Three Circles Theorem and Moments of Riemann Zeta functions

Hamed Mousavi

Ph.D. Student, School of Mathematics
Georgia Institute of Technology

Complex Analysis Project, Georgia Tech, Atlanta, Georgia

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Outline

1. Three Circles Theorem and its Proof
2. Moments of Riemann Zeta functions

Basic Facts

1. A function $f : [a, b] \longrightarrow \mathbb{R}$ is convex, iff for any points $x_1, \cdots, x_n$ in $[a, b]$ and real numbers $t_1, \cdots t_n \geq 0$ with $\sum t_i = 1$

$$f(\sum t_i x_i) \leq \sum t_i f(x_i).$$

2. A set $A \subseteq \mathbb{C}$ is convex iff for any points $z_1, \cdots, z_n$ in A and real numbers $t_1, \cdots t_n \geq 0$ with $\sum t_i = 1$, we conclude that $\sum t_i z_i \in A$.
3. A differentiable function $f : [a, b] \longrightarrow \mathbb{C}$ is convex iff f' is increasing.

Definition

1. A function f is log-convex iff $\log(f(x))$ is convex.
   So exp is log-convex, but x^2 is not.
   If f is convex and g is log-convex, then $g \circ f$ is log-convex.
2. A sequence $\{a_n\}$ is log-convex iff $a_n^2 - a_{n-1}a_{n+1} \leq 0$.
   Partition sequence $p(n)$ is log-concave.
   If a_n, b_n are log-concave, then $a_n b_n$ is log-concave.

Lemma

Let $G = \{z = x + iy | a < x < b\}$ and $f : \overline{G} \longrightarrow \mathbb{C}$ be analytic in G and continuous for $z \in \partial G$ we have $|f(z)| \leq 1$. Then $z \in G$ we have $|f(z)| \leq 1$.

Proof

1. Step one: Let $g_\epsilon(z) = \frac{1}{1-\epsilon(z-a)}$. Then $g_\epsilon(z) \leq 1$.
2. Step two: One can see that $|f(z)g_\epsilon(z)| < \frac{B}{\epsilon y}$ for a constant B.
3. Step three: If $y > \frac{B}{\epsilon}$, use Step two to say $|f(z)| < 1$. If $y < \frac{B}{\epsilon}$, use Maximum Modulus Theorem to say $|f(z)| < 1$.

First Version of Three Circles Theorem

Let $G = \{z = x + iy | a < x < b\}$ and $f : \overline{G} \longrightarrow \mathbb{C}$ be analytic and continuous in ∂G. Also $M : [a, b] \longrightarrow \mathbb{R}$ is defined

$$M(x) = \sup\{|f(x + iy)|, -\infty < y < \infty\}$$

If $|f(z)| < B$, then $M(x)$ is log-convex. i.e. for $a < x < u < y < b$

$$M(u)^{y-x} \leq M(x)^{y-u} M(y)^{u-x}.$$

Proof

1. Step one: $M(a), M(b) \neq 0$.
2. Step two: Let $g(z) = M(a)^{\frac{b-z}{b-a}} M(b)^{\frac{z-a}{b-a}}$ is entire and never vanishes.
3. Step three: $|g(z)|$ is continuous w.r.t x and never vanishes. So $\frac{f}{g}$ is bounded.
4. Step four: $g(a + iy) = M(a)$ and $g(b + iy) = M(b)$. So $|\frac{f}{g}| \leq 1$ in $z \in \partial G$.
5. Step five: Use the Lemma to say $|\frac{f}{g}| \leq 1$ for $z \in G$.

Corollary

Let $G = \{x + iy | a < x < b\}$ and $f : \overline{G} \longrightarrow \mathbb{C}$ be non constant and continuous. Then for $z \in G$ we have $|f(z)| < \sup\{|f(w)| : w \in \partial G\}$.

Remark

Unlike Maximum Modulus Theorem G is not bounded. Also, f need not to be analytic.

Second Version of three circles Theorem

Let $0 < R_1 < R_2 < \infty$ and suppose that f is analytic on annus $(0, R_1, R_2)$. If $R_1 < r < R_2$, define $M(r) = \max\{|f(re^{i\theta})| : 0 < \theta < 2\pi\}$. Then for $R_1 < r_1 \leq r \leq r_2 < R_2$

$$\log(M(r)) \leq \frac{\log(r_2) - \log(r)}{\log(r_2) - \log(r_1)} \log(M(r_1)) + \frac{\log(r) - \log(r_1)}{\log(r_2) - \log(r_1)} \log(M(r_2)). \tag{1}$$

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Ultimate goal

We want to find a bound for

$$I_k(T,\sigma) := \int_1^T |\zeta(\sigma+it)|^{2k} dt$$

where

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \quad Re(s) > 1$$

It is known that $I_k(T) \sim C_k T \log^{k^2}(T)$. We do not know C_k for $k > 10$. But there is a conjecture for all n.
We aim for $I_k(T) << (>>) T \log^{k^2}(T)$. The good news is we can bound this function for specific values of σ.

The reference Theorem

Let f be regular in infinite strip $\alpha < Re(z) < \beta$ and continuous in boundary. Let $f(z) \longrightarrow 0$ as $|Im(z)| \longrightarrow \infty$ uniformly for $\alpha \leq Re(z) \leq \beta$. Then for $q > 0$ and $\alpha \leq \gamma \leq \beta$ we have

$$\int_{-\infty}^{\infty} |f(\gamma + it)|^q dt \leq \left(\int_{-\infty}^{\infty} |f(\alpha + it)|^q dt \right)^{\frac{\beta-\gamma}{\beta-\alpha}} \left(\int_{-\infty}^{\infty} |f(\beta + it)|^q dt \right)^{\frac{\gamma-\alpha}{\beta-\alpha}}$$

The so called application

Let $w(t) = \int_T^{2T} e^{-2k(t-\tau)^2} d\tau$ and $J(\sigma) = \int_{-\infty}^{\infty} |\zeta(\sigma+it)|^{2k} w(t) dt$. Then

1. We have the lower bound
$$J(\frac{1}{2}) << T^{k(\sigma-\frac{1}{2})} J(\sigma) + e^{-\frac{kT^2}{3}}.$$
2. and the upper bound
$$J(\sigma) << T^{\sigma-\frac{1}{2}} J(\frac{1}{2})^{\frac{3}{2}-\sigma} + e^{-\frac{kT^2}{4}}.$$

Sketch of proof

1. Let $f(z) = (z)e^{(z-i\tau)^2}$, and we get three circles $Re(z) = \sigma, \frac{1}{2}, 1-\sigma$.
2. If $Im(z)$ becomes far from τ, then we have no contribution.
3. If not, we use three circles Theorem and functional equation $\zeta(s) = \zeta(1-s)G(s)$ for some controllable G.
4. So the contribution of the inner and outer circle is w.r.t $Re(z) = \sigma$ and the middle one has $Re(z) = \frac{1}{2}$.

References

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Thank You

hmousavi6@gatech.edu