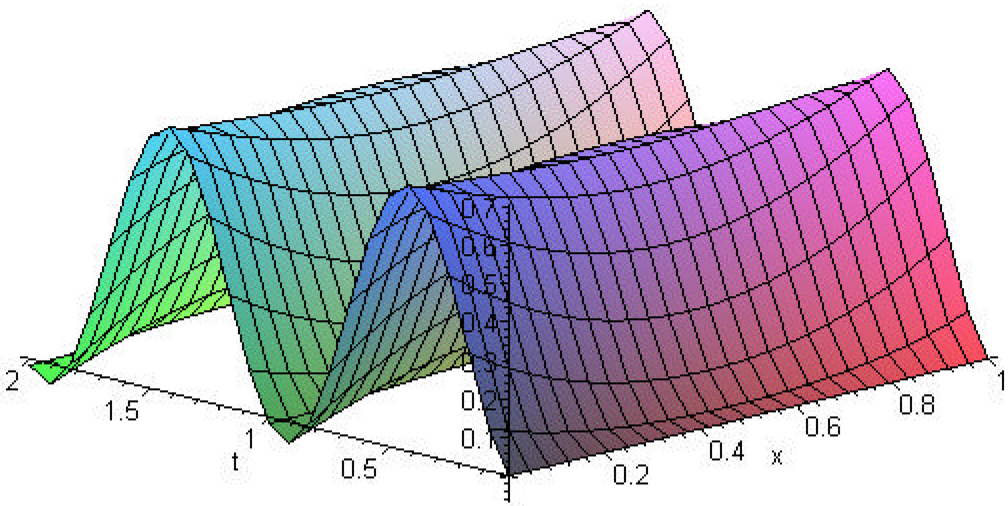
$$F(t, x) = \sum_n f_n(t) \cos(n \pi x)$$

We are led to an infinite number of ordinary differential equations.

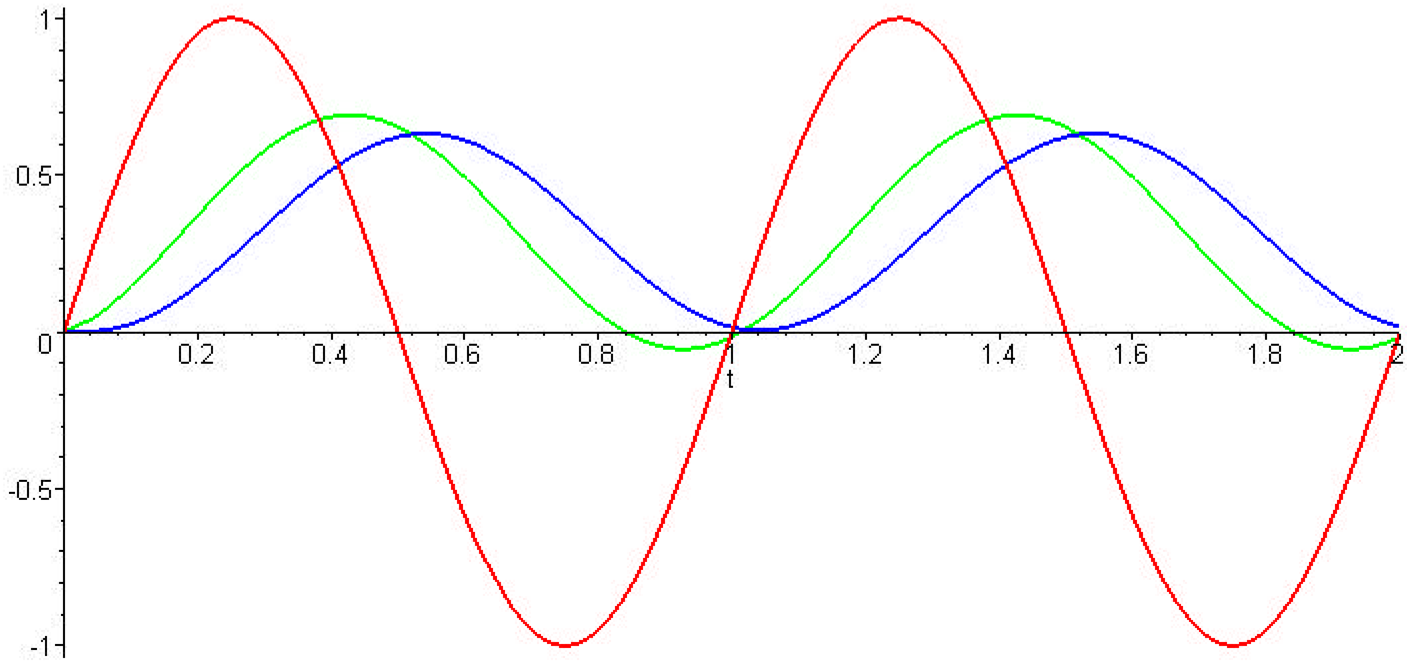
$T_n'(t) = - T_n(t) + f_n(t)$,

$T_n(0)$ comes from the Fourier expansion for u(0, x).

Graph of the solution.



Graph of u(t, 0) and u(t,1/2)



In the notes for this lecture, there is a numerical procedure for doing this same problem.

ADVANTAGE: The messy transformation is not necessary.

DISADVANTAGE: There is no analytic solution.

Be reminded that numerical solutions for PDE's is a part of a set of notes available on the web

SUMMARY:

(a) We considered the diffusion equation with time dependent boundary conditions.

(b) We set this problem in the context of a non - homogeneous PDE with homogeneous boundary conditions.

(c) We also worked the problem using the built in numerical solver available in MAPLE.