A Background for the Weil Conjectures

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Definition

Zeta function Let X_0 be a variety of dimension d in $\mathbb{F}_q$ and $X_n := X_0 \times F_{q^n}$ and $X = X_0 \times \overline{\mathbb{F}_q}$. Then

$$Z_{X_0}(u) := \exp(\sum_{n\geq 1} |X_n| \frac{u^n}{n}). \tag{1}$$

The Euler product form:

$$Z_{X_0,s}(q) = \prod_{x:closeinX_n} \frac{1}{1 - \frac{1}{q^{\deg(x)s}}}. \tag{2}$$

Projective space

Let $X_0 = \mathbb{P}^d(\mathbb{F}_q)$, then

$$|X_n| = 1 + q^n + \cdots + q^{dn}. \tag{3}$$

So

$$\begin{aligned} Z_{X_0}(u) &= \exp(\sum_{n\geq 1}(1 + q^n + \cdots + q^{dn})\frac{u^n}{n}) \\ &= \prod_{i=0}^{d}\exp(\sum_{n\geq 1}\frac{(q^i u)^n}{n}) = \prod_{i=0}^{d}\exp(\log(1 - q^i u)^{-1}) \\ &= \prod_{i=0}^{d}(1 - q^i u)^{-1}. \end{aligned} \tag{4}$$

Definition

Character A character is a nonzero multiplicative function $chi : \mathbb{F}_q \longrightarrow \mathbb{C}$; i.e. $\chi(ab) = \chi(a)\chi(b)$.
The Gauss sum is the discrete Fourier transformation of χ. So

$$g(\chi) = \sum_{t \in \mathbb{F}_p} \chi(t) e^{\frac{2\pi it}{p}} \tag{5}$$

Finally, the Jacobian sum of the character χ, τ (over the same field) is

$$J(\chi, \tau) = \sum_{a+b=1} \chi(a)\tau(b) \tag{6}$$

We have

$$J(\chi, \tau) = \frac{g(\chi)g(\tau)}{g(\chi\tau)}. \tag{7}$$

Lemma

The number of the Affine roots for the polynomial $f(x_1, \cdots, x_n) := \sum_{i=1}^{n} a_i x_i^{m_i} - b$ *over* $\mathbb{F}_q$ *is*

$$N = \sum_{\substack{a_i \in \mathbb{F}_q \\ \sum_i a_i u_i = b}} \sum_{\chi_i^{m_i} = \varepsilon} \chi_i(u_i). \tag{8}$$

Example

Let $X_0 : y^2 = x^3 + 1$ over $\mathbb{F}_q$. Affine solutions

$$
\begin{aligned}
|X_0| &= \sum_{t\in\mathbb{F}_q} \sum_{\chi^3=\varepsilon} \sum_{\rho^2=\varepsilon} \chi(t)\rho(1-t) \\
&= \sum_{t\in\mathbb{F}_q^\times} \left(\varepsilon(t) + \chi(t)\rho(1-t) + \overline{\chi}(t)\rho(1-t) + \chi(t) + \rho(1-t) + \overline{\chi(t)}\right) \\
&= q + (J(\chi,\rho) + J(\overline{\chi},\rho)) \\
&= q + 2Re\left(\chi^2(2)J(\chi,\chi)\right) = q + 2Re\left(\chi^2(2)\sum_{t\in\mathbb{F}_q^\times} \frac{(g(\chi))^2}{g(\chi^2)}\right) \\
&= q + 2Re\left(\chi^2(2)\frac{(g(\chi))^2}{g(\chi^2)}\right) = q + 2Re\left(\chi^2(2)\frac{(g(\chi))^2}{\overline{g(\chi)}}\right). \qquad (9)
\end{aligned}
$$

An important result

If $\chi'(t) = \chi(N(t))$ for every $t \in \mathbb{F}_{p^r}$, then $(-g(\chi))^r = g(\chi')$.

Example

The only Solution of $X_0 : y^2 = x^3 + 1$ at infinity hyperplane:
$x = 0, y = 1, t = 0.$

$$
\begin{aligned}
Z_{X_0}(u) =& \exp\Big(\sum_{n\geq 1}\Big(1 + q^n + \Big(\chi^2 n(2)\frac{(g(\chi))^2 n}{(\overline{g(\chi)})^n}\Big)\Big) + \overline{\Big(\chi^2 n(2)\frac{(g(\chi))^2 n}{(\overline{g(\chi)})^n}\Big)}\Big)\Big)\frac{u^n}{n} \\
=&(1-u)^{-1}(1-qu)^{-1}\exp\Big(\sum_{n\geq 1}\frac{\big(\frac{u\chi^2 g(\chi)^2}{\overline{g(\chi)}}\big)^n + \big(\frac{\overline{u\chi^2 g(\chi)^2}}{g(\chi)}\big)^n}{n}\Big) \\
=&\frac{\big(1-\frac{u\overline{\chi^2 g(\chi)^2}}{g(\chi)}\big)\big(1-\frac{u\chi^2 g(\chi)^2}{\overline{g(\chi)}}\big)}{(1-u)(1-qu)} = \frac{1+au+qu^2}{(1-u)(1-qu)}. \qquad (10)
\end{aligned}
$$

Example

The only Solution of $X_0 : y^2 = x^3 + 1$ at infinity hyperplane:
$x = 0, y = 1, t = 0.$

$$Z_{X_0}(u) = \exp\Big(\sum_{n\geq 1}\big(1 + q^n + (\chi^2 n(2)\frac{(g(\chi))^2 n}{(\overline{g(\chi)})^n})) + \overline{(\chi^2 n(2)\frac{(g(\chi))^2 n}{(\overline{g(\chi)})^n}}))\frac{u^n}{n}$$

$$= (1-u)^{-1}(1-qu)^{-1}\exp\Big(\sum_{n\geq 1}\frac{(\frac{u\chi^2 g(\chi)^2}{\overline{g(\chi)}})^n + (\frac{u\overline{\chi}^2\overline{g(\chi)}^2}{g(\chi)})^n}{n}\Big)$$

$$= \frac{(1 - \frac{u\overline{\chi^2 g(\chi)^2}}{g(\chi)})(1 - \frac{u\chi^2 g(\chi)^2}{\overline{g(\chi)}})}{(1-u)(1-qu)} = \frac{1 + au + qu^2}{(1-u)(1-qu)}. \tag{10}$$

Remarks

1- It seems that zeta function is a rational function.
2- One can see that $|a| \leq 2\sqrt{q}$.

Weil's (Theorem or Conjecture)

If X_0 is a dimension d smooth and projective variety:
1- $Z(X_0)$ is actually a rational function.
2- There exists an integer r and $a = \pm 1$ such that $Z(X_0)$ satisfies a functional equation

$$Z(X_0, d)(u) = aq^{\frac{dr}{2}} u^r Z(X_0, 1)(u). \tag{11}$$

3- We can write $Z(X_0)$ as a rational function of the form

$$Z(X_0) = \frac{\prod_{i=0}^{d-1} P_{2i+1}}{(1-u)(1-q^d u)\prod_{i=1}^{d-1} P_{2i}} \tag{12}$$

4- $P_i \in \mathbb{Z}[x]$ and all the zeroes of P_i are of modulus $q^{-\frac{i}{2}}$.

An application of Deligne's result

If f is a cusp form of weight k, and $f(q) = \sum_{n=1}^{\infty} a_n q^n$, then By Hecke proof, we know that $a_n = O(n^k)$.
Deligne result gives us $a_n = O(n^{k-\frac{1}{2}+\epsilon})$.

cech and etale (1)

if X is of dimension d, then the cech(etale) cohomology $H^i(X, C) = 0$ $(H^i(X, Q_p) = 0)$ for all $i > 2d$.

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cech and etale (2)

$H^i(X, C)$ $(H^i_{et}(X, Q_p))$ is a finite dimensional C-vector (Q_p-vector) space.

cech and etale (3)

if $f : X \longrightarrow Y$ is a morphism then there are associated maps in cohomology for all $i > 0$

$$f* : H^i(Y, C) \longrightarrow H^i(X, C). \tag{13}$$

or

$$f* : H^i_{et}(Y, Q_p) \longrightarrow H^i(X, Q_p). \tag{14}$$

In particular, a map $X \longrightarrow X$ induces endomorphisms of the finite dimensional C-vector (Q_p-vector) spaces $H^i(X, C)$ ($H^i(X, Q_p)$).

cech and etale (4)

Lefschetz trace formula: let $f : X \longrightarrow X$ be a morphism map with isolated fixed points, satisfying a certain separability assumption on $1 - df$ acting on the tangent spaces at the fixed points, and $L(f)$ the number of those. Then we have the equality

$$L(f) = \sum_{i=0}^{2d} (-1)^i Tr(f * |H^i_{et}(X, Q_p)) \tag{15}$$

References

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Kowalski, E. trying to understand delign's proof of the weil conjecture.