Pentagonal Number Theorem and Reimann Hypothesis

Hamed Mousavi

Ph.D. Student, School of Mathematics
Georgia Institute of Technology
HMOUSAVI6@GATECH.EDU

The Legacy of Ramanujan, University of Illinois at Urbana Champaign, Urbana Champaign , Illinois

May 29, 2019

Outline

Pentagonal Number Theorem

Let $p(n)$ be the number of partitions and $G_l = \frac{l(3l-1)}{2}$ be $l-$th pentagonal number. Then

$$\sum_{G_l \leq n} (-1)^l p(n - G_l) = 0.$$

Quick sketch of proof in Berndt Book

The main step of the proof is to show that

$$(q;q)_\infty = \sum_{l=-\infty}^{\infty} (-1)^l q^{G_l}.$$

In order to prove this we find coefficients of $\frac{(az;q)_\infty}{(z;q)_\infty} = \sum_n A_n(a,q)z^n$. The idea is to use the recurrence formula $A_n = f(a,q)A_{n-1}$. Then we literally take $a \longrightarrow \infty$.

Rademacher expression for $p(n)$

Let $\mu_k(n) = \frac{\pi\sqrt{24n-1}}{6k}$. Rademacher-Ramanujan-Hardy proved that

$$p(n) = \frac{\sqrt{12}}{24x-1}\left(\sum_{k=1}^{\infty} A_k(n)\left((1-\frac{1}{\mu_k(n)})e^{\mu_k(n)} + (1+\frac{1}{\mu_k(n)})e^{-\mu_k(n)}\right)\right).$$

where

$$A_k(n) = \sum_{\substack{0\le h<k \\ (h,k)=1}} \omega_{h,k} e^{\frac{2\pi ihn}{k}}.$$

They started with Cauchy integral formula. Then they used Farey dissections to avoid the poles of the generating function. Using modularity, the bound in integral becomes more controllable. A straightforward proof is in Andrew book.

Generalization ...

Define

$$p(x) = \frac{\sqrt{12}}{24x-1}\left(\sum_{k=1}^{\infty} A_k(x)\left((1-\frac{1}{\mu_k(x)})e^{\mu_k(x)} + (1+\frac{1}{\mu_k(x)})e^{-\mu_k(x)}\right)\right).$$

where

$$A_k(x) = \sum_{\substack{0\le h<k \\ (h,k)=1}} \omega_{h,k} e^{\frac{2\pi i h[x]}{k}}.$$

Intuition

Truncation of the first two terms:

$$p(x) = \frac{\sqrt{12}e^{\frac{\pi}{6}\sqrt{24x-1}}}{24x-1}\left(1-\frac{6}{\pi(24x-1)^{\frac{3}{2}}}\right)+O(\sqrt{p(x)}).$$

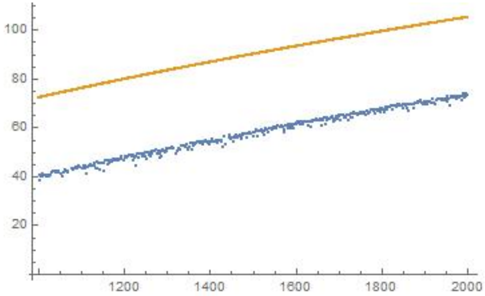


Figure: Comparison between loagarithm of error term of the truncated Rademacher Equation with logarithm of the actual partitions

Why generalization?

Let $p_1(x)$ be the first two terms of the Rademacher formula. Then

$$\sum_{G_l<x} p_1(x-G_l)(-1)^l = O(\sqrt{p_1(x)}).$$

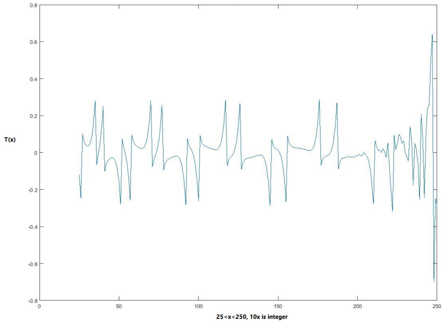


Figure: Error term in Pentagonal Number Theorem for $20 < n < 250$

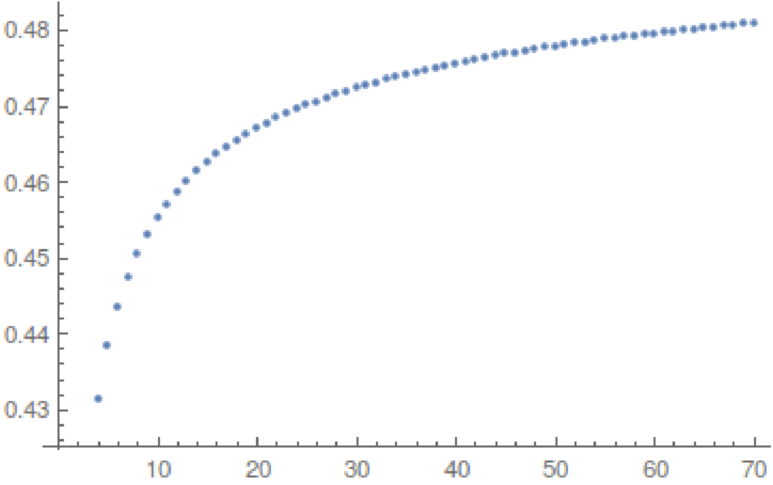


Figure: Error term in Pentagonal Number Theorem for $500 < n < 70000$

Outline

Unexpected issue!

The main difficulty is that we cannot use Cauchy integral formula. i. e.

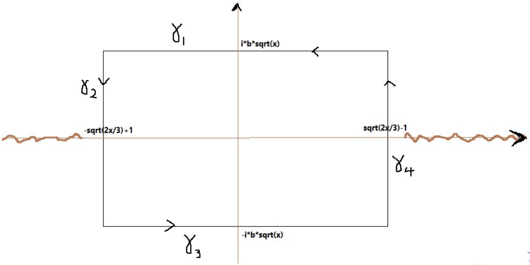
$$p(x) \neq \int_C \frac{P(q)dq}{q^{x+1}}$$

even if we fix the branch cut, the output of integral is not equal to the Rademacher formula numerically. But thy are comparable even for large numbers! Possibly for $A_k(x)$?!
Instead we pick a contour and use Residue Theorem.

Sketch of proof

Define $f(z) = \dfrac{e^{\pi\sqrt{x - \frac{z(3z-1)}{2}}}}{\sin^5(\pi z)}$.

1. First Fact: We do not have branch cut near zero!
2. Second fact: $\frac{1}{\sin^5(\pi z)}$ has poles in integers with residue $\frac{3(-1)^k}{8}$.
3. Third fact: For large enough b (e.g. $0.09\sqrt{x} < b$)we can prove that the contribution of the integral over γ_1, γ_3 is $\sqrt{p(x)}$.
4. Fourth fact: We can prove that for b not very large (e.g. $b < 0.18$) the contribution of γ_2, γ_4 is $\sqrt{p(x)}$.



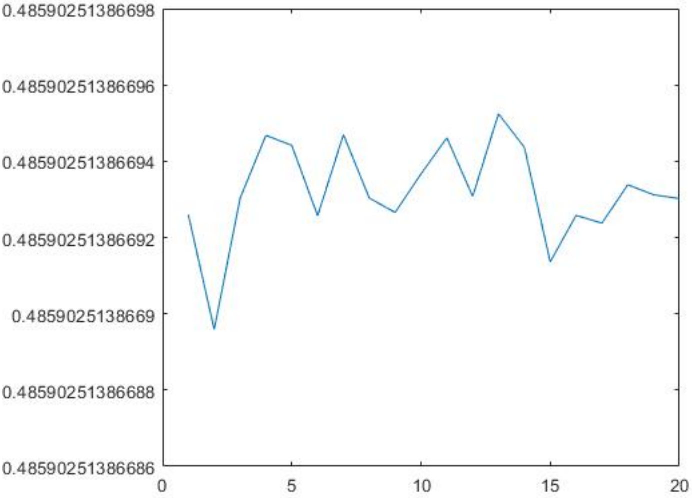


Figure: $\frac{log(integral) - \log(p(x))}{\log(p(x))}$ which should be around 0.5.

Outline

Comparing $\Psi(e^{\frac{\pi}{6}\sqrt{24x-1}})$ and $p(x)$

LEt Ψ be Chebyshev function. Assuming Riemann Hypothesis and Pentagonal number theorem:

$$\sum_{G_l<x}(-1)^l\Psi(e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}})\left(\frac{1}{24(x-G_l)-1}-\frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right)$$

$$=O\left(\frac{e^{\frac{\pi(\frac{1}{2}+\delta)}{6}\sqrt{24x-1}}}{\sqrt{24x-1}}\right)$$

Question: For the above formula, we used the estimation $\Psi(x)=x+\theta(x^{\frac{1}{2}+\delta})$. What if we use the exact amount?

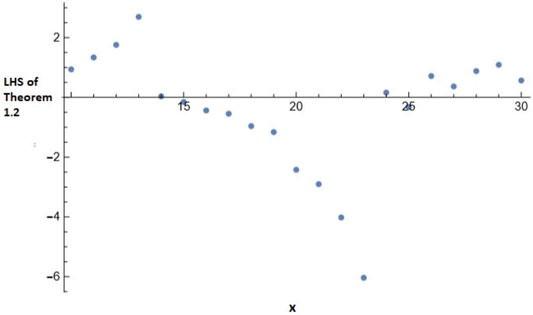


Figure: Error term of the theorem for Chebyshev Ψ function

Observation

This nice behaviour of Chebyshev functions means something deep is going on here; which motivates us to assume the following hypothesis.

The hypothesis: weak version

$$\sum_{G_l<x}\left(\frac{1}{24(x-G_l)-1}-\frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right)(-1)^l\sum_{n\leq e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}}}\frac{\Lambda(n)}{n^s}$$
$$=O\left(e^{\frac{\pi(\frac{1}{2}-\sigma+\delta)}{6}\sqrt{24x-1}}\right).$$

Theorem

1. *For case* $Re(s) = 0$, *for* $\delta > 0$, *and a.e.* t

$$\sum_{G_l < x} (-1)^l \left(\frac{1}{24(x - G_l) - 1} - \frac{6}{\pi (24(x - G_l) - 1)^{\frac{3}{2}}} \right) \times \sum_{n \le e^{\frac{\pi}{6}\sqrt{24(x - G_l) - 1}}} \frac{\Lambda(n)}{n^{it}} = O\left(e^{\frac{\pi\delta}{6}\sqrt{24x - 1}} \right).$$

2. *For case* $Re(s) = \frac{1}{2}$, *for* $\delta > 0$, *and a.e.* t

$$\sum_{G_l < x} (-1)^l \left(\frac{1}{24(x - G_l) - 1} - \frac{6}{\pi (24(x - G_l) - 1)^{\frac{3}{2}}} \right) \times \sum_{n \le e^{\frac{\pi}{6}\sqrt{24(x - G_l) - 1}}} \frac{\Lambda(n)}{n^{\frac{1}{2} + it}} = O\left(e^{\frac{\pi(\frac{1}{4} + \delta)}{6}\sqrt{24x - 1}} \right).$$

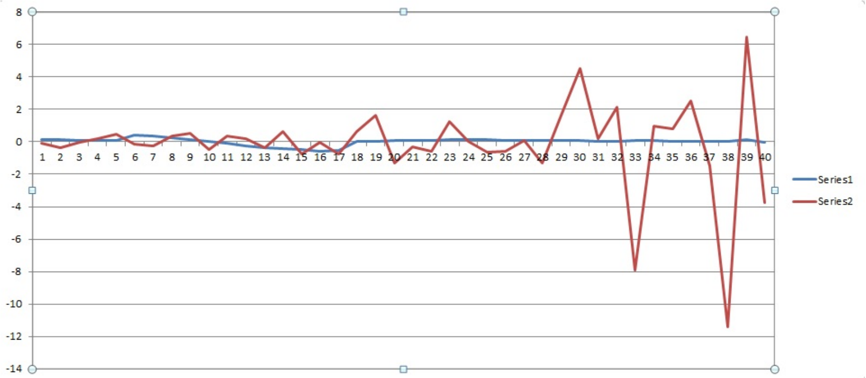


Figure: Error term of the right hand side of hypothesis for $s = 0$ and $s = \frac{1}{2}$.

The Hypothesis: strong version

Let $\epsilon > 0$ and $s = \sigma + it \in \mathbb{C}$. Then there exists a function f with finite Fourier Transform such that

$$\sum_{G_l < x + \frac{1}{24} + \epsilon} (-1)^l \left(\frac{1}{24(x - G_l) - 1} - \frac{6}{\pi(24(x - G_l) - 1)^{\frac{3}{2}}} \right) \times \sum_{n \leq e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}}} \frac{\Lambda(n)}{n^s} = O\left(f(x) e^{\frac{\frac{1}{2} - \sigma}{6}\sqrt{24x-1}} \right).$$

Conjecture

Consider distribution of a random sequence $\{\rho_m\}$ as follows

$$\sum_{0<Im(\rho_m)<T} 1 = \frac{T}{2\pi}\log(\frac{T}{2\pi}) - \frac{T}{2\pi} + O(\log(T)).$$

Then the function that minimize

$$\sum_{G_l<x}(-1)^l\omega(e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}})\left(\frac{1}{24(x-G_l)-1} - \frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right)$$

is chebyshev Ψ function. In particular

$$\sum_{G_l<x}(-1)^l\Psi(e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}})\left(\frac{1}{24(x-G_l)-1} - \frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right)$$
$$= O(f(x)).$$

Outline

Theorem

The strong version of the hypothesis for $\sigma = \frac{1}{2}$ results in Riemann hypothesis.

Sketch of proof

1. Step One: Using Hadamard formula and using the strong hypothesis gives us the Fourier transform of $\mathcal{F}\left(\frac{y^{1-s}}{1-s} + \sum \frac{x^{\rho-s}}{\rho-s}\right) = O(1)$.
2. Use a contour to get $\mathcal{F}\left(\frac{y^{1-s}}{1-s}\right) = O(1)$.
3. If $Re(\rho) - \frac{1}{2} > 0$ for some ρ, and considering the fact that

$$\int_{\epsilon}^{\infty} \frac{e^{\alpha\sqrt{x}}\cos(\pi x)}{x} dx \longrightarrow \infty$$

we get a contradiction with $\mathcal{F}\left(\sum \frac{x^{\rho-s}}{\rho-s}\right) = O(1)$.

Pentagonal Number Theorem and Reimann Hypothesis

Hamed Mousavi

Ph.D. Student, School of Mathematics
Georgia Institute of Technology
hmousavi6@gatech.edu

May 29, 2019

Thank You

hmousavi6@gatech.edu