# Complex Hyperbolic Plane in Cylindrical Coordinates

# About the Real Hyperbolic Plane

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## *Models of the Hyperbolic Plane*

We will begin by reviewing the metric on the hyperbolic plane with curvature $k$.

The unit disc model of the hyperbolic plane has metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$ for $x, y \in \mathbb{R}$, [2, page1]; Note that the metric on a real plane is $ds^2 = dx^2 + dy^2$. As y values approach zero, Hyperbolic distances grow larger while Euclidean distances stay constant. To express this formally, the metric becomes $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$. This is equivalent to

$ds^2 = \dfrac{|dz|^2}{|\operatorname{Im} z|^2}$ where $z = x + iy \in \mathbb{C}$; this transformation arises

from $dz = dx + idy, |dz|^2 = dx^2 + dy^2, \operatorname{Im} z = y, |\operatorname{Im} z|^2 = y^2$. We can obtain the form

$ds^2 = dt^2 + e^{-2t} du^2$ by using the transformation $x = u, y = e^t$ which led to the calculation

$$dx^2 = du^2, dy^2 = e^{2t} dt^2, y^2 = e^{2t}$$

$$ds^2 = \frac{du^2 + e^{2t} dt^2}{e^{2t}} = e^{-2t} du^2 + dt^2 = dt^2 + e^{-2t} du^2 .$$

We can calculate the polar coordinate version for the case where $k = -1$, indicating a complex plane, by starting with the polar coordinate version of the metric $ds^2 = \dfrac{|dz|^2}{|\operatorname{Im} z|^2}$, which is

$$ds^2 = \frac{4 dz d\bar{z}}{(1 - z\bar{z})^2} = \frac{4 |dz|^2}{(1 - |z|^2)^2}, \text{ [2, page7].}$$

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Here, $z = e^{i\theta}\tanh(\tfrac{u}{2})$ and $dz = ie^{i\theta}\tanh(\tfrac{u}{2})d\theta + e^{i\theta}\sec h^2(\tfrac{u}{2})\cdot\tfrac{1}{2}du$ which leads to $|dz|^2 = \tanh^2(\tfrac{u}{2})d\theta^2 + \dfrac{du^2}{4\cosh^4(\tfrac{u}{2})}$ and $(1-|z|^2)^2 = (1-\tanh^2(\tfrac{u}{2}))^2 = \sec h^4(\tfrac{u}{2}) = \dfrac{1}{\cosh^4(\tfrac{u}{2})}$.

This results in the following:

$$\frac{4(\tanh^2(\tfrac{u}{2})d\theta^2 + \tfrac{1}{4}\sec h^2(\tfrac{u}{2})du^2)}{\sec h^4(\tfrac{u}{2})}$$

$$4\sinh^2(\tfrac{u}{2})\cosh^2(\tfrac{u}{2})d\theta^2 + du^2$$

$$(2\sinh(\tfrac{u}{2})\cosh(\tfrac{u}{2}))^2 d\theta^2 + du^2$$

$$\sinh^2(2\tfrac{u}{2})d\theta^2 + du^2$$

$$du^2 + \sinh^2(u)d\theta^2$$

Thus we have derived the polar coordinate version $ds^2 = dr^2 - \tfrac{1}{k}\sinh^2(\sqrt{k}r)d\theta^2$ when the curvature $k = -1$. We can derive $ds^2 = dt^2 + \cosh^2(t)du^2$ by setting $f(t) = \cosh(t)$ and $L = -1$ as defined in the formula in the following referenced paper, [4, p.27-28 Corollary 7.10]. The calculations involved,

$\dfrac{f''(t)}{f(t)} = \dfrac{\cosh(t)}{\cosh(t)} = 1$ and $\dfrac{L - f'^2(t)}{f(t)} = \dfrac{-1-\sinh^2(t)}{\cosh^2(t)} = \dfrac{-\cosh^2(t)}{\cosh^2(t)} = -1$, allow us to apply the generalized metric $g = dt^2 + f^2(t)du^2$ with $f(t) = \cosh(t)$ satisfying the curvature which is -1 for the hyperbolic space. Curvature can be calculated as $k = -{f''(t)}/{f(t)}$. The polar coordinate version $ds^2 = dr^2 - \tfrac{1}{k}\sinh^2(\sqrt{k}r)d\theta^2$ is the measurement of the distance from a point in the hyperbolic space to the origin of the hyperbolic space. The $ds^2 = dt^2 + \cosh^2(t)du^2$ version measures the distance from a point in the hyperbolic space to a line geodesic in the hyperbolic space.

***Linear Algebra***

We are interested in the linear algebra of $\mathbb{R}^4$ as it is associated with $\mathbb{C}^2$; $\mathbb{R}^4$ is equivalent to $\mathbb{C}^2$ when we change the coordinate system from real to complex. . In rectangular coordinates, the $\mathbb{J}$ -*image* of $(\mu,\lambda,\eta,\tau)$ is $(-\lambda,\mu,-\tau,\eta)$; this definition is based on the complex coordinate version of the $\mathbb{J}$ -*image*, which is $i(\mu+i\lambda,\eta+i\tau)=(i\mu-\lambda,i\eta-\tau)$ if the original is $(\mu+i\lambda,\eta+i\tau)$; in both cases, $\mu,\lambda,\eta,\tau\in\mathbb{R}$. When we find the complex version of the $\mathbb{J}$ -image of $\mathbb{R}^4$, we see that we are actually recovering $\mathbb{C}^2$. The inner product for $\mathbb{R}^4$ is $((\upsilon_1,\upsilon_2,\upsilon_3,\upsilon_4),(\omega_1,\omega_2,\omega_3,\omega_4))=\upsilon_1\omega_1+\upsilon_2\omega_2+\upsilon_3\omega_3+\upsilon_4\omega_4$ in real coordinates and is $(\upsilon,\omega)=((\upsilon_1+i\upsilon_2),(\omega_1+i\omega_2))=((\upsilon_1,\upsilon_2),(\omega_1,\omega_2))=\upsilon_1\bar{\omega}_1+\upsilon_2\bar{\omega}_2$ in complex coordinates.

The basis $\{e_1=(1,0,0,0),e_2=(0,1,0,0),e_3=(0,0,1,0),e_4=(0,0,0,1)\}$ which will be important later, other than being the basis of $\mathbb{R}^4$, is derived by fixing $\upsilon=(1,0,0,0)$ and finding $\omega's$ in which the $span(\upsilon,\omega)$ is called *totally real* or *complex*. The $span(\upsilon,\omega)$ is called *totally real* if the inner product of $span(\upsilon,\omega)$ is orthogonal to its $\mathbb{J}$ -image. The $span(\upsilon,\omega)$ is called *complex* if it is $\mathbb{J}$ -invariant, which means that a vector in the $\mathbb{J}$ -image of the $span(\upsilon,\omega)$ is in the $span(\upsilon,\omega)$; that is $\mathbb{J}(span(\upsilon,\omega))=span(\upsilon,\omega)$.

We can compute the stabilizers of $\mathbb{R}^2$ in $U(2)$ as $\{\begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\}$ where $\alpha\in[0,2\pi]$; since orthogonality is preserved by matrices in $U(2)$, the same matrices stabilize $\mathbb{J}\mathbb{R}^2$. We find the stabilizers of $\mathbb{R}^2$ in $U(2)$ by calculating the union of the stabilizers of $\mathbb{R}^2$ in $U(1)\subset U(2)$ and the stabilizers of $\mathbb{R}^2$ in $SU(2)\subset U(2)$ since $U(2)=U(1)\cup SU(2)$. The matrices $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\}$ stabilize $\mathbb{R}^2$ in $U(1)$ and the

matrices

$$\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} r & \rho \\ \rho & r \end{pmatrix}, \begin{pmatrix} -r & -\rho \\ -\rho & -r \end{pmatrix}, \begin{pmatrix} r & -\rho \\ -\rho & r \end{pmatrix}, \begin{pmatrix} -r & \rho \\ \rho & -r \end{pmatrix}\}$$

where $r, \rho \in \mathbb{R}$. Since the union of these sets of matrices covers all possible matrices with real entries, we can express the entries with cosine and sine in the form mentioned above.

The stabilizer of $\mathbb{R}^2$ in $\mathbb{C}^2$ is $\{\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\} \subset U(2)$ where $\alpha \in [0, 2\pi]$. Let's see how it acts on $\mathbb{C}^2$. We begin by taking $x+iy, a+ib \in \mathbb{C}$ and observing $\begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\begin{pmatrix} x+iy \\ a+ib \end{pmatrix} = \begin{pmatrix} x\cos\alpha + iy\cos\alpha + a\sin\alpha + ib\sin\alpha \\ -x\sin\alpha - iy\sin\alpha + a\cos\alpha + ib\cos\alpha \end{pmatrix}$ which equals $\begin{pmatrix} x\cos\alpha + a\sin\alpha + i(y\cos\alpha + b\sin\alpha) \\ -x\sin\alpha + a\cos\alpha + i(b\cos\alpha - y\sin\alpha) \end{pmatrix}$. We can see that this stabilizer set's elements are rotations on $\mathbb{R}^2$ and they are also simultaneous rotations of vectors orthogonal to vectors in $\mathbb{R}^2$; $\mathbb{J}\mathbb{R}^2$ is formed by these orthogonal vectors. The stabilizer preserves angles, so each vector in $\mathbb{J}\mathbb{R}^2$ is rotated by the same angle. Thus, this stabilizer acts as rotations simultaneously on $\mathbb{R}^2$ and $\mathbb{J}\mathbb{R}^2$.

***Complex Hyperbolic Geometry***

We will be working with the ball model of the complex hyperbolic plane. This means we will be investigating $\mathbb{RH}^2 \subset \mathbb{CH}^2$ where we will model $\mathbb{CH}^2$ as the unit ball in $\mathbb{C}^2$ and $\mathbb{RH}^2 = \mathbb{R}^2 \cap \mathbb{CH}^2$. $\mathbb{RH}^2 = \{x \in \mathbb{R}^3 : \| x \|^2 = -1\}$ is a sphere of unit radius in the Lorentzian space of $\mathbb{R}^3$, $\mathbb{R}^{2,1}$, and within the Lorentzian space $\mathbb{R}^{2,1}$ the sphere is an imaginary sphere with the negative curvature [3, page 63].

As a totally real geodesic two-dimensional plane, $\mathbb{RH}^2$ is stabilized by matrices in $PU(2,1)$ which are conjugate to matrices in $SO(2,1)$, [1, page 2]. Similarly to how elements of the special orthogonal group $SO(3)$ of 3x3 matrices are isometries of $\mathbb{R}^3$ which preserve the origin, we have that elements of the special orthogonal group of 3x3 matrices $SO(2,1)$ are isometries of $\mathbb{R}^{2,1}$ which preserve the origin. The Lorentzian space $\mathbb{R}^{2,1}$ is the inner product space which is $\mathbb{R}^3$ with a different inner product, the Lorentzian inner product which is $< x, y >= x_1 y_1 + x_2 y_2 - x_3 y_3$, [3, page 56]. Note that the typical inner product which $\mathbb{R}^3$ has is $< x, y >= x_1 y_1 + x_2 y_2 + x_3 y_3$. Both groups, $SO(3)$ and $SO(2,1)$, consist of orthogonal matrices with determinants equal to 1. Just as $SO(3)$ does, the matrices in the group $SO(2,1)$ act as rotations about the origin. So, these matrices will be isometries which act as translations along $\mathbb{RH}^2$ and rotations around the origin of $\mathbb{RH}^2$; the translations actually come from a simultaneous rotation within $\mathbb{JRH}^2$ around $\mathbb{RH}^2 \cap \mathbb{JRH}^2$. The ball model of $\mathbb{CH}^2$ is stabilized at the origin by $U(2) \subset PU(2,1)$. In order to see what stabilizes $\mathbb{RH}^2$ and $\mathbb{JRH}^2$ in $\mathbb{CH}^2$, we need to see what stabilizes $\mathbb{R}^2$ and $\mathbb{JR}^2$ in $\mathbb{C}^2$. We can refer to the linear algebra section where we computed these stabilizers and see how these rotations occur.

***The Metric on the Complex Hyperbolic Plane as defined by the Real Hyperbolic Plane***

We are observing the metric on $\mathbb{RH}^2 \subset \mathbb{CH}^2$. If we take an r-tube $F(r)$ around $\mathbb{RH}^2$, since $\mathbb{RH}^2$ is convex in $\mathbb{CH}^2$ and the simply connected space $\mathbb{CH}^2$ has non-positive curvature we know a projection exists from $\mathbb{CH}^2$ to $\mathbb{RH}^2$ such that a point $v \in F(r) \subset \mathbb{CH}^2$ maps to a unique point $z \in \mathbb{RH}^2$. This point $v \in F(r) \subset \mathbb{CH}^2$ can be moved by isometries, those reviewed above, to a point on the radial geodesic centered about the origin of $\mathbb{RH}^2$. So, we look at this single point $v \in F(r) \subset \mathbb{CH}^2$ which is on the radial geodesic centered about the origin of $\mathbb{RH}^2$; then we find the metric on the tangent space, $\mathbb{TCH}^2$ of $\mathbb{CH}^2$ at $v$. We see that $\mathbb{CH}^2$ decomposes uniquely into $span(\frac{\partial}{\partial r}) \oplus \mathbb{T}F(r)$ where $\mathbb{T}F(r)$ is the three dimensional tangent space to $F(r)$ and $span(\frac{\partial}{\partial r})$ is a projective space of $\frac{\partial}{\partial r}$ which is the one-dimensional orthogonal complement of $\mathbb{T}F(r)$. The metric on $\mathbb{CH}^2$ can be described by the sum of the metric on $span(\frac{\partial}{\partial r})$ which is simply $dr^2$ and the Riemannian metric on $\mathbb{T}F(r)$.

Since the curvature of a space is defined by its two-dimensional planes, we can find the metric on $\mathbb{T}F(r)$ by finding the metrics on two-dimensional planes. The basis $\{e_1 = (1,0,0,0), e_2 = (0,1,0,0), e_3 = (0,0,1,0), e_4 = (0,0,0,1)\}$ for $\mathbb{R}^4$ is helpful in our situation because it is also a basis of vectors in which pairs of these vectors lead to only totally real and complex subspaces since the $span(e_i, e_j)$ is either totally real or complex for $i \neq j$ and $i, j = 1,2,3,4$. Note that each $span(e_i, e_j)$ is a two-dimensional plane. If we fix $e_1 = (1,0,0,0)$, then there are three spans which we can observe which should be either totally real or complex. This is an important quality since we know the curvatures of totally real and complex spaces as well as knowing that these two-dimensional spans can be modeled by the hyperbolic plane models reviewed above. By calculations based on the review above, we find the $span(e_1, e_2)$ and $span(e_3, e_4)$ are complex; we find the $span(e_1, e_3), span(e_1, e_4), span(e_2, e_3), span(e_2, e_4)$ to be totally real. As an example of how these calculations work, we will look into the details of one of the complex and one of the totally real situations. Let's first look at the $span(e_1, e_2)$:

$$e_1 = (1,0,0,0) \text{ and } e_2 = (0,1,0,0)$$
$$span(e_1, e_2) = (\mu, \lambda, 0, 0) \text{ for some } \mu, \lambda \in \mathbb{R}$$

The $\mathbb{J}$-image is $(-\lambda, \mu, 0, 0)$, which is still in the $span(e_1, e_2)$ since $-\lambda, \mu \in \mathbb{R}$. We can see that this calculation shows us that the $span(e_1, e_2)$ is a complex plane by observing how the complex coordinates work.

$$span(e_1, e_2) = (\mu + i\lambda, 0) \text{ with } \mathbb{J}\text{-image } i(\mu + i\lambda, 0) = (i\mu - \lambda, 0)$$

This indicates that the map from the span to its $\mathbb{J}$-image is a map from $\mathbb{C} \to \mathbb{C}$.

Now, if we look at the $span(e_1, e_3)$:

$$e_1 = (1,0,0,0) \text{ and } e_3 = (0,0,1,0)$$
$$span(e_1, e_3) = (\eta, 0, \tau, 0) \text{ for some } \eta, \tau \in \mathbb{R}$$

The $\mathbb{J}$-image is $(0, \eta, 0, \tau)$, which is orthogonal to the $span(e_1, e_3)$.

If we look at a specific set of $\eta, \tau \in \mathbb{R}$, then we have two vectors which are in the $span(e_1, e_3)$ and its $\mathbb{J}$-image, respectively. We can see that these two vectors are orthogonal by taking the typical inner product as seen here:

$$(\eta, 0, \tau, 0) \bullet (0, \eta, 0, \tau) = \eta \cdot 0 + 0 \cdot \eta + \tau \cdot 0 + 0 \cdot \tau = 0$$

Since this inner product equals 0, we see that any two vectors from these spaces are orthogonal by observing that these two vectors are orthogonal. The same type of calculations can be done for each of the other four spans formed by this basis.

Since we have fixed the position of $e_1$, we only need to observe three of these six spans due to symmetry. If we observe the three spans $span(e_1,e_2), span(e_1,e_3), span(e_1,e_4)$, we have one complex subspace of $\mathbb{R}^4$ which is $span(e_1,e_2)$ and two totally real subspaces of $\mathbb{R}^4$ which are $span(e_1,e_3), span(e_1,e_4)$. Let this basis of $\mathbb{TCH}^2$ specifically at the point $v \in F(r) \subset \mathbb{CH}^2$ be $\{\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial t}, \frac{\partial}{\partial m}\}$ where $\dfrac{\partial}{\partial r}$ is in the direction of the radius of the circle, $\dfrac{\partial}{\partial \theta}$ is in the direction of the angle of rotation of the circle, $\dfrac{\partial}{\partial t}$ and $\dfrac{\partial}{\partial m}$ are orthogonal to each other and both of those. If we observe a radial geodesic of $\mathbb{JRH}^2$ in the direction of $e_1 = (1,0,0,0)$, then $\dfrac{\partial}{\partial r}$ is the vector field, in $\mathbb{JRH}^2$, tangent to this radial geodesic. If we intersect each of the three two-dimensional subspaces of $\mathbb{R}^4$, found as spans of basis vectors of $\mathbb{R}^4$, with $F(r)$ then we will have three curves in $\mathbb{R}^4$ with which the other orthogonal vector fields forming the basis of $\mathbb{T}_v\mathbb{CH}^2$ will be tangent. Thus, we have that $\dfrac{\partial}{\partial m}$ tangent to the curve $span(e_1,e_2) \cap F(r)$, $\dfrac{\partial}{\partial \theta}$ tangent to the curve $span(e_1,e_3) \cap F(r)$, and $\dfrac{\partial}{\partial t}$ tangent to the curve $span(e_1,e_4) \cap F(r)$ or $\dfrac{\partial}{\partial m}$ tangent to the curve $span(e_1,e_2) \cap F(r)$, $\dfrac{\partial}{\partial t}$ tangent to the curve $span(e_1,e_3) \cap F(r)$, $\dfrac{\partial}{\partial \theta}$ tangent to the curve $span(e_1,e_4) \cap F(r)$.

Let's look at the totally real plane $\mathbb{JRH}^2$ defined by the $span(e_1,e_3)$ or $span(e_1,e_4)$. Since $\mathbb{JRH}^2$ intersects $\mathbb{RH}^2$ in a single point, then we use the polar coordinates model of the hyperbolic plane as reviewed above to measure this totally real plane and this leads to the metric of this plane being $\sinh^2(\frac{r}{2})d\theta^2$; note that $r/2$ is necessary to achieve the curvature of this totally real plane which is - $1/4$. Then we need to look at $span(e_1, e_2)$ and the remaining plane of the $span(e_1, e_3)$ or $span(e_1, e_4)$. These other two subspaces are two-dimensional planes which intersect $\mathbb{RH}^2$ in a geodesic which is a line; thus the metric for these two planes are $\cosh^2(\frac{r}{2})ds^2$ to measure the totally real plane with curvature equal to - $1/4$ and $\cosh^2(r)dt^2$ for the complex plane, $span(e_1, e_2)$ with curvature equal to -1. All of this together gives us the metric for $\mathbb{RH}^2 \subset \mathbb{CH}^2$ as

$$dr^2 + \sinh^2(\tfrac{r}{2})d\theta^2 + \cosh^2(\tfrac{r}{2})ds^2 + \cosh^2(r)dt^2 .$$

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## *References*

1. "COMPLEX HYPERBOLIC MANIFOLDS: HOMOTOPY EQUIVALENT TO A RIEMANN SURFACE" by William M. Goldman, Michael Kapovich, and Bernhard Leeb.
2. Fuchsian Groups by Svetlana Katok; The University of Chicago, 1992.
3. Graduate Texts In Mathematics: Foundations of Hyperbolic Manifolds by John G. Ratcliffe; 1994 Springer-Verlag New York, Inc.
4. "Manifolds of Negative Curvature" by R.L. Bishop and B. O'Neil. *Transactions of the American Mathematical Society*, Vol. 145. (Nov., 1969), pp.1-49.