Number of partitions with parts of the form $pt + a$

Seyyed Hamed Mousavi

Ph.D. Student, School of Mathematics, Georgia Institute of Technology

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Outline

1. Introduction
2. Exact formula for the case $T \in \Gamma_0(p)$
3. Exact formula for the case $T \in \Gamma(1) \setminus \Gamma_0(p)$
4. Remark, Corollary, and Meinardus Theorem

Definition

Generating function For a positive integer n, prime number p, and $0 \leq a \leq p-1$ let $p_a(n)$ be the number of partitions.

$$F_a(x) = \prod_{m=0}^{\infty} \frac{1}{1-x^{pm+a}} = \frac{1}{(q^a;q^p)_\infty} := \sum_{n=0}^{\infty} p_a(n)x^n.$$

Rademacher formula for number of Partitions

$$p(n) = \frac{1}{\pi\sqrt{2}} \sum_{k=1}^{\infty} A_k(n)\sqrt{k} \left[\frac{d}{dx} \frac{\sinh\left(\frac{\pi}{k}\sqrt{\frac{2}{3}(x-\frac{1}{24})}\right)}{\sqrt{x-\frac{1}{24}}} \right]_{x=n}.$$

We plan to find an exact formula for $p_a(n)$.

Literature after Rademacher

1. Hao 1940 partition with parts $2t + 1$
2. Haberzetle 1941 partition with parts divisable by p or q and $24|(p-1)(q-1)$
3. Lehner 1942 partition with parts $5t \pm a$
4. Lingwood 1945 partition with parts $pt \pm a$
5. Meinardus 1954 asymptotic formula for generating functions ($L-$functions)
6. Isako 1957 partition with parts $Mt \pm a$
7. Brendt 1973 modular transformation of general generating functions
8. Ono 2000 Congruence equation for $p(\frac{m^k l^3 n+1}{24})$ mod m
9. Laughlin 2010 partitions into parts which are coprime with both numbers r, s simultanously

Definition

For $a, b, m \in \mathbb{N}$, the Kloosterman sum is

$$K(a, b; m) = \sum_{\substack{0 \le h \le m-1 \\ gcd(h,m)=1}} e^{\frac{2\pi i}{m}(ah+bh')}.$$

Definition

The Bessel function of order α is

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha}.$$

In particular, the Bessel function of order zero is

$$J_0(x) = \frac{1}{2} \int_0^{\infty} e^{-t+\frac{z^2}{4t}} \frac{d}{dt}.$$

Symmetric subset of Residue modulo m

Let $m \in \mathbb{N}$; then a symmetric subset A of $\mathbb{Z}_m$ is a subset in which if $a \in A$, then $-a \in A$. All of similar previous results are working in symmetric set. We plan to find a way to discuss for non symmetric sets.

Definition

Farey dissection is a recurrence sequences of numbers.
Let $\frac{a_0}{b_0} = \frac{0}{1}$ and $\frac{c_0}{d_0} = \frac{1}{1}$.
$\frac{a_n}{b_n}, \frac{c_n}{d_n} \in A_n \longrightarrow \frac{a_{n+1}}{b_{n+1}} = \frac{a_n+c_n}{b_n+d_n} \in A_{n+1}$.

$$A_0 = \{\frac{0}{1}, \frac{1}{1}\}$$
$$A_1 = \{\frac{0}{1}, \frac{1}{2}, \frac{1}{1}\}$$
$$A_2 = \{\frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}\}$$
$$A_3 = \{\frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}\}$$

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Step one: Modular Transformation

Define

$$G_a(x) = \log(F_a(x)) = -\sum_{m=0}^{\infty} \log(1 - x^{pm+a}) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{x^{(pm+a)n}}{n}.$$

Then

$$G_a(x) = G_b(x') - 2\pi i(R_1 + R_2).$$

where R_1 and R_2 are the residue of the following functions, respectively.

$$Res\left(\frac{1}{4\pi i k^2} \sum_{\substack{\mu_b \equiv ha \equiv b(mod p) \\ 0 \le \nu, \lambda, \mu_b < k}} \cos(\frac{2\pi\mu\nu}{k}) \cos(\frac{2\pi h'\lambda\mu}{k}) \frac{\zeta(1+s, \frac{\lambda}{k})\zeta(1-s, \frac{\nu}{k})}{z^{-s}\cos(\frac{\pi s}{2})}\right)$$

$$Res\left(\frac{1}{4\pi k^2} \sum_{\substack{\mu_b \equiv ha \equiv b(mod p) \\ 0 \le \nu, \lambda, \mu_b < k}} \sin(\frac{2\pi\mu\nu}{k}) \sin(\frac{2\pi h'\lambda\mu}{k}) \frac{\zeta(1+s, \frac{\lambda}{k})\zeta(1-s, \frac{\nu}{k})}{z^{-s}\sin(\frac{\pi s}{2})}\right).$$

In modular forms Notation

Let $G(x) = [G_1(x), \cdots G_p(x)]$, Then if $\gamma = \begin{bmatrix} -h' & \frac{hh'-1}{k} \\ k & -h \end{bmatrix} \in \Gamma_0(p)$ (i.e. $p|k$), then

$$G|_\gamma(x) = AG(x)$$

where

$$A_{ab} = e^{2\pi i(R_1+R_2)}$$

where R_1, R_2 are defined in the previous slide.

Second step: Finding Residues

We have

$$R_1 + R_2 = \frac{z(p^2 - 6pa + 6a^2)}{48pki} - \frac{p^2 - 6pb + 6b^2}{48pkiz} + \frac{1}{2} \sum_{\mu_a \equiv a \pmod p} \left(\left(\frac{\mu_a}{k}\right)\right) \left(\left(\frac{h\mu_a}{k}\right)\right).$$

First point: Using the following equations for Hurwitz-zeta function,

$$\zeta(s, \frac{\mu}{k}) = \frac{2\Gamma(1-s)}{(2\pi k)^{1-s}} \Bigg(\sin(\frac{\pi s}{2}) \sum_{\lambda=0}^{k-1} \cos(\frac{2\pi\lambda\mu}{k}) \zeta(1-s, \frac{\lambda}{k}) + \cos(\frac{\pi s}{2}) \sum_{\lambda=0}^{k-1} \sin(\frac{2\pi\lambda\mu}{k}) \zeta(1-s, \frac{\lambda}{k}) \Bigg)$$

Second point: $\sum_{\mu_b \equiv ha} \{h\mu_b, k\}^2 = \sum_{\mu_a \equiv a} \{h\mu_a, k\}^2$

Third step: Computing R_2

Let $\omega_a(h,k) = e^{2\pi i R_2}$. Then

$$\omega_a(h,k) = e^{\frac{\pi i}{12kp}(h(12a^2 - f\phi u + 12(k-p)(2k-p)) + h' f \phi u)} \times e^{\frac{\pi i (6kf\phi - 6gk\gamma + 6k)\{\frac{ha}{p}\}}{12k}} \times e^{\frac{\pi i D}{12kp}}$$

where

$$D = 3akp - 12kpr - 3k(k-p) - 6ka - 3f\phi kp - 6kga\gamma - 6kgp\gamma + 3gk\gamma.$$

To find a congruent equation for R_2 modulo 1, we need $\sum_\mu \frac{h\mu}{k}\left[\frac{h\mu}{k}\right]$. It can be find from the fact that $\sum_{\mu_a}(\{\frac{h\mu_a}{k}\} - \frac{1}{2})^2 = \sum_{\mu_b}(\frac{\mu_b}{k} - \frac{1}{2})^2$.

Fourth step: Incomplete Kloosterman's sum

Assume that

$$A(n, \nu, k, d, \sigma_1, \sigma_2, a) := \sum \omega_a(h, k) e^{-\frac{2\pi i (hn - h'\nu)}{k}}.$$

where the sum is over $0 \leq h \leq k$, $h \equiv d \pmod{p}$, $0 \leq \sigma_1 \leq h' < \sigma_2 \leq k$. Then $A(n, \nu, k, d, \sigma_1, \sigma_2, a) = O(k^{\frac{2}{3}} n^{\frac{1}{3}})$.

First point: We change A to a complete Kloosterman's sum by multiplying a characteristic function $a(h') = \begin{cases} 1 & \sigma_1 \leq h' < \sigma_2 \\ 0 & O.W. \end{cases}$. Then we find Fourier Transform of $a(h')$.

Second Point: We need to change $e^{\pi i \{\frac{ha}{k}\}}$ into the form $e^{\pi i (ah + bh' + c)}$. To do this we use Fourier Transform. If $e^{\pi i \{\frac{ha}{k}\}} = \sum_{r=0}^{\frac{k}{(a,k)} - 1} b_r e^{\frac{2\pi i r (a,k)}{k}}$, then

$$b_r = \frac{2}{k(1 - e^{\frac{\pi i (1-2t)(a,k)}{k}})}.$$

Third point: Because of orthognality, we have symetry for all $1 \leq d \leq p$.

Last step: Circle method and Integral computation

Let $D_r = \frac{-\pi^2}{24k^2p}(p^2 - 6pa + 6a^2 + 24pn)(2r + \frac{p^2-6pb+6b^2}{24p})$ and $E_r = \frac{\pi}{k^2}(2r - \frac{p^2-6pb+6b^2}{24p})$.

$$p_a^1(n) = \sum_{\substack{0<k\leq N\\ p|k}} \left(\sum_{0\leq r<[-\frac{p^2-6pb+6b^2}{48p}]} \frac{2\pi i J_1(2\sqrt{D_r})}{24kp} \sqrt{\frac{48pr - p^2 - 6pb + 6b^2}{p^2 - 6pa - 6a^2 + 24pn}} \times \sum_{\substack{0\leq h<k\\ h\equiv d \pmod p\\ 0<h'\leq k}} \omega_a(h,k) e^{\frac{-\pi i(-2nh+2h'r+k)}{k}} \right).$$

where $J_1(\cdot)$ is the Bessel function of the first degree.

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First step: Modular Transformation

Let $p \nmid k$, $K = kp$, $k\alpha \equiv a \pmod{p}$, and

$$J_a(x) = \log\left(\prod_{0<n<\infty} \frac{1}{1 - e^{\frac{2\pi i\alpha}{K}} x^n}\right).$$

Then

$$G_a(x) = J_a|_\gamma(x) - 2\pi i(R_1 + R_2).$$

where R_1, R_2 are the residues of the following functions

$$\frac{1}{4\pi ikK} \sum_{\substack{\mu_a \equiv a \pmod{p} \\ 0 \le \nu, \lambda < k}} \cos\Big(\frac{2\pi\mu'\nu}{k}\Big) \cos\Big(\frac{2\pi\lambda\mu}{K}\Big) \frac{\zeta(1+s, \frac{\lambda}{K})\zeta(1-s, \frac{\nu}{k})ds}{z^{-s}\cos(\frac{\pi s}{2})}$$

$$\frac{1}{4\pi kK} \sum_{\substack{\mu_a \equiv a \pmod{p} \\ 0 \le \nu, \lambda < k}} \sin\Big(\frac{2\pi\mu'\nu}{k}\Big) \sin\Big(\frac{2\pi\lambda\mu}{K}\Big) \frac{\zeta(1+s, \frac{\lambda}{K})\zeta(1-s, \frac{\nu}{k})ds}{z^{-s}\sin(\frac{\pi s}{2})}$$

Second and Third steps: Computing Residues

We have

$$2\pi i(R_1 + R_2) = \frac{1}{48ikp}\left(z(p^2 - 6pa + 6a^2) - \frac{1}{z}\right) - 4\pi ikp\varsigma(k, a, p) + \pi i\kappa(h, k, a, p).$$

where

$$\varsigma(k, a, p) = (\log(2kp\pi) + \gamma)\left(\frac{1}{2} - \frac{\alpha}{p}\right) + \log\left(\frac{\alpha}{p}\right) - \frac{1}{2}\log(2\pi).$$

and

$$\kappa(h, k, a, p) = \sum_{\mu \equiv a \pmod p} \left(\left(\frac{\mu}{K}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

Fourth step: Incomplete Kloosterman's sum

We define an analogous A for this case and we have

$$A(n, v, u, k, d, \sigma_1, \sigma_2, a) = O((pk)^{\frac{2}{3}+\epsilon}).$$

The main difference: We need to deal with $\sum_{t=0}^{k'} \{\frac{hp(2t+1+l)}{2k'+1}\}$ and $\sum_{t=0}^{k'} \{\frac{hp(2t+1-l)}{k}\}$. We compute a congruence equation for these sums to change them in the form of $ah + bh' + c$.

Last step: Circle method

We have

$$P_a^2(n) = -\sum_{\substack{0 \le h < k \le N \\ h \equiv d \pmod{p}}}{}' \Bigg(e^{-\frac{2\pi}{k}\left(hn + k\left((\log(2\pi pk)+\gamma)(\frac{1}{2}) + \log(\frac{\alpha}{p}) - \frac{\log(2\pi)}{2}\right)\right)} \omega_a(h,k)$$
$$\times \frac{\sqrt{\pi(p^2 - 6pa + 6a^2 - 48pn)}}{24pk}$$
$$\times I_1\Big(\frac{\sqrt{\pi(p^2 - 6pa + 6a^2 - 48pn)}}{48pk}\Big)\Bigg) + O(e^{N^{-2}}).$$

where I_1 is the modified Bessel function of the first kind and order one.
So we can find $p_a(n) = p_a^1(n) + p_a^2(n)$.

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Remark

The expansion for $p_a^1(n)$ is in the form of sum over multiplication of Kloosterman's sums and Bessel functions. This exactly the same as the Fourier expansion of Poincare series. In particular, assume that $\phi_m(\tau) = q^m$, $k > 2$ be a half integer and $P_{k,m,N}(\tau) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \phi_m|_k \gamma(\tau)$. Also, if $P_{k,m,N} = \sum_{n=1}^{\infty} b_m(n) q^n$ then

$$b_m(n) = \left(\frac{n}{m}\right)^{\frac{k-1}{2}} \left(\delta_{m,n} + 2\pi i^{-k} \sum_{c>0, N|c} \frac{K_k(m,n,c)}{c} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{c}\right)\right).$$

Comparing this and the formula for partitions give us a clue that all of such generating functions can be represented as a series of Poincare series.

Corollary

Let $p_s(n) := p_s^1(n) + p_s^2(n)$ *be the number of ways to represent* n *into parts of the form* $pt + a^2$*. Then*

$$p_s^2 = \sum_{1\le d\le p} \sum{}'_{\substack{0\le h<k\le N \\ h\equiv d \pmod p}} e^{\frac{-2\pi i}{k}(hn+\frac{k(\log(2\pi pk)+\gamma)}{2}-\frac{k\log(2\pi)}{2})}\omega_0(ph,k)(pk)^{\frac{1}{2}}$$

$$\left(\prod_{a=1}^{\frac{p-1}{2}} \omega_{a^2}(h,k)\right)\left(\prod_{a=1}^{\frac{p-1}{2}} e^{-2\pi i(\log(\frac{\{a^2k^{-1}\}_p}{p}))}\right.$$

$$\times \left(3k^2(\frac{U_n}{\pi})^{\frac{3}{2}}\frac{d}{dx}\left(\frac{2\cosh(\sqrt{U_xV_x})}{V_x'\sqrt{U_x}}\right)_{x=n}\right.$$

$$\left.\left. - \frac{-\pi}{2\sqrt{U_n}}\frac{d}{dx}\left(\frac{2e^{-2\sqrt{U_xV_x}}}{V_x'\sqrt{U_x}}\right)_{x=n} + O(\frac{1}{kN})\right)\right)$$

where $\prod_{a=1}^{\frac{p-1}{2}} \omega_{a^2}(h,k)$, U_n, V_n *can be computed. Also* $\omega_0(h,k)$ *is defined in the Rademacher proof and* $\{a^2k^{-1}\}_p$ *is* $a^2k^{-1} \pmod p$.

Corollary

Also

$$p_s^1(n) = \sum_d \sideset{}{'}\sum_{\substack{0 \le h < k \le N \\ \vartheta' = \vartheta'(h,k) \\ \vartheta'' = \vartheta''(h,k) \\ h \equiv d \pmod p}} e^{\frac{-2\pi inh}{k}} \prod_{a=1}^{\frac{p-1}{2}} \left(\sum_r \sum_{r'} P_{tot}(r) P_0(r') e^{\frac{2\pi ir}{k} + \frac{2\pi ir'h'}{k}} \omega_{a^2}(h,k) \right)$$

$$\times \omega_0(h,k) 3k^{5/2} (\frac{Z_n}{\pi})^{\frac{3}{2}} \frac{d}{dx} \left(\frac{2\cosh(\sqrt{Z_x Y_x})}{Y_x' \sqrt{Z_x}} \right)_{x=n}$$

$$- \frac{-\pi}{2\sqrt{Z_n}} \frac{d}{dx} \left(\frac{2e^{-2\sqrt{Z_x Y_x}}}{Y_x' \sqrt{Z_x}} \right)_{x=n} + O(\frac{1}{kN})$$

where we have $r + r' \le \frac{1}{2\pi} \left(\frac{p}{24} \sum_{a=1}^{\frac{p-1}{2}} \left(1 - 6\{\frac{ha^2}{p}\} + 6\{\frac{ha^2}{p}\}^2 \right) + \frac{\pi}{12} \right).$

Meinardus

Let $f(z)=\sum_{n=0}^{\infty} c_n z^n = \prod_{k=0}^{\infty}(1-z^k)^{-b_k}$. Assume that $D(s)=\sum_{n=1}^{\infty} b_k k^{-s}$. If first: D is convergent for $\sigma > r$ and has analytic continuation for $\sigma > -C_0$ for $1 \geq C_0 > 0$.
Second: There is a constant C_1 such that $D(s)=O(|t|^{C_1})$ uniformly when $\sigma > -C_0$ and $t \longrightarrow \infty$.
Third: There exists C_2, ϵ such that $Re(G(e^{-\sigma-it})) - G(e^{-\sigma}) \leq -C_2\sigma^{-\epsilon}$ for $|arg(\sigma+it)| < \frac{\pi}{4}$, $0 \neq |t| < \pi$, and $\sigma \longrightarrow 0$. Then there exists k_1, k_2, C such that

$$c_n = Cn^{k_1} e^{n^{\frac{r}{r+1}}\left(1+\frac{1}{r}\right)\left(A\Gamma(r+1)\zeta(r+1)\right)^{\frac{1}{r+1}}}$$

Corollary

Let $D(s) = \sum_{n=1}^{\infty} \frac{1}{(np+a)^s} = p^{-s}\zeta(s,a/p)$. *Then* $G(z) = z^{a+p}(1-z^p)^{-1}$. *We can check that all three conditions hold and* $p_a(n) \sim e^{D'(0)}(4\pi)^{-1/2}(\pi^2/6)^{2a/p} n^{-1/2-a/2p} \exp\left(\pi(2n/3)^{1/2}\right)$.

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THANK YOU.