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Discrete versions of the Kadomtsev-Petviashvili hierarchy and the bispectral problem

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Contents

Introduction

The Korteweg-de Vries equation (in short KdV)

$$u_t = 6uu_x + u_{xxx} \tag{1}$$

arose in the 19$^{\text{th}}$ century in connection with the theory of waves in shallow water. It describes also the propagation of waves with weak dispersion in various nonlinear media.

In 1967, Gardner, Green, Kruskal and Miura proposed the method of the inverse scattering problem for solving the KdV equation [28]. This procedure was further developed by numerous authors. In [59], Lax discovered that the KdV equation can be rewritten in the operator form

$$\frac{dL}{dt} = [A, L] \tag{2}$$

where

$$L = \frac{d^2}{dx^2} + u \tag{3}$$

and

$$A = \frac{d^3}{dx^3} + \frac{3}{4}\left(u\frac{d}{dx} + \frac{d}{dx} \cdot u\right). \tag{4}$$

From this representation, it follows that the spectrum of the operator L gives constants of the motion for the KdV equation. This simple fact reveals the algebraic structure and turns out to be extremely useful not only for the KdV equation, but for the application of the method to many other problems. In particular for each n there exists a skew-symmetric operator A_n of order $2n+1$ ($A_1 = A$ defined by (4)), whose coefficients are differential polynomials of $u(x)$, and such that $[A_n, L]$ is a zero order operator. Thus one can define "higher order" KdV equations by

$$\frac{\partial L}{\partial t_n} = [A_n, L], \qquad n \in \mathbb{Z}_+. \tag{5}$$

An important feature of the family of equations (5) is that the flows $\partial/\partial t_n$ commute with each other. Thus each of the evolution equations (5) generates a symmetry of all other equations. For this reason, the set of equations (5) is called a hierarchy.

Gardner [27] and Zakharov-Faddeev [80] have shown that the KdV hierarchy can be represented as a Hamiltonian system with respect to a certain Poisson bracket. Gelfand

and Dickey [30] generalized the KdV hierarchy to the case when L is an operator of arbitrary order and proved that the equations (5) are Hamiltonian. In [61], Magri discovered that the KdV equation is even bi-Hamiltonian. The second Poisson structure in the general case was found by Adler [1].

The Cauchy problem for the KdV equation with periodic initial condition was studied by Novikov [68], Dubrovin and Novikov [22], Its and Matveev [47], McKean and van Moerbeke [64] and McKean and Trubowitz [63]. The common eigenfunction to the Schrödinger operator (3) and the operator of translation by a period, known in quantum physics as the Bloch function, is in general defined on a Riemann surface of infinite genus. This corresponds to the fact that generally there are infinitely many gaps in the spectrum of the Schrödinger operator.

Novikov [68] observed that in the special case where there are only finitely many gaps in the spectrum, say g gaps, there exists an operator

$$B = A_g + \sum_{k=0}^{g-1} c_k A_k$$

of order $2g + 1$ commuting with L, that is

$$[L, B] = 0. \tag{6}$$

The equations of the form (6) are called the stationary equations. They are, in fact, ordinary differential equations for the potential $u(x)$. Hence the manifold of their solutions is finite dimensional. It is remarkable that if a solution $u(x)$ of these equations is taken as an initial condition for the KdV hierarchy, at each moment t, $u(x, t)$ remains a solution (as a function of x) of the stationary equation. Thus, finite-dimensional invariant submanifolds in the infinite dimensional phase space of the non stationary equation can be obtained. This yields finite dimensional classes of algebro-geometric solutions, which include in particular the soliton solutions.

The stationary equations can also be considered as finite dimensional Hamiltonian systems. They have sufficiently many first integrals in involution to be integrated using only quadratures, according to the Liouville-Arnold theorem. Moreover, one can write explicit formulae for these solutions, in general in terms of theta functions.

In a series of papers, Krichever proposed an algebro-geometrical construction of a broad class of periodic and conditionally periodic solutions of the general Zakharov-Shabat equation

$$\frac{\partial L}{\partial t_1} - \frac{\partial A}{\partial t_2} = [A, L], \tag{7}$$

which, in particular, contains the physically important Kadomtsev-Petviashvili equation (KP in short). More precisely, Krichever associates a solution of (7) to a non-singular algebraic curve X, with distinguished point P_∞ and a non-special divisor $D = P_1 + \cdots + P_g$ on $X \setminus P_\infty$, where g is the genus of X. In addition, this construction led him to the apparently unrelated problem of classifying commutative rings of ordinary differential operators, which contain a pair of operators with relatively prime orders, see the survey [56]. Mumford [66] pointed out that the construction still applies more or less unchanged if we allow X to be any complete irreducible complex algebraic curve (possibly singular).

The inclusion of singular curves is important since the n-soliton solutions correspond to rational curves with n double points and the rational solutions correspond to rational curves with only cusp-like singularities.

The most powerful tool for studying the KP hierarchy was developed by the Kyoto school, starting with the works of Sato [71] and Date, Jimbo, Kashiwara and Miwa [17]. It consists of assigning a solution of the KP hierarchy to each point of a certain infinite dimensional Grassmannian Gr, called today the Sato Grassmannian. The class of solutions obtained in this way includes the explicit algebro-geometric solutions, see [72]. One of the main ideas of the Japanese mathematicians (which goes back to the earlier work of Hirota [42]) is to represent the solutions of the KP hierarchy in terms of some universal function, called the tau function, which can be thought of as a generalization of the classical theta function.

One of the most puzzling discoveries in the theory of the integrable systems is the connection between the motion of the poles of certain rational solutions of integrable partial differential equations and completely integrable systems of ordinary differential equations. In our case, concerning the rational solutions of the KdV and the KP hierarchy, the relevant classical dynamical system is the Calogero-Moser hierarchy. The Calogero-Moser system represents the motion of N particles on the line under the influence of an inverse square potential. More precisely, if $x_1 < x_2 < \cdots < x_N$ denote the positions and $y_1, y_2, \ldots, y_N$ denote the velocities, the motion is governed by the Hamiltonian system

$$\begin{aligned} \dot{x}_j &= \frac{\partial H}{\partial y_j}, \\ \dot{y}_j &= -\frac{\partial H}{\partial x_j}, \end{aligned}$$

with Hamiltonian

$$H = \frac{1}{2}\sum_{j=1}^{N} y_j^2 + \sum_{j<k} \frac{1}{(x_j - x_k)^2}.$$

In [16] Calogero found that the corresponding quantum theoretical problem of N mass points on a line can be be solved completely and conjectured that the classical problem might be integrable. In [65] Moser confirmed this conjecture by proving that if Y denotes the $N \times N$ matrix

$$\begin{aligned} Y_{jj} &= y_j \\ Y_{kl} &= \frac{i}{x_k - x_l}, \qquad k \neq l, \end{aligned} \tag{8}$$

then

i) the eigenvalues of Y are constants of the motion;

ii) they are in involution.

Let us denote by H_n the trace of Y^n, i.e.

$$H_n = \mathrm{tr}(Y^n). \tag{9}$$

Obviously $H_2 = 2H$. The involutive character of the eigenvalues of Y implies that the Hamiltonian flows

$$\frac{\partial}{\partial t_n}\begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} \partial H_n/\partial y_j \\ -\partial H_n/\partial x_j \end{pmatrix}, \qquad n \in \mathbb{Z}_+ \tag{10}$$

commute with each other. Thus we naturally call the hierarchy of equations (10) the Calogero-Moser hierarchy. In the rest of the thesis, we shall refer to the first two non-trivial equations (10) for $n = 2$ and $n = 3$ as the Calogero-Moser system.

In [7] Airault, McKean, and Moser discovered an amazing relation between the equations of KdV type and the Calogero-Moser system. Namely, they showed that the motion of the poles of a rational solution to the KdV or Boussinesq equation[1] that vanishes at infinity, is described by the Calogero-Moser system with some constraint on the configuration of poles. Krichever [57] observed that the poles of the rational solutions of the KP equation that vanish at $x = \infty$, move according to the Calogero-Moser system with no constraints. Finally, Shiota [74] extended this phenomenon to the whole KP hierarchy.

It is interesting that the rational solutions of the KP hierarchy appear in the (seemingly unrelated) bispectral problem. As originally formulated by Duistermaat and Grünbaum [23], this problem asks for which ordinary differential operators $L(x, d/dx)$ there exists a family of eigenfunctions $\Psi(x, z)$ that are also eigenfunctions for another differential operator $B(z, d/dz)$ in the "spectral parameter" z, i.e.

$$L(x, d/dx)\Psi(x, z) = f(z)\Psi(x, z), \tag{11}$$

$$B(z, d/dz)\Psi(x, z) = \theta(x)\Psi(x, z) \tag{12}$$

for some functions $f(z)$ and $\theta(x)$. This problem first appeared in [31] in connection with "limited angle tomography". The first general result in the direction of classifying bispectral operators belongs to Duistermaat and Grünbaum [23]. They determined all second order bispectral operators L. The answer is as follows. If we write the operator L in the standard Schrödinger form

$$L = \frac{d^2}{dx^2} + u(x),$$

the bispectral potentials $u(x)$ are given (up to translations and rescalings of x and z) by the following list:

$$u(x) = x \qquad \text{(Airy);} \tag{13}$$

$$u(x) = cx^{-2},\ c \in \mathbb{C} \qquad \text{(Bessel);} \tag{14}$$

$u(x)$, which can be obtained by finitely many rational Darboux transformations from $u(x) = 0$; (15)

$u(x)$, which can be obtained by finitely many rational Darboux transformations from $u(x) = \dfrac{1}{4x^2}$. (16)

As it was already shown by Adler and Moser [4], the family (15) coincides with the rational solutions of the KdV hierarchy vanishing as $x \to \infty$. These solutions can be

[1] The Boussinesq equation [14] is the first non-trivial equation in the third Gelfand-Dickey hierarchy.

obtained also by applying the higher KdV flows to the potentials $u_k(x) = -k(k+1)/x^2$, $k \in \mathbb{Z}_+$. The second family (16) was more mysterious at the time of writing [23]; Magri and Zubelli [82] succeeded to interpret it as potentials invariant under the flows of the "master symmetries" or "Virasoro flows" of the KdV hierarchy.

A decisive new step in the study of the bispectral problem was undertaken by G. Wilson [78] who proposed to extend the problem to commutative rings of differential operators. Such a ring will be called bispectral, when there is a joint eigenfunction of the operators in the ring that is also a joint eigenfunction of a ring of differential operators in the spectral variable. Notice that a bispectral operator always belongs to a bispectral ring, namely the ring generated by its powers, so the interest lies in the maximal commutative ring which contains it. An important invariant of such a ring is its rank, meaning the dimension of the common space of eigenfunctions to the operators belonging to the ring. According to Burchnall-Chaundy-Krichever classification [15, 56], there is a dictionary connecting these rings to the theory of algebraic curves. Specifically, in the case of rank 1, the spectrum of the ring in the sense of algebraic geometry is an affine irreducible complex curve which completes by adding one point at infinity. The common eigenfunction is the so-called Baker-Akhiezer function; when the curve is non-singular it can be expressed in terms of Riemann's theta function. G. Wilson proved that a maximal rank 1 commutative ring of differential operators is bispectral if and only if the corresponding curve is unicursal, that is there is a bijective map from the complex projective line to the curve. This condition is equivalent to the fact that the Baker-Akhiezer function is a wave function for a rational solution to the KP hierarchy, which we mentioned above. His beautiful idea was that exchanging the rôle of the "space" and the "spectral" variables in the Baker-Akhiezer function of such a curve produces a Baker-Akhiezer function of another curve of the same type. In the framework of the Segal-Wilson Grassmannian [72], the tau functions of these curves are parametrized by a sub-Grassmannian Gr^{ad} that he called the adelic Grassmannian. The bispectrality is a consequence of the following symmetry in Gr^{ad}: if $W \in Gr^{ad}$ then there is a plane $W' \in Gr^{ad}$ such that the stationary wave functions of W and W' satisfy

$$\bar{\Psi}_W(x,z) = \bar{\Psi}_{W'}(z,x).$$

The map $b: W \to W'$ is called the *bispectral involution*.

In an attempt to make the bispectral involution more transparent, Wilson [79] re-examined the connection between the rational solutions of the KP hierarchy and the Calogero-Moser systems. Inspired by the beautiful geometric approach of Kazhdan, Kostant and Sternberg [50] to the Calogero-Moser hierarchy, he gave the following description of the adelic Grassmannian. Let $\tilde{\mathcal{C}}_N$ be the space of all pairs (X,Z) of $N \times N$ complex matrices such that $[X,Z]+E_N$ has rank one, with E_N - the identity matrix. Denote by $\mathcal{C}_N$ the quotient space $\tilde{\mathcal{C}}_N/GL(N,\mathbb{C})$, where $GL(N,\mathbb{C})$ acts on pairs of matrices by simultaneous conjugation. The Calogero-Moser flows, are the flows on $\mathcal{C}_N$, induced by the very simple $GL(N,\mathbb{C})$-invariant flows

$$(X,Z) \to (X + nt_n(-Z)^{n-1}, Z), \tag{17}$$

on $\tilde{C}_N$. Indeed, in the case when $X = (x_1,\ldots,x_N)$ is diagonalizable, we can choose coordinates $\{x_j, y_j\}_{j=1}^N$ on $\mathcal{C}_N$, where

$$Z_{jj} = y_j$$
$$Z_{kl} = \frac{1}{x_k - x_l}, \qquad k \neq l.$$

Applying the symplectic reduction procedure, one can see that the flows (17) translate exactly into the Calogero-Moser flows (10), modulo some rescalings[2]. Thus, the variety $\mathcal{C}_N$ can be thought of as a completed phase space for the N-particle Calogero-Moser hierarchy, see [50, 79] for details.

For each pair $(X, Z) \in \mathcal{C}_N$ the function

$$\bar{\Psi}(x, z) = \det \left\{ E_N - (xE_N + X)^{-1}(zE_N + Z)^{-1} \right\} e^{xz}$$

is the stationary wave function of a plane W from Gr^{ad}. Moreover, this construction defines a bijective map

$$\bigsqcup_{N \geq 0} \mathcal{C}_N \to Gr^{ad}.$$

In particular, this bijection gives a very simple explanation of the bispectral involution: on pairs of matrices it corresponds to the map

$$\beta : (X, Z) \to (Z^t, X^t).$$

The classification of the bispectral commutative rings of ordinary differential operators of rank higher than 1 seems to be much more subtle. A broad class of examples was constructed by Bakalov, Horozov and Yakimov [10, 11] and Kasman and Rothstein [49], but the complete answer is still unknown.

In a series of papers [32, 33, 34] Grünbaum and Haine started a study of a discrete version of the original problem by replacing L by a doubly infinite tridiagonal matrix. If one imposes special boundary conditions on the joint eigenfunctions, this problem contains the classical problem of classifying orthogonal polynomials which are eigenfunctions of a differential operator (of arbitrary order), going back to the works of Bochner [13] and Krall [54, 55]. In [35], the authors went even further by replacing B by a second order q-difference operator. The solution is described in terms of an arbitrary solution of a q-analogue of Gauss' hypergeometric equation depending on five free parameters and extends the four dimensional family of solutions given by the well known Askey-Wilson polynomials [8]. For a comprehensive review of the "difference, differential (q-difference)" bispectral problem we refer the reader to the recent survey paper [37].

In the present thesis we shall consider different q-difference and difference versions of the KP hierarchy. In [25], Frenkel proposed a q-deformation of the N-th KdV hierarchy by replacing the differential operator d/dx by the q-shift operator D, acting on functions of x by

$$Df(x) = f(qx).$$

[2] Note that Z differs slightly from the matrix Y used before.

He proved that the corresponding hierarchy is Hamiltonian with respect to the quantum Poisson algebra $\mathcal{W}_q(\mathfrak{sl}_N)$ defined in [26]. A similar deformation of the KP hierarchy was obtained by Khesin, Lyubashenko and Roger [51], who considered a certain q-deformation of the Lie algebra of pseudo-differential operators on the circle of the form

$$L = \tilde{D} + \sum_{j=0}^{\infty} a_j \tilde{D}^{-j},$$

where

$$\tilde{D}f(x) = \frac{f(qx) - f(x)}{q-1}.$$

Thus, the operator L defined above differs by an unessential (constant) factor $(q-1)$ from the one used by Frenkel. In the rest of the thesis, we shall refer to this hierarchy as FKLR hierarchy, see also [69].

In [40], a slightly different deformation of the KP hierarchy was proposed; the differential operator was replaced by the q-derivative $D_{q,x}$, acting on functions of x as

$$D_{q,x}f(x) = \frac{f(qx) - f(x)}{x(q-1)}.$$

It was shown that by making an appropriate shift in the arguments of the classical Schur polynomials, one obtains rational solutions of the deformed hierarchy. In [43], we defined a q-analogue of the classical tau function and proved that the same shift in any classical tau function leads to a solution of the deformed hierarchy. This result was independently obtained in [3], using the correspondence with the Toda lattice hierarchy, see also [6]. Finally, in [45, 46] we have shown that this shift characterizes the q-tau functions in the ring of formal power series.

Theorem 1 ([45, 46]). *A formal power series $\tau_q(x,t) \in \mathbb{C}[[x, t_1, t_2, \dots]]$ is a tau function for the q-KP hierarchy if and only if, up to an unessential factor depending only on x, we have*

$$\tau_q(x,t) = \tilde{\tau}(t + [x]_q), \tag{18}$$

where $\tilde{\tau}(t) \in \mathbb{C}[[t_1, t_2, \dots]]$ is a tau function for the classical KP hierarchy, and

$$[x]_q = \left(x, \frac{(1-q)^2}{2(1-q^2)}x^2, \frac{(1-q)^3}{3(1-q^3)}x^3, \dots \right).$$

Using this result, we can construct rank 1 commutative rings $\mathcal{A}_W^q$ of q-difference operators from any plane $W \in Gr$, corresponding to an algebro-geometrical solution of the KP hierarchy. In Chapter 3, we define a deformation of Wilson's adelic Grassmannian Gr_q^{ad} and show that the corresponding rings are bispectral. Precisely, if A_W denotes the set of polynomials that leave W invariant, we have

Theorem 2 ([45, 46]). *For any plane $W \in Gr_q^{ad}$ the commutative ring of q-difference operators $\mathcal{A}_W^q$ is bispectral. More precisely, the stationary q-wave function $\bar{\Psi}_W^q(x,z)$ satisfies*

$$L_f(x, D_{q,x})\bar{\Psi}_W^q(x,z) = f(z)\bar{\Psi}_W^q(x,z) \tag{19}$$

for any $f(z) \in A_W$ *and, for any polynomial* $\theta(x)$ *such that* $D_{q,x}\theta(x)$ *is divisible by* $\tau_W^q(xq,t)|_{t=0}$*, there exists a* q*-difference operator in* z*,* $B_\theta(z, D_{q,z})$ *independent of* x *such that*

$$B_\theta(z, D_{q,z})\bar{\Psi}_W^q(x,z) = \theta(x)\bar{\Psi}_W^q(x,z). \tag{20}$$

The q-deformed Grassmannian Gr_q^{ad} is still contained in the sub-Grassmannian Gr^{rat} which parametrizes the solutions of the KP hierarchy arising from rational algebraic curves. For $W \in Gr_q^{ad}$, the corresponding q-tau function is a polynomial in x. Thus, Gr_q^{ad} parametrizes rational solutions (in x) to the q-KP hierarchy. The intersection $Gr_q^{ad} \cap Gr^{ad}$ coincides with the sub-Grassmannian Gr_0 whose tau functions are polynomials in only finitely many time variables $t_1, t_2, \dots$. As a consequence, the rational curves corresponding to planes $W \in Gr_q^{ad} \backslash Gr_0$ must have *at least one node as a singular point*, (cf. Example 3.2.3).

In the first section of Chapter 4, we consider a "generic" plane $W \in Gr_q^{ad}$ which corresponds to a N-soliton solution. We show that in this case, W is determined by a pair of (Calogero-Moser) matrices X and Y, satisfying the condition

$$\operatorname{rank}([X,Y]_q + E) = 1, \tag{21}$$

where E is the identity matrix and $[X,Y]_q$ denotes the q-commutator $XY - qYX$. Using this representation, we can give a simple description of the bispectral involution, extending the symmetry in Gr^{ad} to the $q \neq 1$ case.

Theorem 3 ([45, 46]). *Let* (X,Y) *be a pair of matrices which have* q*-different eigenvalues*[3] *and such that* (21) *holds. Let* W *and* W' *denote the planes in* Gr_q^{ad}*, corresponding to the pairs of matrices* (X,Y) *and* $(-qY^t, -q^{-1}X^t)$*, respectively. Then, the stationary wave functions* $\bar{\Psi}_W^q$ *and* $\bar{\Psi}_{W'}^q$ *satisfy*

$$\bar{\Psi}_W^q(x,z) = \bar{\Psi}_{W'}^q(z,x),$$

that is, on pair of matrices, the bispectral involution corresponds to the map

$$\beta : (X,Y) \to (-qY^t, -q^{-1}X^t).$$

The dynamics of the poles of the rational (in x) solutions are studied in the second section of Chapter 4. Since the poles come from zeros of the tau function, we put

$$\tau_q(x,t) = \prod_{j=1}^{N}(x - x_j(t)).$$

and define a certain matrix Y, which is a deformation of the Calogero-Moser matrix.

Theorem 4 ([45, 46]). *Let* $\tau_q(x,t) = \prod_{i=1}^N (x - x_i(t))$ *be a tau function of the* q*-KP hierarchy, which is a monic polynomial in* x*. Then the motion of the zeros of* τ_q *is governed by a hierarchy of Hamiltonian systems, which is a* q*-deformation of the Calogero-Moser hierarchy. Precisely, if we define*

$$H_n = (-1)^n \frac{[n]_q}{n} \operatorname{tr}(Y^n),$$

[3] that is, $\lambda_i \neq \lambda_j$ and $\lambda_i \neq q\lambda_j$

we have

$$\frac{\partial}{\partial t_n}\begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \partial H_n/\partial y_i \\ -\partial H_n/\partial x_i \end{pmatrix}, \qquad n = 1, 2, \ldots \tag{22}$$

The FKLR hierarchy was studied in [44], using the q-difference operator

$$\Delta_q f(x) = f(qx) - f(x).$$

The corresponding shift in the tau function is given by the following theorem.

Theorem 5 ([44]). *Let $\tau(t)$ be a tau-function of the classical KP hierarchy. Then*

$$\tau_q(x;t) = \tau(t + \{x\}_q), \tag{23}$$

where

$$\{x\}_q = \left(\frac{\log x}{\log q}, -\frac{1}{2}\frac{\log x}{\log q}, \frac{1}{3}\frac{\log x}{\log q}, \ldots\right) \tag{24}$$

is a tau-function of the FKLR hierarchy.

Note that the absence of x in the denominator of Δ_q, makes it essentially equivalent to the standard difference operator Δ

$$\Delta f(n) = f(n+1) - f(n).$$

under the substitution $x = q^n$. Thus the previous theorem naturally leads us to a construction assigning a solution of the Discrete KP hierarchy (DKP) to any solution of the (continuous) KP hierarchy. We adopt the point of view of [41], viewing the discrete KP hierarchy as the classical KP hierarchy, with the continuous derivative $\partial = d/dx$ replaced by the discrete derivative Δ. This has the effect that instead of building a Sato type theory for this hierarchy by conjugating first order pseudo-difference operators to the shift operator $\Lambda f(n) = f(n+1)$, as it is usually done, we build the theory by conjugating these operators to the discrete derivative operator Δ. Notice that although equivalent, these two approaches lead to a different definition of the tau function.

Theorem 6 ([41]). *Let $\tau(t)$ be a tau function for the continuous KP hierarchy. Then,*

$$\tau(n;t) = \tau(t_1 + n, t_2 - n/2, t_3 + n/3, \ldots) \tag{25}$$

is a tau function for the discrete KP hierarchy.

Using this theorem, we show that rank 1 commutative rings of difference operators can be systematically constructed from their differential analogues, making the shift (25). The study of rank 1 commutative rings of difference operators was done by P. van Moerbeke and D. Mumford [77], see also [58]. The case of singular curves was treated very completely in the related paper of Mumford [66]. An essential difference between the theory of rank 1 commutative rings of difference operators and the one of differential operators, is that in the difference case the spectrum of the ring is an irreducible complex affine curve which completes by adding *two* non-singular points at infinity, instead of just *one* non-singular point in the case of differential operators, see [58], [77] and [66]. In [41] it was shown that this is precisely the geometric meaning of the shift performed in the arguments of the tau function in (25). Starting with a tau function solution of the continuous KP hierarchy corresponding to a non-singular curve, that is starting with Riemann's theta function multiplied by the exponential of a suitable quadratic form (see Proposition 1.3.5 for a precise formula), we show that the wave function of the discrete KP hierarchy corresponding to the tau function (25) is exactly the Baker-Akhiezer function of a maximal rank 1 commutative ring of difference operators. This is the content of the following theorem.

Theorem 7 ([41]). *Let $\tau(t)$ be the tau function for an algebro-geometrical solution of the KP hierarchy arising from a non-singular curve X with distinguished point P_∞, and let $\tau(n;t)$ be the discrete KP tau function defined in Theorem 6. Then, the corresponding wave function of the discrete KP hierarchy coincides precisely with the Baker-Akhiezer function of the rank 1 commutative ring of difference operators associated with the affine curve $X \setminus \{P_\infty, Q_\infty\}$, with P_∞ and Q_∞ the points with respective coordinates $z = \infty$ and $z = -1$.*

Next, we consider explicitly the case of rational curves and prove that the same conclusion is true. To formulate the precise statement, note that starting with a plane $W \in Gr^{rat}$ the sequence of tau functions defined by (25) corresponds to a flag $\mathcal{W}$ of nested subspaces

$$\mathcal{W}: \qquad \cdots \subset W_{n+1} \subset W_n \subset W_{n-1} \subset \cdots ,$$

with

$$W_n = \operatorname{span}\{\Psi(n;0,z), \Psi(n+1;0,z), \dots\},$$

where $\Psi(n,t,z)$ is the wave function of the DKP hierarchy, corresponding to $\tau(n,t)$. Let us introduce the ring $A_{\mathcal{W}}$ of rational functions which preserve the flag $\mathcal{W}$ (see Chapter 5 for details). Then we have

Theorem 8 ([41]). *For each $f \in A_{\mathcal{W}}$, there is a finite band operator L_f with i diagonals above the main diagonal and j diagonals below it, with i and j denoting respectively the order of the poles of f at $z = \infty$ and $z = -1$, such that*

$$L_f \Psi(n;t,z) = f(z)\Psi(n;t,z). \tag{26}$$

Moreover,

$$\operatorname{Spec}(A_{\mathcal{W}}) = \operatorname{Spec}(A_W) \setminus \{Q_\infty\}, \tag{27}$$

where Q_∞ is the point with coordinate $z = -1$ on the complete irreducible rational curve $X = \operatorname{Spec}(A_W) \cup \{P_\infty\}$, $z(P_\infty) = \infty$.

In view of the connection between the bispectral problem and the theory of orthogonal polynomials mentioned above, it is natural to ask for a *difference* version of Wilson's result. Interestingly enough, the first section of Mumford's paper [66] ends up with the following comment: "And if X is unicursal,..., then we apparently get solutions in which the entries A_{ij} of the matrix are rational functions of i, j. This has not been fully worked out as yet." We show that the situation can be completely described by making the shift (25) in any tau function $\tau(t)$ constructed from a plane belonging to Wilson's adelic

Grassmannian. This sequence of tau functions corresponds to a flag of nested subspaces $\mathcal{W}$ each of which belongs to Gr^{ad}; thus we naturally call the manifold obtained in this way an adelic flag manifold. In fact the corresponding rings of difference operators enjoy a bispectral property reminiscent of the one satisfied by the classical orthogonal polynomials. We show that the common eigenfunction $\Psi(n,z)$ of the operators in the ring is also the common eigenfunction of a maximal rank 1 commutative ring of *differential* operators in the spectral variable z. The curve associated with this ring of differential operators is not unicursal in general, but it is a *rational curve* leading to a wave function $\Psi'(x,z)$ of *solitonic* type. The bispectral symmetry is expressed by

$$\Psi'(x,z) = \Psi(z, e^x - 1),$$

instead of $\Psi'(x,z) = \Psi(z,x)$ in the situation considered by G. Wilson. The appearance of $e^x - 1$ on the right-hand side of the above equation can be intuitively explained by the fact that the discrete derivative operator Δ can be symbolically written as $\Delta = e^{\partial} - 1$, with $\partial = d/dx$.

Theorem 9 ([41]). *Let $W \in Gr^{ad}$ and let $\bar{\Psi}(n,z)$ be the corresponding discrete KP wave function evaluated at $t = 0$. There exists a plane $W' \in Gr^{rat}$ whose stationary wave function $\bar{\Psi}_{W'}(x,z)$ satisfies*

$$\bar{\Psi}_{W'}(x,z) = \bar{\Psi}(z, e^x - 1). \tag{28}$$

As a consequence, the common eigenfunction $\bar{\Psi}(n,z)$ of the maximal rank 1 commutative ring of difference operators $A_{\mathcal{W}}$ constructed in Theorem 8, satisfying

$$L_f \bar{\Psi}(n,z) = f(z)\bar{\Psi}(n,z), \qquad \forall f \in A_{\mathcal{W}}, \tag{29}$$

is also the common eigenfunction of a maximal rank 1 commutative ring of differential operators $\mathcal{A}_{W'}$ (in the variable z), that is

$$B_\theta(z, d/dz)\bar{\Psi}(n,z) = \theta(n)\bar{\Psi}(n,z), \qquad \forall \theta \in A_{W'}. \tag{30}$$

Up to now, all solutions of the "difference, differential" bispectral problem were obtained by means of repeated Darboux transformations starting from the "basic" solutions - the classical orthogonal polynomials. Indeed, in [32] Grünbaum and Haine have shown that the Krall polynomials, which are eigenfunctions of a fourth order differential operator, could be obtained by application of the (matrix) Darboux transformation to some instances of the Laguerre and the Jacobi polynomials. In [36] it was shown that successive (matrix) Darboux transformations from the Laguerre polynomials lead to an extension of Koornwinder's generalized Laguerre polynomials [52], with weight distributions involving not only the delta function, but also its derivatives.

It is worth to observe that all these solutions of the bispectral problem correspond to rank 2 commutative rings both on the difference and the differential sides. As already suggested in [33, pp. 169-170], a natural way to obtain rank 1 solutions is to look for the tridiagonal matrices which are obtained by iteration of the (matrix) Darboux transformation from the discrete second derivative operator

$$L_0 = \Lambda - 2I + \Lambda^{-1} = \begin{pmatrix} \ddots & \ddots & & & & \\ \ddots & -2 & 1 & & & \\ & 1 & -2 & 1 & & \\ & & 1 & -2 & 1 & \\ & & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{31}$$

hoping to find the "true analogue" of the rational solutions of the Korteweg-de Vries hierarchy, which are obtained by iteration of the Darboux transformation from d^2/dx^2, see [4]. In [33] Grünbaum and Haine, computed the first few iterations of the Darboux transformation from L_0 and checked that, by choosing appropriately the Darboux free parameters, one obtains in this way rational solutions of the Toda lattice hierarchy and that the resulting operators provide solutions to the discrete-continuous version of the bispectral problem. Here we shall show that the resulting tridiagonal matrices can be obtained from Theorem 9 applied to a suitable plane $W \in Gr^{ad}$. The tau function of this plane $\tau_W(t)$ turns out to be a *polynomial* tau function of the KP hierarchy but, contrary to the case of the Korteweg-de Vries hierarchy, it is *not* a Schur polynomial. Moreover, the corresponding wave functions (thought of as a family of functions of z) satisfy an orthogonality relation on the circle analogous to that of the classical orthogonal polynomials (see (37)). Finally, the spectral curves of the rank 1 commutative rings constructed above are given by (35).

Theorem 10 ([41]). *The tridiagonal operator L, obtained by performing K elementary Darboux transformations from the discrete second derivative operator L_0 can be expressed in terms of a tau function $\tau(n,t)$ via the formula*

$$L = \Lambda + \left(-2 + \frac{\partial}{\partial t_1}\log\frac{\tau(n+1;t)}{\tau(n;t)}\right) I + \frac{\tau(n-1;t)\tau(n+1;t)}{\tau(n;t)^2}\Lambda^{-1}, \tag{32}$$

where $\tau(n,t)$ is constructed from the classical tau function $\tau_W(t)$, for some plane[4] *$W \in Gr^{ad}$ according to Theorem 6. After the change of variables*

$$t_i = s_i + \sum_{j=i+1}^{\infty} (-1)^{i+j}\binom{2j-i-1}{j-1} s_j, \tag{33}$$

L provides a rational solution (in n) to the Toda lattice hierarchy

$$\frac{\partial L}{\partial s_i} = [(L^i)_+, L]. \tag{34}$$

The operator L belongs to a maximal rank 1 *commutative ring of difference operators $A_{\mathcal{W}}$ with the affine curve* $\mathrm{Spec}(A_{\mathcal{W}})$ *defined by the equation*

$$\mathrm{Spec}(A_{\mathcal{W}}): \qquad y^2 = r^{2K+1}(r+1). \tag{35}$$

[4] see Chapter 5 for an explicit definition of W.

As a consequence, the function

$$p(n,x) \equiv \Psi(n;t,x-1)\exp\left(-\sum_{i=1}^{\infty} t_i(x-1)^i\right), \tag{36}$$

which is the common eigenfunction of the operators belonging to the ring $A_{\mathcal{W}}$, is also the common eigenfunction of a maximal rank 1 commutative ring of differential operators in the variable x, corresponding to some plane $W' \in Gr^{rat}$. Moreover, viewed as a family of functions of x, the functions $p(n;x)$ satisfy the orthogonality relations

$$\frac{1}{2\pi i}\oint p(n,x)p(m,x^{-1})\frac{dx}{x} = \frac{\tau(n+1;t)}{\tau(n;t)}\delta_{nm}, \ \textit{for all } n,m \in \mathbb{Z}, \tag{37}$$

where the integral is taken along any circle in the complex plane of center $x=0$ and radius $R \neq 1$.

Chapter 1

The KP hierarchy and an adelic Grassmannian

In this introductory chapter, we have collected some basic facts concerning the "classical" KP hierarchy, the bispectral problem and the Calogero-Moser hierarchy, which we shall need in the rest of the thesis. At the beginning of each section we give a set of references where more details can be found.

1.1 Formal pseudo-differential operators

In this section we define the ring $R\{\partial\}$ of formal pseudo-differential operators, which allows us to construct easily the whole KP hierarchy, see for example [18].

A formal pseudo-differential operator is, by definition, a formal series of the form

$$R = \sum_{i=-\infty}^{m} r_i(x)\partial^i, \qquad \partial = d/dx, \tag{1.1.1}$$

where $m \in \mathbb{Z}$ and $r_i(x)$ are some function of x. If $r_m \neq 0$, then $m = \text{ord}R$ is the order of R. We denote by $R\{\partial\}$ the set of all pseudo-differential operators. The product of two elements of $R\{\partial\}$ is defined by the following generalization of the Leibniz rule:

$$\partial^k \cdot r(x) = \sum_{i=0}^{\infty} \binom{k}{i} \frac{d^i r(x)}{dx^i} \partial^{k-i},$$

for any $k \in \mathbb{Z}$. It is straightforward to check that with the product defined above $R\{\partial\}$ becomes an associative ring.

An important feature of $R\{\partial\}$ is that we can define m-th root and inverse of any pseudo-differential operator R of the form (1.1.1) with $r_m = 1$. Indeed, one can easily show that there exist unique pseudo-differential operators $R^{1/m}$ and R^{-1} of the form

$$R^{1/m} = \partial + \sum_{i\leq 0} p_i\partial^i$$

$$R^{-1} = \partial^{-m} + \sum_{j<-m} q_j\partial^j,$$

such that

$$R = \left(R^{1/m}\right)^m \qquad \text{and} \qquad RR^{-1} = 1.$$

The operators R, R^{-1}, and $R^{1/m}$ commute with each other.

For a formal pseudo-differential operator R we shall write R_+ for the *differential operator part*

$$R_+ = \sum_{i\geq 0} r_i(x)\partial^i,$$

and R_- for the *integral (or Volterra) part*

$$R_- = \sum_{i<0} r_i(x)\partial^i.$$

Now, we are ready to give the definition of the KP hierarchy. Let

$$L = \partial + \sum_{i\geq 1} u_i(x,t)\partial^{-i}, \tag{1.1.2}$$

be a first order pseudo-differential operator, depending on an additional parameter $t = (t_1, t_2, \dots)$.

Definition 1.1.1. The Kadomtsev-Petviashvili hierarchy (in short KP hierarchy) is defined by the following Lax equations

$$\frac{\partial L}{\partial t_n} = [B_n, L], \qquad n = 1, 2, \dots, \tag{1.1.3}$$

where $B_n = (L^n)_+$.

The KP hierarchy is equivalent to the following set of Zakharov-Shabat equations

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m], \qquad m, n = 1, 2, \dots, \tag{1.1.4}$$

also called *zero curvature equations*. Using this representation, one can easily show that the vector fields $\partial_n = \partial/\partial t_n$ of the KP hierarchy commute with each other. Thus, each of the equations (1.1.3) generates a symmetry of every other equations, i.e. translation of a solution of one of these equations along another vector field transforms this solution into another solution of the same equation.

Example 1.1.2. From (1.1.2) one easily computes

$$B_2 = (L^2)_+ = \partial^2 + 2u_1$$
$$B_3 = (L^3)_+ = \partial^3 + 3u_1\partial + 3(u_1' + u_2),$$

where the prime stands for the derivation with respect to x. Now, equation (1.1.4) for $m = 2$, $n = 3$ gives the system of equations

$$\partial_2 u_1 = u_1'' + 2u_2'$$
$$\partial_2 u_1' + \partial_2 u_2 - \frac{2}{3}\partial_3 u_1 = \frac{u_1'''}{3} + u_2'' - 2u_1u_1'.$$

Eliminating u_2 we obtain

$$3\partial_2^2 u_1 = \left(4\partial_3 u_1 - u_1''' - 12u_1u_1'\right)',$$

which is nothing but the Kadomtsev-Petviashvili equation for $u = 2u_1$:

$$3\partial_2^2 u = \left(4\partial_3 u - u''' - 6uu'\right)'. \tag{1.1.5}$$

Remark 1.1.3. The first equation in (1.1.3) reads

$$\frac{\partial L}{\partial t_1} = [\partial, L] = \frac{\partial L}{\partial x}.$$

Thus, the first KP flow acts as translation in x, and we have $u_i(x,t) = u_i(t_1+x, t_2, t_3, \dots)$. Using this simple dependence of x, we shall identify t_1 and x.

Important reductions of the KP hierarchy are the KdV hierarchies. Suppose that there exists an integer N, such that the N-th power of L is a differential operator[1], i.e.

$$\tilde{L} := L^N = \partial^N + \tilde{u}_{N-2}\partial^{N-2} + \cdots + \tilde{u}_0. \tag{1.1.6}$$

From (1.1.3) it follows that the operator $\tilde{L}$ satisfies the equations

$$\frac{\partial \tilde{L}}{\partial t_n} = [(\tilde{L}^{n/N})_+, \tilde{L}], \qquad n = 1, 2, \dots. \tag{1.1.7}$$

Definition 1.1.4. The set of equations (1.1.7) is called N-th KdV (or N-th Gelfand-Dickey) hierarchy.

Example 1.1.5. If $N = 2$ we have $\tilde{L} = \partial^2 + u$. A short computation shows that

$$(\tilde{L}^{3/2})_+ = \partial^3 + \frac{3}{2}u\partial + \frac{3}{4}u',$$

and thus for $n = 3$ (1.1.7) gives

$$4\partial_3 u = u''' + 6uu', \tag{1.1.8}$$

which is the famous Korteweg-de Vries (KdV) equation. It can be looked up as a reduction of the KP equation if u does not depend on t_2 (see Example 1.1.2).

1.2 Wave functions and tau functions

In this section, we introduce the main objects connected to any solution of the KP hierarchy - the wave function $\Psi(t,z)$, the adjoint wave function $\Psi^*(t,z)$ and the tau function $\tau(t)$. For details we refer the reader to the original papers [71, 17] or the book [18].

[1]this condition is compatible with the KP hierarchy, in a sense that if L^N is a differential operator at some initial point t^0, it will stay a differential operator after flowing L along the KP vector fields.

The main idea in the study of the KP hierarchy is to follow the time evolution of the eigenfunctions of L by comparing them with the eigenfunctions of the constant operator ∂. For this aim, we look for an operator S of the form

$$S = 1 + \sum_{i=1}^{\infty} \psi_i \partial^{-i}, \tag{1.2.1}$$

which conjugates L to ∂

$$L = S\partial S^{-1}. \tag{1.2.2}$$

Rewriting equation (1.2.2) as

$$LS = S\partial$$

and comparing the coefficients of powers of ∂ we get equations of the form

$$\psi_i' = -u_i + \{\text{expression containing } u_j \text{ and } \psi_j \text{ with } j < i\}.$$

We can therefore solve these equations successively, proving thus the existence of an operator S such that (1.2.2) holds. Since only the constant coefficient operators commute with ∂, it is clear that S is unique up to right multiplication by a constant coefficient operator of the form

$$1 + c_1\partial^{-1} + c_2\partial^{-2} + \cdots .$$

The representation (1.2.2) is called *dressing* of the operator ∂ and S is called the *wave operator.*

The KP vector fields ∂_n can be extended by

$$\partial_n S = -(L^n)_- S, \tag{1.2.3}$$

and they remain commutative. Introducing a formal parameter z, we define an action of $R\{\partial\}$ on the space of formal oscillating functions $\{f(x,z)e^{xz}\}$ by the following two properties

i) $\left(\sum_i r_i \partial^i\right) e^{xz} = \sum_i r_i z^i e^{xz}$

ii) $R_1 \left(R_2 e^{xz}\right) = (R_1 R_2) e^{xz}$.

Definition 1.2.1. The wave function (or the Baker-Akhiezer function) is[2]

$$\begin{aligned} \Psi(t,z) = Se^{\sum_{i=1}^{\infty} t_i z^i} &= \left(1 + \sum_{j=1}^{\infty} \psi_j(t) z^{-j}\right) e^{\sum_{i=1}^{\infty} t_i z^i} \\ &= \psi(t,z) e^{\sum_{i=1}^{\infty} t_i z^i}, \end{aligned} \tag{1.2.4}$$

where S is the wave operator.

[2]Remember that we identify x and t_1.

From equation (1.2.3) and (1.2.4) it follows that the wave function satisfies the equations

$$L\Psi = z\Psi \tag{1.2.5}$$
$$\partial_n \Psi = (L^n)_+\Psi = B_n\Psi. \tag{1.2.6}$$

For an operator $R = \sum_i r_i \partial^i$ we denote by R^* the *formal adjoint operator*, defined by

$$R^* = \sum (-\partial)^i \cdot r_i.$$

For any two operators R_1 and R_2 one has

$$(R_1R_2)^* = R_2^* R_1^*,$$

i.e. taking the adjoint is an anti-isomorphism of $R\{\partial\}$.

Definition 1.2.2. The adjoint wave function is

$$\begin{aligned}\Psi^*(t,z) = (S^*)^{-1}\, e^{-\sum_{i=1}^\infty t_i z^i} &= \left(1 + \sum_{j=1}^\infty \psi_j^*(t) z^{-j}\right) e^{-\sum_{i=1}^\infty t_i z^i} \\ &= \psi^*(t,z) e^{-\sum_{i=1}^\infty t_i z^i}.\end{aligned} \tag{1.2.7}$$

The adjoint wave function satisfies the equations

$$L^*\Psi^* = z\Psi^* \tag{1.2.8}$$
$$\partial_n \Psi^* = -(L^n)_+^* \Psi^*. \tag{1.2.9}$$

Let us denote as usual

$$\mathrm{res}_z \left(\sum a_i z^i\right) = a_{-1} \qquad \text{and} \qquad \mathrm{res}_\partial \left(\sum r_i \partial^i\right) = r_{-1}.$$

It is not difficult to check that for any two pseudo-differential operators R_1 and R_2 one has

$$\mathrm{res}_z \left[(R_1 e^{xz})(R_2^* e^{-xz})\right] = \mathrm{res}_\partial (R_1 R_2).$$

Using the wave function $\Psi(t,z)$ and the adjoint wave function $\Psi^*(t,z)$ it is possible to characterize the solutions of the KP hierarchy by the following bilinear identity.

Theorem 1.2.3 (Bilinear identity). *The following conditions are equivalent:*
1. $\Psi(t,z)$ and $\Psi^*(t,z)$ are the wave and the adjoint wave functions for a solution of the KP hierarchy;
2. $\Psi(t,z)$ and $\Psi^*(t,z)$ are formal power series of the form

$$\Psi(t,z) = \left(1 + \sum_{i=1}^\infty \psi_i(t) z^{-i}\right) e^{\sum_{i=1}^\infty t_i z^i} \tag{1.2.10}$$
$$\Psi^*(t,z) = \left(1 + \sum_{i=1}^\infty \psi_i^*(t) z^{-i}\right) e^{-\sum_{i=1}^\infty t_i z^i}, \tag{1.2.11}$$

for some functions $\psi_i(t)$ *and* $\psi_i^*(t)$*, and for any multi-index* $\alpha = (\alpha_1, \dots, \alpha_m)$*, with* $\alpha_s \in \mathbb{Z}_+$ *the following identity*

$$\operatorname{res}_z \left[(\partial^\alpha \Psi(t,z))\, \Psi^*(t,z) \right] = 0 \tag{1.2.12}$$

holds with $\partial^\alpha = \partial_{t_1}^{\alpha_1} \cdots \partial_{t_m}^{\alpha_m}$.

The most striking consequence of the bilinear identity is that the wave function $\Psi(t,z)$ and the adjoint wave function $\Psi^*(t,z)$ can be be represented in terms of a single function, called *tau function* (see [17], [18, pp. 95-97]).

Theorem 1.2.4. *If* $\Psi(t,z)$ *and* $\Psi^*(t,z)$ *satisfy the equivalent conditions in Theorem 1.2.3, then there exists a function* $\tau(t_1, t_2, \dots)$ *such that*

$$\Psi(t,z) = \frac{\tau(t - [z^{-1}])}{\tau(t)} e^{\sum_{i=1}^\infty t_i z^i} \tag{1.2.13}$$

$$\Psi^*(t,z) = \frac{\tau(t + [z^{-1}])}{\tau(t)} e^{-\sum_{i=1}^\infty t_i z^i}, \tag{1.2.14}$$

where

$$[z] = \left(z, z^2/2, z^3/3, \dots\right). \tag{1.2.15}$$

Remark 1.2.5. Let us denote by Γ_- the multiplicative group

$$\Gamma_- = \left\{ 1 + \sum_{j\geq 1} c_j z^{-j}, \quad c_j = \text{const} \right\}.$$

As we saw above, Γ_- acts on the space of wave functions by scalar multiplication, without changing the solution, i.e. if $\gamma(z) \in \Gamma_-$, then $\Psi(t,z)$ and $\tilde{\Psi}(t,z) = \gamma(z)\Psi(t,z)$ are wave functions for the same solution. These transformations are sometimes called *gauge transformations*. If $\tau(t)$ and $\tilde{\tau}(t)$ are the tau functions corresponding to $\Psi(t,z)$ and $\tilde{\Psi}(t,z)$ respectively, we have

$$\tilde{\tau}(t) = \tau(t) e^{\sum_{j\geq 1} \alpha_j t_j},$$

where α_j are constants defined by

$$\gamma(z) = \exp\left(-\sum_{j\geq 1} \frac{\alpha_j}{j z^j} \right).$$

From Theorem 1.2.4 we can express all coefficients of the wave and the adjoint wave functions in terms of the tau function. Let $S_n(t)$ be the elementary Schur polynomials

$$\sum_{n\geq 0} S_n(t) z^n = \exp\left(\sum_{j=1}^\infty t_j z^j \right), \tag{1.2.16}$$

and denote

$$\tilde{\partial} = \left(\partial_1, \frac{\partial_2}{2}, \frac{\partial_3}{3}, \dots\right).$$

From (1.2.13) and (1.2.14) we can write

$$\Psi(t,z) = \left(1 + \sum_{n\geq 1} \frac{S_n(-\tilde{\partial})\tau(t)}{\tau(t)} \frac{1}{z^n}\right) e^{\sum_{i=1}^{\infty} t_i z^i},$$
$$\Psi^*(t,z) = \left(1 + \sum_{n\geq 1} \frac{S_n(\tilde{\partial})\tau(t)}{\tau(t)} \frac{1}{z^n}\right) e^{-\sum_{i=1}^{\infty} t_i z^i},$$

which shows that

$$S = 1 + \sum_{n\geq 1} \frac{S_n(-\tilde{\partial})\tau(t)}{\tau(t)} \partial^{-n}$$

and

$$S^{-1} = 1 + \sum_{m\geq 1} \partial^{-m} \cdot \frac{S_m(\tilde{\partial})\tau(t)}{\tau(t)}.$$

Thus

$$L = S\partial S^{-1} = \sum_{m,n\geq 0} \frac{S_n(-\tilde{\partial})\tau(t)}{\tau(t)} \partial^{-m-n+1} \cdot \frac{S_m(\tilde{\partial})\tau(t)}{\tau(t)}$$
$$= \partial + (\log \tau(t))'' \partial^{-1} + O(\partial^{-2}).$$

In particular, the solution of the KP equation (cf. Example 1.1.2) is given by

$$u = 2u_1 = 2\left(\log \tau(t)\right)''. \qquad (1.2.17)$$

1.3 Algebro-geometric solutions of Krichever

In this section we present Krichever's beautiful construction which assigns a solution of the KP hierarchy to any non-singular complex curve X of genus g with distinguished point P_∞ and non-special divisor $D = P_1 + \cdots + P_g$.

1.3.1 Preliminaries

In this subsection we briefly recall some basic facts about the Riemann surfaces, following the book of Dubrovin [21], see also [20].

Let X be a non-singular complex algebraic curve of genus g. We fix a canonical basis $\{\alpha_j, \beta_j\}$, $j = 1, \dots, g$ for $H_1(X, \mathbb{Z})$, that is, such that $\alpha_j \circ \beta_j = 1$ and all other intersections are zero.

The space of holomorphic 1-forms is g-dimensional. We define a basis ω_j, $j = 1, \ldots, g$ of this space normalized by the condition

$$\oint_{\alpha_j} \omega_k = \delta_{jk},$$

and denote by B the matrix of β-periods

$$B_{jk} = \oint_{\beta_j} \omega_k.$$

By Riemann's bilinear relations, it follows that the matrix B satisfies the following conditions:

i) $B^t = B$, i.e. it is symmetric;

ii) $\mathfrak{Im} B$ is a positive definite matrix.

The torus $\mathfrak{J}(X) = \mathbb{C}^g/\{\mathbb{Z}^g + B\mathbb{Z}^g\}$ is called Jacobian of the curve X. The Abel map is an application $A : X \to \mathfrak{J}(X)$, defined by

$$A(P) = \left(\int_{P_0}^{P} \omega_1, \int_{P_0}^{P} \omega_2, \ldots, \int_{P_0}^{P} \omega_g \right), \tag{1.3.1}$$

where P_0 is some fixed point on the Riemann surface, and the path of integration is the same in all integrals. A change of the path of integration in (1.3.1) amounts to adding a vector from $\mathbb{Z}^g + B\mathbb{Z}^g$, and thus (1.3.1) is well defined on $\mathfrak{J}(X)$.

Let D be a divisor on X, i.e. a linear combination with integer coefficients of a finite number of points

$$D = \sum_{j=1}^{N} n_j P_j, \qquad P_j \in X, \quad n_j \in \mathbb{Z}.$$

The degree of D is the integer $\deg D = \sum n_j$. D is called positive if $n_j \geq 0$, and in this case we write $D \geq 0$. For a meromorphic function f with zeros $P_1, \ldots, P_k$ and poles $Q_1, \ldots, Q_l$ with multiplicities $m_1, \ldots, m_k$ and $n_1, \ldots, n_l$ respectively, we denote by (f) the divisor

$$(f) = \sum_{j=1}^{k} m_j P_j - \sum_{j=1}^{l} n_j Q_j.$$

In the same way we define (ω) for a meromorphic 1-form ω.

Two divisors D_1 and D_2 are called equivalent if there exists a meromorphic function f, such that $D_1 - D_2 = (f)$. Thus any two Abel differentials are equivalent. This class of equivalence is called *canonical class* and is denoted by K_X.

A divisor D is called effective if it is equivalent to a positive divisor.

For any divisor D we denote by $\mathfrak{L}(D)$ the space of meromorphic functions f on X, such that $(f) + D \geq 0$, i.e.

$$\mathfrak{L}(D) = \{f : (f) + D \geq 0\}.$$

The Riemann-Roch theorem asserts that for any divisor D we have

$$\dim \mathfrak{L}(D) = 1 - g + \deg D + \dim \mathfrak{L}(K_X - D),$$

and as a consequence

$$\dim \mathfrak{L}(D) \geq 1 - g + \deg D. \tag{1.3.2}$$

D is called non-special if (1.3.2) is in fact an equality.

From now, we specialize to divisors of the form

$$D = P_1 + P_2 + \cdots + P_g. \tag{1.3.3}$$

These divisors correspond to points of the g-th symmetric power $S^g X$ of X. The Abel map extends to a map

$$A^{(g)} : S^g X \to \mathfrak{J}(X), \tag{1.3.4}$$

by linearity

$$A^{(g)}(D) = A(P_1) + \cdots + A(P_g).$$

D is a critical point of the Abel map $A^{(g)}$ if and only if $\mathfrak{L}(D - K_X) \neq 0$, i.e. if D is a special divisor. Thus, for g points in general position the divisor (1.3.3) is non-special.

The problem of finding the inverse of the Abel map (1.3.4) is called *Jacobi inverse problem*. It can be solved using theta functions.

Definition 1.3.1. The Riemann theta function is defined by

$$\theta(z) = \theta(z|B) = \sum_{N \in \mathbb{Z}^g} \exp\left(\pi i \langle BN, N\rangle + 2\pi i \langle N, z\rangle\right),$$

where $z \in \mathbb{Z}^g$, B is the matrix of β-periods, and $\langle a, b\rangle = \sum_{j=1}^N a_j b_j$ for $a, b \in \mathbb{Z}^g$.

For any $M, K \in \mathbb{Z}^g$, one can check that

$$\theta(z + K + BM) = \exp\left(-\pi i \langle BM, M\rangle - 2\pi i \langle M, z\rangle\right) \theta(z).$$

A theorem of Riemann guarantees, that there exists a constant vector $\mathcal{K} = (\mathcal{K}_1, \ldots, \mathcal{K}_g)$, called the vector of Riemann constants, such that for any non-special divisor $D = P_1 + \cdots + P_g$ the function

$$F(P) = \theta(A(P) - A^{(g)}(D) - \mathcal{K})$$

has exactly g zeros[3] on X, $P = P_1, \ldots, P = P_g$. The constants $\mathcal{K}_j$ are given by the formulae

$$\mathcal{K}_j = \frac{B_{jj} + 1}{2} - \sum_{\substack{l=1 \\ l \neq j}}^{g} \oint_{\alpha_l} \omega_l \int_{P_0}^{P} \omega_j, \qquad j = 1, 2, \ldots, g.$$

Now we come to the main object of this subsection the Baker-Akhiezer function. Let (X, P_∞, z, D, q) be a set of data defined as follows:

[3]Note that $F(P)$ is multivalued on X, but its zeros are well defined since the distinct branches differ by a (non-zero) exponential term.

i) X a non-singular algebraic curve over $\mathbb{C}$;

ii) $P_\infty \in X$ a fixed point, and z^{-1} a local parameter near P_∞, $z(P_\infty) = \infty$;

iii) $D = P_1 + \cdots + P_g$ a non-special effective divisor on $X \setminus P_\infty$;

iv) $q = q(z)$ a polynomial in z.

Definition 1.3.2. The Baker-Akhiezer function corresponding to (X, P_∞, z, D, q) is a function $\Psi(P)$ on X, having the following properties

i) It is a meromorphic function on $X \setminus P_\infty$ with poles at $P_1, \ldots, P_g$ (i.e. $(\Psi)|_{X\setminus P_\infty} + D \geq 0$);

ii) Near P_∞ it has the form

$$\Psi(z) = \left(1 + \sum_{j=1}^{\infty} \frac{\psi_j}{z^j}\right) e^{q(z)}. \tag{1.3.5}$$

For a set of data (X, P_∞, z, D, q) as above, one can show that there exists a unique Baker-Akhiezer function. Moreover, one can write an explicit formula for $\Psi(P)$ in terms of the Riemann theta function.

Let η^q be the unique normalized[4] Abel differential of second kind with principal part $dq(z)$. Thus we have

$$\int_{P_0}^{P} \eta^q = q(z) + \sum_{j=0}^{\infty} c_j z^{-j}.$$

Denote by $2\pi i U_q$ the vector of the β-periods of η^q, i.e.

$$(U_q)_j = \frac{1}{2\pi i} \oint_{\beta_j} \eta^q.$$

With these notations we can write the Baker-Akhiezer function in the form

$$\Psi(P) = \exp\left(\int_{P_0}^{P} \eta^q - c_0\right) \frac{\theta(A(P) - A^{(g)}(D) + U_q - \mathcal{K})}{\theta(A(P_\infty) - A^{(g)}(D) + U_q - \mathcal{K})} \frac{\theta(A(P_\infty) - A^{(g)}(D) - \mathcal{K})}{\theta(A(P) - A^{(g)}(D) - \mathcal{K})}. \tag{1.3.6}$$

Indeed, using the properties of the theta functions, it easily follows that the function defined by (1.3.6) is single-valued function on the Riemann surface. From the Riemann's theorem it has exactly g-poles $P_1, \ldots, P_g$ on $X \setminus P_\infty$. Finally, if $P \to P_\infty$ the quotient of the theta functions tends to 1 and the exponent can be represented in the form (1.3.5), which shows that (1.3.6) is the Baker-Akhiezer function corresponding to (X, P_∞, z, D, q).

[4]That is, $\oint_{\alpha_j} \eta^q = 0$, for $j = 1, \ldots, g$.

1.3.2 Krichever's construction

Let X, P_∞, z, D be as in the previous subsection, i.e.

i) X a non-singular algebraic curve over $\mathbb{C}$;

ii) $P_\infty \in X$ a fixed point, and z^{-1} a local parameter near P_∞;

iii) $D = P_1 + \cdots + P_g$ a non-special effective divisor on $X \setminus P_\infty$.

We denote by $q(z)$ the time dependent function of z, given by

$$q(z) = \sum_{j=1}^{\infty} t_j z^j.$$

Finally, let $\eta^{(n)}$ be the unique normalized Abel differential of second kind with a pole of order $(n+1)$ of the form

$$\eta^{(n)} = dz^n + (\text{holomorphic part}),$$

with vector of β-periods $(2\pi i)U_n$:

$$(U_n)_j = \frac{1}{2\pi i} \oint_{\beta_j} \eta^{(n)}.$$

For P close to P_∞ we have

$$\int_{P_0}^{P} \eta^{(n)} = z^n + c_{n0} + \sum_{j=1}^{\infty} \frac{c_{nj}}{j} z^{-j}. \tag{1.3.7}$$

The Baker-Akhiezer function, corresponding to this data is

$$\begin{aligned} \Psi(P) = \exp &\left(\sum_{n=1}^{\infty} t_n \left(\int_{P_0}^{P} \eta^{(n)} - c_{n0} \right) \right) \times \\ &\frac{\theta(A(P) + \sum t_n U_n - A^{(g)}(D) - \mathcal{K})}{\theta(A(P_\infty) + \sum t_n U_n - A^{(g)}(D) - \mathcal{K})} \frac{\theta(A(P_\infty) - A^{(g)}(D) - \mathcal{K})}{\theta(A(P) - A^{(g)}(D) - \mathcal{K})}. \end{aligned} \tag{1.3.8}$$

If $\{\psi_j\}_{j=1}^{\infty}$ are the coefficients of the expansion of $\Psi(P)$ around P_∞

$$\Psi(t,z) = \left(1 + \sum_{j=1}^{\infty} \frac{\psi_j}{z^j} \right) e^{\sum_{i=1}^{\infty} t_i z^i}, \tag{1.3.9}$$

we set

$$S = 1 + \sum_{j=1}^{\infty} \psi_j \partial^{-j}. \tag{1.3.10}$$

Theorem 1.3.3 (Krichever). *The Baker-Akhiezer function Ψ constructed above is a wave function for a solution of the KP hierarchy, that is, the operator $L = S\partial S^{-1}$ satisfies the equations* (1.1.3)*.*

Proof. First we show that for any $n \in \mathbb{N}$, there exists a unique differential operator P_n of the form

$$P_n = \partial^n + \sum_{j=0}^{n-1} a_j \partial^j,$$

such that

$$\frac{\partial \Psi}{\partial t_n} = P_n \Psi. \tag{1.3.11}$$

Indeed, from (1.3.9) we have

$$\frac{\partial \Psi}{\partial t_n} = \left(z^n + O(z^{n-1})\right) e^{\sum_{i=1}^{\infty} t_i z^i}.$$

On the other hand, we have

$$\partial^k \Psi = \Big(z^k + O(z^{k-1})\Big) e^{\sum_{i=1}^{\infty} t_i z^i}. \tag{1.3.12}$$

Taking appropriate linear combinations of the equations (1.3.12) for $k = 0, \ldots, n$ we see that there is a unique operator P_n of the form stated, such that

$$\frac{\partial \Psi}{\partial t_n} - P_n \Psi = \left(O(z^{-1})\right) e^{\sum_{i=1}^{\infty} t_i z^i}.$$

Consider the function $\partial\Psi/\partial t_n - P_n\Psi$. It satisfies condition i) of Definition 1.3.2. However, the expansion of the regular part near P_∞ is of the form $O(z^{-1})$. From the uniqueness of the Baker-Akhiezer function, it follows that this function vanishes, i.e. (1.3.11) holds.

The equation (1.3.11) can be rewritten in terms of the formal pseudo-differential operator S, defined by (1.3.10), as

$$\frac{\partial S}{\partial t_n} + S\partial^n = P_n S, \tag{1.3.13}$$

or equivalently

$$\frac{\partial S}{\partial t_n} S^{-1} + S\partial^n S^{-1} = P_n.$$

Since $\partial S/\partial t_n S^{-1} = O(\partial^{-1})$, taking the differential part of the last equality gives

$$P_n = \left(S\partial^n S^{-1}\right)_+,$$

which combined with (1.3.13) shows that S satisfies (1.2.3) and finishes the proof. $\square$

To write an explicit formula for the tau function we shall need the following lemma.

Lemma 1.3.4. *i) The constants c_{nj} defined by* (1.3.7) *satisfy*

$$c_{nj} = c_{jn}, \qquad \text{for } j, n \geq 1; \tag{1.3.14}$$

ii) The $(U_n)_j$ are given by the coefficients of the Taylor series

$$\omega_j = -\sum_{n=1}^{\infty} (U_n)_j z^{-n+1}\, dz^{-1} = -\sum_{n=1}^{\infty} (U_n)_j \frac{dz^{-n}}{n}. \tag{1.3.15}$$

Lemma 1.3.4 can be easily proved by using Riemann bilinear relations, see for example [21, Problem 3, p. 44, and Lemma 3, p. 42]. Let us denote

$$\Omega(t) = -\frac{1}{2}\sum_{j,k\geq 1} c_{jk}t_jt_k + \sum_{j\geq 1}\lambda_j t_j,$$

for some constants λ_j, which we define below. From Lemma 1.3.4 i) it follows that

$$\begin{aligned}&\exp\left(\sum_{n=1}^{\infty} t_n\left(\int_{P_0}^{P}\eta^{(n)} - c_{n0}\right)\right)\frac{\theta(A(P_\infty) - A^{(g)}(D) - \mathcal{K})}{\theta(A(P) - A^{(g)}(D) - \mathcal{K})}\\ &\quad = \exp\left(\Omega(t-[z^{-1}]) - \Omega(t)\right)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right),\end{aligned} \tag{1.3.16}$$

if λ_j satisfy

$$\exp\left(\sum \frac{\lambda_j}{jz^j}\right) = \frac{\theta(A(P) - A^{(g)}(D) - \mathcal{K})}{\theta(A(P_\infty) - A^{(g)}(D) - \mathcal{K})}\exp\left(-\frac{1}{2}\sum_{j,k\geq 1}\frac{1}{kjz^{k+j}}\right). \tag{1.3.17}$$

From ii) we see that

$$A(P) = A(P_\infty) - \sum_{n=1}^{\infty} U_n \frac{1}{nz^n}. \tag{1.3.18}$$

Combining (1.3.16) and (1.3.18) we can write the following explicit formula for $\tau(t)$, corresponding to the solution constructed in Theorem 1.3.3.

Proposition 1.3.5. *The tau function has the form*

$$\tau(t) = \exp\left(-\frac{1}{2}\sum_{j,k\geq 1} c_{jk}t_jt_k + \sum_{j\geq 1}\lambda_j t_j\right)\theta(A(P_\infty) + \sum_{j\geq 1} t_j U_j - A^{(g)}(D) - \mathcal{K}), \tag{1.3.19}$$

where the constants λ_j are defined by formula (1.3.17).

1.4 Sato Grassmannian

In this section we recall briefly the Sato Grassmannian Gr and the general construction assigning a solution of the KP hierarchy to any plane $W \in Gr$. For details we refer the reader to the original papers by M. Sato and S. Sato [71] and Date et al. [17]. For the analytical theory and the precise connection with the algebro-geometric solutions of Krichever (including also the case of singular curves), see the Segal-Wilson modification of Sato Grassmannian [72]. Our exposition is close to the more recent survey papers [38, 76].

Sato Grassmannian Gr consists of linear spaces W of formal power series in large z, having the property that W possesses an algebraic basis

$$W = \{w_0(z), w_1(z), \dots\}$$

such that

$$w_n = z^{s_n} + \sum_{j<s_n} a_{nj} z^j$$

and the sequence of the orders $s_W = (s_0, s_1, \dots)$ satisfies

$$s_0 < s_1 < \cdots \qquad \text{and} \qquad s_n = n \text{ for large } n.$$

The big stratum is the set of all linear spaces $W \in Gr$, such that $s_W = (0, 1, 2, \dots)$. We denote by H_+ the space

$$H_+ = \{1, z, z^2 \dots\} \in Gr.$$

The tau function $\tau_W(t) \in \mathbb{C}[[t_1, t_2, \dots]]$, corresponding to W is defined by

$$\tau_W(t) = \det\left(\text{Proj} : e^{-\sum_{i=1}^{\infty} t_i z^i} W \to H_+\right).$$

The wave function $\Psi_W(t, z)$ is connected to the tau function by the Sato formula

$$\begin{aligned}\Psi_W(t, z) = \frac{\tau_W(t - [z^{-1}])}{\tau_W(t)} e^{\sum_{i=1}^{\infty} t_i z^i} &= \left(1 + \sum_{j=1}^{\infty} \psi_j z^{-j}\right) e^{\sum_{i=1}^{\infty} t_i z^i} \\ &= \psi_W(t, z) e^{\sum_{i=1}^{\infty} t_i z^i}.\end{aligned}$$

Remark 1.4.1. Let S^1 be the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and let H denote the Hilbert space $H = L^2(S^1, \mathbb{C})$. We split H as the direct sum $H = H_+ \oplus H_-$, where H_+ (resp. H_-) consists of the functions whose Fourier series involves only non-negative (resp. only negative) powers of z. Then the Segal-Wilson Grassmannian is the Grassmannian of all closed subspaces W of H such that

i) the projection $W \to H_+$ is a Fredholm operator of index zero (hence generically an isomorphism);

ii) the projection $W \to H_-$ is a compact operator.

An important feature for the planes W from Segal-Wilson Grassmannian is that the tau function is an analytical function in a neighbourhood of the origin, see [72]. In the rest of the thesis we shall refer to this modification of Sato Grassmannian as the Segal-Wilson Grassmannian.

The plane W is uniquely determined by its wave function. Suppose for simplicity that W belongs to the big stratum of Gr, that is, $\tau(t)|_{t=0} \neq 0$ (otherwise, we can shift the first variable $t_1 \to t_1 + \text{const}$). Then for any $n \geq 0$ the functions

$$w_n = \partial_x^n \Psi(t, z)|_{t=0} = z^n + O(z^{n-1})$$

form a basis of W. Equivalently, one can characterize $\Psi(t, z)$ as the unique function $\Psi(t, z) \in W$, of the form

$$\Psi(t, z) = \left(1 + \sum_{j=1}^{\infty} \frac{\psi_j(t)}{z^j}\right) e^{\sum_{i=1}^{\infty} t_i z^i}.$$

Using this simple description of $\Psi_W(t,z)$ it is not difficult to adapt Krichever's argument and show that $\Psi_W(t,z)$ effectively leads to a solution $L = S_W \partial S_W^{-1}$ of the KP hierarchy.

Let us denote by $Gr^{(N)}$ the submanifold of Gr, consisting of planes W, invariant under the multiplication by z^N:

$$Gr^{(N)} = \{W \in Gr : z^N W \subset W\}.$$

This sub-Grassmannian parametrizes the solutions of the N-th KdV hierarchy, i.e. L^N is a differential operator if and only if $W \in Gr^{(N)}$. In this case, ψ_j do not depend on the time variables $\{t_N, t_{2N}, \dots\}$ and the tau function $\tau_W(t)$ is independent of $t_N, t_{2N}, \dots$ modulo an unessential factor of the form

$$\exp\left(\sum_j c_{Nj} t_{Nj}\right),$$

where $\{c_{Nj}\}_{j=1}^{\infty}$ are some constants.

An important object connected to W is the ring A_W of meromorphic functions $f(z)$ with poles only at $z = \infty$ that leave W invariant

$$A_W = \{f(z) : f(z)W \subset W\}. \tag{1.4.1}$$

For any $f(z) \in A_W$ there exists a unique differential operator $L_f(t, \partial/\partial t_1)$ of order equal to the order of $f(z)$, such that

$$L_f(t, \partial/\partial t_1)\Psi_W(t,z) = f(z)\Psi(t,z). \tag{1.4.2}$$

We denote by $\mathcal{A}_W$ the commutative ring of these operators at $t_2 = t_3 = \cdots = 0$

$$\mathcal{A}_W = \{L_f|_{t_2=t_3=\cdots=0} : f(z) \in A_W\}. \tag{1.4.3}$$

$\mathcal{A}_W$ is a commutative ring of differential operators with common eigenfunction the stationary wave function $\bar{\Psi}(x,z)$, given by

$$\bar{\Psi}(x,z) = \Psi_W(t,z)|_{t_2=t_3=\cdots=0}$$

Obviously, A_W and $\mathcal{A}_W$ are isomorphic. However, in general, these rings are trivial, i.e. $A_W = \mathbb{C}$. The spaces W arising from algebro-geometrical data are precisely those, such that A_W contains an element of any sufficiently large order. The SpecA_W is an algebraic curve, called *spectral curve* corresponding to W. If this curve is non-singular, it corresponds to the curve X in Krichever's construction.

1.5 Rational solutions to KP hierarchy and Calogero-Moser hierarchy

In this section we give the precise connection between the rational solutions of the KP hierarchy and the Calogero-Moser hierarchy, discovered by Airault, McKean, and Moser [7] in the KdV case and extended by Krichever [57] and Shiota [74] to the KP equation and KP hierarchy respectively. The statements below are taken from the work of Shiota

[74], where we refer the reader for proofs. In the next section, we shall come back to the rational solutions of the KP hierarchy from a slightly different point of view. Let $u = u(t_1, t_2, t_3)$ be a rational solution of the KP equation (see Example 1.1.2, $u = u_1/2$)

$$3\partial_2^2 u = \left(4\partial_3 u - u''' - 6uu'\right)'. \tag{1.5.1}$$

It is easy to see that any solution that is rational in t_1 and vanishes as $t_1 \to \infty$ must have the form

$$u(t_1, t_2, t_3) = -2\sum_{j=1}^{N} \frac{1}{(t_1 - x_j(t_2, t_3))^2}, \tag{1.5.2}$$

for some $N \in \mathbb{N}$.

Theorem 1.5.1. *1. Let $u(t_1, t_2, t_3)$ be a solution of the KP equation* (1.5.1) *of the form* (1.5.2)*. The function*

$$\tau(t_1, t_2, t_3) = \prod_{j=1}^{N} (t_1 - x_j(t_2, t_3))$$

can be uniquely extended to a tau function of the KP hierarchy, which is also a monic polynomial in t_1.

2. Let $\tau(t) = \prod_{j=1}^{N}(t_1 - x_j(t'))$ be a monic polynomial in t_1 whose coefficients depend on $t' = (t_2, t_3, \dots)$. Then $\tau(t)$ is a tau function of the KP hierarchy if and only if the motion of the zeros of τ is governed by a hierarchy of Calogero-Moser systems

$$\frac{\partial}{\partial t_n}\begin{pmatrix} x_j \\ \xi_j \end{pmatrix} = (-1)^n \begin{pmatrix} \partial H_n/\partial \xi_j \\ -\partial H_n/\partial x_j \end{pmatrix}, \qquad n = 2, 3, \dots \tag{1.5.3}$$

where $\xi_j = (1/2)\partial x_j/\partial t_2$, $H_n = \mathrm{tr}(Y^n)$ and Y is the Calogero-Moser $N \times N$ matrix

$$Y_{ij} = \frac{1}{x_i - x_j} \qquad \text{for } i \neq j$$
$$Y_{ii} = \xi_i.$$

Corollary 1.5.2. *Let $\tau(t) = \prod_{j=1}^{N}(t_1 - x_j(t'))$ be a tau function of a rational solution of the KP hierarchy. Denoting $X_0 = \mathrm{diag}(x_1(0), \dots, x_N(0))$ and Y_0 - the Calogero-Moser matrix at $t' = 0$, we have*

$$\tau(t) = \det\left(-X_0 + \sum_{n=1}^{\infty} n t_n (-Y_0)^{n-1}\right). \tag{1.5.4}$$

Corollary 1.5.3. *A solution of the KP hierarchy is rational if and only if it arises from an algebraic curve X, which is rational and unicursal (i.e. with only cusp-like singularities).*

1.6 The adelic Grassmannian Gr^{ad}

An ordinary differential operator $L(x, \partial_x)$ is called bispectral if it has a family of eigenfunctions $\Psi(x, z)$ that is also a family of eigenfunctions for some differential operator $B(z, \partial_z)$ in the spectral parameter z, i.e.

$$L(x, \partial_x)\Psi(x, z) = f(z)\Psi(x, z) \tag{1.6.1}$$

$$B(z, \partial_z)\Psi(x, z) = \theta(x)\Psi(x, z). \tag{1.6.2}$$

In view of the works of Burchnall-Chaundy [15] and Krichever [56], one may consider any operator $L(x, \partial_x)$ as an element of a maximal commutative ring $\mathcal{A}$ of differential operators. Thus, it is natural to consider the more general question of classifying bispectral commutative rings of differential operators. An important invariant of such ring is its *rank*, that is, the greatest common divisor of the orders of the operators in the ring. In this section we sketch the description of the solution of the bispectral problem in rank one case following G. Wilson [78].

We can choose the parameters x and z in such a way, that the operators L and B are monic. From (1.6.1) and (1.6.2) one easily deduces the following "ad-conditions"

$$(\text{ad}L)^{m+1}\theta = 0 \text{ and } (\text{ad}B)^{n+1}f = 0, \tag{1.6.3}$$

where $m = \text{ord}B$ and $n = \text{ord}L$. Equality (1.6.3) implies that $f(z)$ and $\theta(x)$ are polynomials of degree at most n and m respectively.

Now, from Burchnall-Chaundy-Krichever classification (see also [66]) it follows that the curve Spec$\mathcal{A}$ is rational. Conjugating L by an appropriate function, depending only on x, we can normalize L so that the second coefficient (in front of ∂^{n-1}) is a constant too. Then there is a unique choice of z, such that the common eigenfunction is a rational Baker-Akhiezer function, i.e. it has the form

$$\Psi(x, z) = \left(1 + \sum_{j=1}^{\infty} \frac{\psi_j(x)}{z^j}\right) e^{xz},$$

where $\psi_j(x)$ are rational functions of x.

According to Segal-Wilson [72], all rank one commutative rings can be put in the framework of Segal-Wilson Grassmannian. In the notations of Section 1.4, we have that $\mathcal{A} = \mathcal{A}_W$, for some plane W corresponding to an algebro-geometrical solution of the KP hierarchy, arising from a rational curve, with rational stationary wave (Baker-Akhiezer) function. Let us denote by Gr^{rat} the set of planes $W \in Gr$, corresponding to rational algebraic curves[5]. Gr^{rat} can be simply described as follows (see [72, Proposition 7.1, p. 44]).

Definition 1.6.1. A plane W belongs to Gr^{rat} if there exist polynomials $p(z)$ and $r(z)$ such that

$$p(z)H_+ \subset W \subset r(z)^{-1}H_+.$$

[5] In [72] this sub-Grassmannian is denoted by Gr_1.

We shall give another equivalent definition of Gr^{rat}, which is very useful in computing the tau function and the wave function. Following Wilson [78] we denote by $e(m,\lambda)$ the linear functional

$$\langle e(m,\lambda), g\rangle = g^{(m)}(\lambda)$$

on $\mathbb{C}[z]$ for $m \geq 0, \lambda \in \mathbb{C}$ and by $\mathcal{C}$ the infinite dimensional vector space over $\mathbb{C}$, generated by $e(m,\lambda)$. For each finite dimensional subspace $C \subset \mathcal{C}$ we set

$$V_C = \{g \in \mathbb{C}[z] : \ \langle c, g\rangle = 0 \text{ for } c \in C\}.$$

With these notations, we can give an equivalent definition of Gr^{rat} as follows.

Definition 1.6.2. A plane W belongs to Gr^{rat} if there is a finite dimensional subspace C of $\mathcal{C}$ and a polynomial $r(z)$, of degree equal to the dimension of C, such that $W = r^{-1}V_C$.

Let us fix a basis $\{c_1, c_2, \dots, c_N\}$ of C, where $N = \deg r(z)$. We can think of $\{c_i\}$'s as *conditions* which are posed on the plane $W = r^{-1}V_C$. The corresponding stationary wave function is of the form (we suppose $r(z)$ monic polynomial)

$$\bar{\Psi}_W(x,z) = \frac{1}{r(z)}\left(z^N + \sum_{i=1}^{N}\alpha_i(x)z^{N-i}\right)e^{xz}, \tag{1.6.4}$$

where the coefficients $\alpha_i(x)$ are determined by the linear system of equations

$$\langle c_i, r(z)\bar{\Psi}_W(x,z)\rangle = 0.$$

A linear functional c is called a *one point condition* if it involves derivatives at only one point λ, i.e. if c can be written in the form

$$c = \sum a_i e(i,\lambda).$$

With these definitions we can state the following theorem from [78] p. 188:

Theorem 1.6.3. *The following conditions on a space $W \in Gr^{rat}$ are equivalent:*

i) the coefficients $\alpha_i(x)$ in the expansion (1.6.4) *of Ψ_W are rational;*

ii) W has the form $r^{-1}V_C$, for some $C \subset \mathcal{C}$ that has a basis consisting of one-point conditions;

iii) the curve $\mathrm{Spec}\mathcal{A}_W$ *is unicursal.*

Notice that the ring $\mathcal{A}_W$ does not depend on the polynomial $r(z)$. Indeed, if we fix C and $r_1(z)$ is any other polynomial of degree $\dim C$, the wave functions of $W = r^{-1}V_C$ and $W_1 = r_1^{-1}V_C$ differ only by the factor $\gamma(z) = r(z)/r_1(z) \in \Gamma_-$. This gives us some freedom in choosing $r(z)$.

Now we are ready to give the definition of the adelic Grassmannian, by fixing $r(z)$ in such a way, that $\lim_{x\to\infty}\bar{\Psi}(x,z)e^{-xz} = 1$. This condition is justified by Theorem 1.6.5 below and is equivalent to the fact that $\tau_W(t)$ is a polynomial in t_1 with constant leading coefficient.

Definition 1.6.4. A space $W \in Gr^{rat}$ belongs to Gr^{ad} if W has the form $W = r^{-1}V_C$, for some C that has a basis consisting of one point conditions and $r(z) = \prod(z-\lambda_i)^{N_i}$, where N_i is the number of the conditions at the point λ_i.

Thus, up to now, we have seen that the only possible candidates for rank one rings are those constructed from unicursal rational curves, i.e. among the rings $\mathcal{A}_W$ with $W \in Gr^{ad}$. It is a pleasant surprise to see, that all these rings are really bispectral. In fact, the bispectral property is a consequence of an amazing symmetry in Gr^{ad} ([78], Theorem 2.):

Theorem 1.6.5. *For each $W \in Gr^{ad}$ there is a plane $W' \in Gr^{ad}$ such that*

$$\bar{\Psi}_W(x,z) = \bar{\Psi}_{W'}(z,x).$$

The map $b: W \to W'$ is called the *bispectral involution*. According to [79], the adelic Grassmannian Gr^{ad} can be described as follows. Let $\tilde{\mathcal{C}}_N$ be the space of all pairs (X,Y) of $N \times N$ complex matrices such that $[X,Y]+E_N$ has rank one, with E_N - the identity matrix. Denote by $\mathcal{C}_N$ the quotient space $\tilde{\mathcal{C}}_N/GL(N,\mathbb{C})$, where $GL(N,\mathbb{C})$ acts on pairs of matrices by simultaneous conjugation. As we explained in the introduction, the variety $\mathcal{C}_N$ can be thought of as a completed phase space for the N-particle Calogero-Moser system, see [50, 79]. For each pair (X,Y) the function[6]

$$\bar{\Psi}(x,z) = \det\left\{E_N - (xE_N - X)^{-1}(zE_N+Y)^{-1}\right\} e^{xz}$$

is the stationary wave function of a plane W from Gr^{ad}. Moreover, this construction defines a bijective map

$$\bigsqcup_{N\geq 0} \mathcal{C}_N \to Gr^{ad}.$$

In particular, this bijection gives a very simple explanation of the bispectral involution: on pairs of matrices it corresponds to the map

$$\beta: (X,Y) \to (-Y^t, -X^t).$$

The corresponding tau function is given by (cf. formula (1.5.4))

$$\tau(t) = \det\left(-X + \sum_{j=1}^{\infty} jt_j(-Y)^{j-1}\right). \tag{1.6.5}$$

[6]We have changed slightly Wilson's notations in order to unify them with the notations in the previous section.

Chapter 2

q-deformations of the KP hierarchy

In this chapter we consider different q-deformations of the KP hierarchy. In the first three sections we develop the necessary machinery in the spirit of Sato theory, which allows us to introduce appropriate analogues of the tau functions. The exposition follows closely [43] and aims at solving the q-KP hierarchy introduced in [40]. In Section 2.4 we give simple characterization of the q-tau functions in terms of the classical tau functions, see [45, 46]. The last section shows how this technique applies to construct some solutions to the q-deformation of the KP hierarchy that was considered by Frenkel [25] and Khesin et al. [51].

2.1 q-Pseudo-difference operators

We shall denote by D and $D_{q,x}$, respectively, the q-shift and the q-derivative operators, acting on functions of x by

$$Df(x) = f(xq) \tag{2.1.1}$$

$$D_{q,x}f(x) = \frac{f(xq) - f(x)}{x(q-1)}. \tag{2.1.2}$$

Thus, for $k \in \mathbb{Z}$ we have

$$D^k f(x) = f(xq^k). \tag{2.1.3}$$

We shall use the standard definitions (see for instance [29, 53]) of the q-shifted factorials

$$\begin{aligned}(a;q)_0 = 1, \qquad (a;q)_k &= \prod_{s=0}^{k-1}(1 - aq^s) \text{ for } k = 1, 2, \ldots, \\ (a;q)_\infty &= \prod_{s=0}^{\infty}(1 - aq^s) \text{ for } |q| < 1,\end{aligned} \tag{2.1.4}$$

and of the q-binomial coefficients for $n \in \mathbb{Z}$, $k \in \mathbb{Z}_+$

$$\binom{n}{0}_q = 1, \qquad \binom{n}{k}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)}. \tag{2.1.5}$$

We denote by $R\{D_{q,x}\}$ the set of all q-pseudo-difference operators of the form

$$R = \sum_{i=-\infty}^{m} r_i(x) D_{q,x}^{-i}.$$

Using the fact that for $n \in \mathbb{Z}_+$

$$D_{q,x}^n(fg) = \sum_{k=0}^{n} \binom{n}{k}_q (D^{n-k} D_{q,x}^k f)\, D_{q,x}^{n-k} g,$$

we define $D_{q,x}^n \cdot f$ for any $n \in \mathbb{Z}$ as the formal q-pseudo-difference operator

$$D_{q,x}^n \cdot f = \sum_{k=0}^{\infty} \binom{n}{k}_q (D^{n-k} D_{q,x}^k f)\, D_{q,x}^{n-k}. \tag{2.1.6}$$

The following proposition shows that this definition makes $R\{D_{q,x}\}$ into an associative ring.

Proposition 2.1.1. *For $m, n \in \mathbb{Z}$ we have*

$$D_{q,x}^m \cdot (D_{q,x}^n \cdot f) = D_{q,x}^{m+n} \cdot f. \tag{2.1.7}$$

Proof. From the definitions (2.1.1)-(2.1.5) one obtains immediately that

$$D_{q,x}^k\, D^s = q^{ks}\, D^s\, D_{q,x}^k \tag{2.1.8}$$

and

$$\binom{n}{k}_q = (-1)^k q^{k(2n-k+1)/2}\, \frac{(q^{-n};q)_k}{(q;q)_k}, \tag{2.1.9}$$

for $n, s, k \in \mathbb{Z}$, $k \geq 0$. Using these equalities and comparing the coefficients of $D_{q,x}^{m+n-l}$ in both sides of (2.1.7), we see that the validity of this identity amounts to checking that

$$\sum_{\substack{k+s=l \\ k,s\geq 0}} \frac{(q^{-n};q)_k}{(q;q)_k} \frac{(q^{-m};q)_s}{(q;q)_s}\, \frac{1}{q^{mk}} = \frac{(q^{-m-n};q)_l}{(q;q)_l}. \tag{2.1.10}$$

The q-Chu-Vandermonde identity (see [29, p. 236, formula (II.6)]) asserts that

$$ {}_2\phi_1(a, q^{-l}; c; q, q) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} \frac{(q^{-l};q)_k}{(c;q)_k} q^k = \frac{(c/a;q)_l}{(c;q)_l} a^l. \tag{2.1.11}$$

Notice that the left-hand side of equation (2.1.10) can be rewritten as

$$\frac{(q^{-m};q)_l}{(q;q)_l}\, {}_2\phi_1(q^{-n}, q^{-l}; q^{m-l+1}; q, q),$$

which combined with (2.1.11) gives exactly (2.1.10). □

Definition 2.1.2. We define the formal adjoint to the operator $P = \sum a_i \, D_{q,x}^i$ to be $P^* = \sum (D_{q,x}^*)^i \, a_i$, where $D_{q,x}^* = -\frac{1}{q} D_{1/q,x}$.

The next proposition shows that with this definition, taking the adjoint is an anti-isomorphism from $R\{D_{q,x}\}$ onto $R\{D_{1/q,x}\}$.

Proposition 2.1.3. *For P, Q - q-pseudo-difference operators the following identity*

$$(P\,Q)^* = Q^* P^*.$$

holds.

Proof. We have to prove that

$$(aD_{q,x}^m)^* \, (bD_{q,x}^n)^* = (bD_{q,x}^n \cdot aD_{q,x}^m)^*,$$

or equivalently:

$$\sum_{k=0}^{\infty} (-q)^k \binom{n}{k}_q D_{1/q,x}^{n-k} \cdot (D^{n-k} D_{q,x}^k a) = aD_{1/q,x}^n \, .$$

Denoting the left-hand side by A and using that (here again $D^k f = f(xq^k)$)

$$D_{1/q,x}^k \, D^n = q^{kn} \, D^n \, D_{1/q,x}^k, \qquad D^s \, D_{1/q,x}^s = q^{-s(s-1)/2} \, D_{q,x}^s \, ,$$

for $k, s \in \mathbb{Z}_+, n \in \mathbb{Z}$, we obtain:

$$\begin{aligned} A &= \sum_{k=0}^{\infty} \sum_{s=0}^{\infty} (-q)^k \binom{n}{k}_q \binom{n-k}{s}_{1/q} (D^{-(n-k-s)} D_{1/q,x}^s D^{n-k} D_{q,x}^k a) \, D_{1/q,x}^{n-k-s} \\ &= \sum_{k,s=0}^{\infty} (-1)^k q^{k+s(n-k)-s(s-1)/2} \binom{n}{k}_q \binom{n-k}{s}_{1/q} (D_{q,x}^{k+s} a) \, D_{1/q,x}^{n-k-s}. \end{aligned}$$

From (2.1.9) and

$$\binom{n-k}{s}_{1/q} = \frac{q^{s(s+1)/2}}{q^{s(2n-2k-s+1)/2}} \, \frac{(q^{n-k-s+1};q)_s}{(q;q)_s}$$

it follows that

$$A = \sum_{l=0}^{\infty} q^l A_l (D_{q,x}^l a) \, D_{1/q,x}^{n-l},$$

where

$$\begin{aligned} A_l &= \sum_{k+s=l} q^{(s-k)(l-1)/2+nk} \, \frac{(q^{-n};q)_k}{(q;q)_k} \, \frac{(q^{n-l+1};q)_s}{(q;q)_s} \\ &= q^{l(l-1)/2} \frac{(q^{n-l+1};q)_l}{(q;q)_l} \sum_{k=0}^{l} \frac{(q^{-l};q)_k}{(q;q)_k} q^k. \end{aligned}$$

Using the q-binomial theorem (see [29, 53])

$$ {}_1\phi_0(a;-;q,z) = \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = \frac{(az;q)_\infty}{(z;q)_\infty}, $$

with $a = q^{-l}$ and $z = q$ we obtain

$$ A_l = q^{l(l-1)/2} \frac{(q^{n-l+1};q)_l}{(q;q)_l} \frac{(q^{1-l};q)_\infty}{(q;q)_\infty} = \begin{cases} 0 & \text{if } l \geq 1 \\ 1 & \text{if } l = 0, \end{cases} $$

which establishes the proposition. □

2.2 q-KP, wave and adjoint wave functions

In this section we introduce a q-analogue of the Kadomtsev-Petviashvili (KP) hierarchy. The presentation follows closely the classical way of introducing the wave and adjoint wave functions [71, 17, 72, 18].

Consider the formal q-pseudo-difference operator

$$ L = D_{q,x} + a_0 + \sum_{i=1}^{\infty} a_i D_{q,x}^{-i}. \tag{2.2.1} $$

The q-deformed KP hierarchy is defined by the Lax equations

$$ \frac{\partial L}{\partial t_j} = [(L^j)_+, L], \tag{2.2.2} $$

where $(L^j)_+$ denotes the positive part of the pseudo-difference operator L^j. In this deformation of the KP hierarchy we use an operator L which differs by factor x from the one given in [25, 51].

Let us denote by S a pseudo-difference operator $S = 1 + \sum_{k=1}^{\infty} \psi_k D_{q,x}^{-k}$ (wave operator) which conjugates L to $D_{q,x}$:

$$ L = S D_{q,x} S^{-1}. \tag{2.2.3} $$

To prove the existence of S, we simply compare the coefficients of $D_{q,x}^{-k+1}$ in the equality $LS = SD_{q,x}$, for $k = 1, 2, \ldots$ Thus we get equations of the form $\psi_k(xq) - \psi_k(x) =$ terms involving $\psi_i(xq^s)$ with $i < k$. We can therefore solve (at least formally for $q \neq 1$) these equations.

The vector fields $\partial/\partial t_j$ can be extended by

$$ \frac{\partial S}{\partial t_j} = -(L^j)_- S \tag{2.2.4} $$

and they remain commutative.

The q-exponential function is defined by

$$ \exp_q(x) = \sum_{k=0}^{\infty} \frac{(1-q)^k}{(q;q)_k} x^k. \tag{2.2.5} $$

Clearly,

$$D_{q,x}\exp_q(xz) = z\exp_q(xz).$$

Using the last equality, one can deduce the following formula, which will be crucial for us

$$\exp_q(x) = \exp\left(\sum_{k=1}^{\infty}\frac{(1-q)^k}{k(1-q^k)}\,x^k\right). \tag{2.2.6}$$

For a q-pseudo-difference operator $P=\sum_i p_i\,D^i_{q,x}$ we introduce the following notation

$$P|_{x/t} = \sum_i p_i(x/t)t^i D^i_{q,x},$$

which corresponds to the linear change of variable $x = y/t$ in the operator P.

In analogy with the classical case, we define action of $R\{D_{q,x}\}$ on the space of formal oscillating functions $\{f(x,z)e_q^{xz}\}$ by the following properties

i) $\left(\sum_i r_i D^i_{q,x}\right)e_q^{xz} = \sum_i r_i z^i e_q^{xz}$

ii) $R_1\left(R_2 e_q^{xz}\right) = (R_1R_2)e_q^{xz}$.

Definition 2.2.1. The q-wave function $\Psi_q(x,t,z)$ and the q-adjoint wave function $\Psi_q^*(x,t,z)$ for the equation (2.2.2) with S as in (2.2.3) are defined by

$$\Psi_q(x,t,z) = S\,\exp_q(xz)\exp\left(\sum_{i=1}^{\infty}t_iz^i\right) \tag{2.2.7}$$

and

$$\Psi_q^*(x,t,z) = (S^*)^{-1}|_{x/q}\exp_{1/q}(-xz)\exp\left(-\sum_{i=1}^{\infty}t_iz^i\right) \tag{2.2.8}$$

where $t = (t_1,t_2,\ldots)$.

Using the definitions above, one can show that

$$L\Psi_q = z\Psi_q, \qquad \partial_m\Psi_q = (L^m)_+\Psi_q, \tag{2.2.9}$$

$$L^*|_{x/q}\Psi_q^* = z\Psi_q^*, \qquad \partial_m\Psi_q^* = -(L^m|_{x/q})_+^*\Psi_q^*. \tag{2.2.10}$$

In what follows we shall use also the notation

$$\mathrm{res}_{D_{q,x}}\left(\sum_i b_iD^i_{q,x}\right) = b_{-1}.$$

The following simple proposition is the key ingredient towards the bilinear identity which characterizes the q-KP hierarchy.

Proposition 2.2.2. *Let P and R be two q-pseudo-difference operators. Then*

$$\mathrm{res}_z\Big(P\exp_q(xz)\,R^*|_{x/q}\exp_{1/q}(-xz)\Big) = \mathrm{res}_{D_{q,x}}(PR). \tag{2.2.11}$$

Proof. Let $P = \sum p_i\, D^i_{q,x}$ and $R^* = \sum r_j\, D^j_{1/q,x}$. On the one hand we have

$$\mathrm{res}_z \left(P \exp_q(xz)\, R^*|_{x/q} \exp_{1/q}(-xz)\right) = \sum_{i+j=-1} (-q)^j p_i(x) r_j(x/q),$$

and on the other hand we compute

$$\mathrm{res}_{D_{q,x}}(PR) = \mathrm{res}_{D_{q,x}} \left(\sum p_i(x) D^i_{q,x} (-q)^j D^j_{q,x} \cdot r_j(x)\right) = \sum_{i+j=-1} (-q)^j p_i(x) r_j(x/q),$$

which establishes the proposition. □

2.3 Bilinear identity and tau function

In this section we formulate the q-analogue of the "bilinear identity". This allows us to define q-tau functions.

Proposition 2.3.1 (Bilinear identity). *If L is a solution of the q-KP hierarchy* (2.2.2) *with S as in* (2.2.3) *(or equivalently* (2.2.4)*), then for any $n \in \mathbb{Z}_+$ and $\alpha = (\alpha_1, \alpha_2, \ldots)$ - multi-index with $\alpha_i \in \mathbb{Z}_+$ we have*

$$\mathrm{res}_z \left[(D^n_{q,x} \partial^\alpha \Psi_q(x,t,z))\; \Psi^*_q(x,t,z)\right] = 0. \tag{2.3.1}$$

The converse of this proposition is also true.

Proposition 2.3.2. *Let*

$$\Psi_q(x,t,z) = \left(1 + \sum_{i=1}^{\infty} \psi_i(x,t) z^{-i}\right) \exp_q(xz) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \tag{2.3.2}$$

and

$$\Psi^*_q(x,t,z) = \left(1 + \sum_{i=1}^{\infty} \psi^*_i(x,t) z^{-i}\right) \exp_{1/q}(-xz) \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right) \tag{2.3.3}$$

be formal series, with ψ_i and ψ^*_i functions of $(x, t_1, t_2, \ldots)$, and assume that (2.3.1) *holds for any $n \in \mathbb{Z}_+$ and for any multi-index α with non-negative components α_i. Then the operator $L = S D_{q,x} S^{-1}$, where $S = 1 + \sum_{i=1}^{\infty} \psi_i D^{-i}_{q,x}$, is a solution of the q-KP hierarchy with wave and adjoint wave functions given respectively by Ψ_q and Ψ^*_q.*

Once Proposition 2.2.2 is established, these propositions can be proved following exactly the same path as in Dickey [18], see also [17].

As a consequence, one can easily prove the existence of a q-analogue of the τ-function. Indeed, for L a solution of the q-KP hierarchy, let

$$\tilde{\Psi}_q = \left[\exp_q(xz)\right]^{-1} \Psi_q \qquad \text{and} \qquad \tilde{\Psi}^*_q = \left[\exp_{1/q}(-xz)\right]^{-1} \Psi^*_q,$$

with Ψ_q, Ψ^*_q as in (2.2.7) and (2.2.8). If we fix x and consider $\tilde{\Psi}_q$ and $\tilde{\Psi}^*_q$ only as functions of $(t_1, t_2, \ldots)$, then (2.3.1) yields

$$\mathrm{res}_z(\partial^\alpha \tilde{\Psi}_q\; \tilde{\Psi}^*_q) = 0,$$

for any non-negative multi-index α. But this simply means, that (for any x-fixed) $\tilde{\Psi}_q$ and $\tilde{\Psi}_q^*$ are wave and adjoint wave functions for the classical KP hierarchy. From Sato theory it follows that there exists a function $\tau_q(x;t)$ such that:

$$\tilde{\Psi}_q = \frac{\tau_q(x;t-[z^{-1}])}{\tau_q(x;t)} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right),$$
$$\tilde{\Psi}_q^* = \frac{\tau_q(x;t+[z^{-1}])}{\tau_q(x;t)} \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right),$$

where as in the previous chapter $[z]$ denotes

$$[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \dots\right). \tag{2.3.4}$$

Thus, we can write $\Psi_q(x,t,z)$ and $\Psi_q^*(x,t,z)$ as

$$\Psi_q(x,z,t) = \frac{\tau_q(x;t-[z^{-1}])}{\tau_q(x;t)} \exp_q(xz) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right), \tag{2.3.5}$$

$$\Psi_q^*(x,t,z) = \frac{\tau_q(x;t+[z^{-1}])}{\tau_q(x;t)} \exp_{1/q}(-xz) \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right). \tag{2.3.6}$$

Combining formula (2.3.5) with (2.2.3), we can express all coefficients $\{a_i(x,t)\}$ in terms of the tau function $\tau_q(x,t)$. In particular, we have

$$a_0(x,t) = \frac{\partial}{\partial t_1} \log \frac{\tau_q(xq,t)}{\tau_q(x,t)}. \tag{2.3.7}$$

From the last equality and (2.2.9) for $m=1$ we get

$$\frac{\partial \Psi_q}{\partial t_1}(x,t,z) = \left(D_{q,x} + \frac{\partial}{\partial t_1} \log \frac{\tau_q(xq,t)}{\tau_q(x,t)}\right) \Psi_q(x,t,z). \tag{2.3.8}$$

The last equality, combined with the fact that $\tau_q(x,t)$ is a tau function in the sense of Kyoto school for any x fixed, characterizes completely the q-tau functions. In the next section shall give a simple explicit description of the q-tau functions in terms of the classical tau functions.

2.4 q-tau functions as deformations of the classical tau-functions

Now we come to the main point - the characterization of the q-tau functions in the ring of formal power series. The next theorem gives a simple description of the q-tau functions in terms of the classical tau functions. The "if part" of this theorem was first proved in [40] in the case of Schur polynomials and extended in [3, 43] to arbitrary tau function. The "only if" part was proved in [45, 46].

We can characterize the q-tau functions in the ring of formal power series by the following theorem.

Theorem 2.4.1. *A formal power series $\tau_q(x,t) \in \mathbb{C}[[x,t_1,t_2,\ldots]]$ is a tau function for the q-KP hierarchy if and only if, up to unessential factor depending only on x, we have*

$$\tau_q(x,t) = \tilde{\tau}(t + [x]_q), \tag{2.4.1}$$

where $\tilde{\tau}(t) \in \mathbb{C}[[t_1,t_2,\ldots]]$ is a tau function for the classical KP hierarchy, and

$$[x]_q = \left(x, \frac{(1-q)^2}{2(1-q^2)}x^2, \frac{(1-q)^3}{3(1-q^3)}x^3, \ldots\right). \tag{2.4.2}$$

Proof of the "if" part. Let us denote by $\tau(t)$, $\Psi(t,z)$ and $\Psi^*(t,z)$ the tau function, the wave and the adjoint wave functions for a solution of the classical KP hierarchy. If we define $\tau(x,t)$, $\Psi_q(x,t,z)$ and $\Psi_q^*(x,t,z)$ by formulae (2.4.1), (2.3.5) and (2.3.6), respectively, then using (2.2.6) we can write

$$\Psi_q(x;t) = \Psi(t+[x]_q), \qquad \Psi_q^*(x;t) = \Psi^*(t+[x]_q).$$

By definition, it is obvious that $\Psi_q(x,t,z)$ and $\Psi_q^*(x,t,z)$ are of the form (2.3.2) and (2.3.3) respectively. Thus, to prove that $\tau_q(x,t)$ is a tau function for the q-KP hierarchy, it remains to establish that (2.3.1) holds for any $n \in \mathbb{Z}_+$ and α - non-negative multi-index, or equivalently

$$\mathrm{res}_z(D^n \partial^\alpha \Psi_q \, \Psi_q^*) = 0.$$

But this follows at once if we develop the left-hand side with respect to x and use bilinear identities for the classical KP hierarchy. □

Proof of the "only if" part. From (2.3.5) it is clear that if we multiply τ_q by a function which depends only on x, we get another tau function for the same solution. Thus, without any restriction, we may suppose that $\tau_q(0,t) \not\equiv 0$. Plugging (2.3.5) in (2.3.8) and canceling the exponential part we get

$$\begin{aligned}\left(z + \frac{1}{x(q-1)}\right)&\left(\frac{\tau_q(xq, t-[z^{-1}])}{\tau_q(xq,t)} - \frac{\tau_q(x,t-[z^{-1}])}{\tau_q(x,t)}\right)\\ &= \frac{\tau_q(x,t-[z^{-1}])}{\tau_q(x,t)}\left(\frac{\partial}{\partial t_1}\log \tau_q(x,t-[z^{-1}]) - \frac{\partial}{\partial t_1}\log\tau_q(xq,t)\right).\end{aligned} \tag{2.4.3}$$

From the last equality it follows that

$$\frac{\partial}{\partial t_1}\log\tau_q(x, t-[x(1-q)]) = \frac{\partial}{\partial t_1}\log\tau_q(xq,t),$$

and we can rewrite (2.4.3) as

$$\begin{aligned}\frac{\tau_q(xq,t-[z^{-1}])}{\tau_q(xq,t)} &- \frac{\tau_q(x,t-[z^{-1}])}{\tau_q(x,t)}\\ &= \left(z + \frac{1}{x(q-1)}\right)^{-1} \frac{\{\tau_q(x,t-[z^{-1}]), \tau_q(x,t-[x(1-q)])\}}{\tau_q(x,t)\tau_q(x,t-[x(1-q)])},\end{aligned} \tag{2.4.4}$$

with $\{f,g\}$ denoting the Wronskian of g and f with respect to t_1

$$\{f,g\} := \frac{\partial f}{\partial t_1}g - f\frac{\partial g}{\partial t_1}.$$

Since $\tau_q(x,t)$ is a classical tau function in $t_1, t_2, \ldots$, it satisfies the differential Fay identity due to Adler-van Moerbeke [5]

$$\frac{\{\tau_q(x,t-[z^{-1}]),\tau_q(x,t-[y^{-1}])\}}{z-y} = -\,\tau_q(x,t-[z^{-1}])\tau_q(x,t-[y^{-1}]) + \tau_q(x,t)\tau_q(x,t-[z^{-1}]-[y^{-1}]). \tag{2.4.5}$$

For $y^{-1} = x(1-q)$ from (2.4.4) and (2.4.5) we get

$$\frac{\tau_q(xq,t-[z^{-1}])}{\tau_q(xq,t)} = \frac{\tau_q(x,t-[z^{-1}]-[x(1-q)])}{\tau_q(x,t-[x(1-q)])}. \tag{2.4.6}$$

Let us consider a new tau function $\tilde{\tau}(x,t) := \tau_q(x,t-[x]_q)$. Replacing t par $t-[xq]_q$ in (2.4.6) and using that $[xq]_q + [x(1-q)] = [x]_q$ we obtain

$$\frac{\tilde{\tau}(xq,t-[z^{-1}])}{\tilde{\tau}(xq,t)} = \frac{\tilde{\tau}(x,t-[z^{-1}])}{\tilde{\tau}(x,t)}.$$

The last equality simply means that the ratio $\tilde{\tau}(x,t-[z^{-1}])/\tilde{\tau}(x,t)$ does not depend on x, and so we have

$$\frac{\tilde{\tau}(x,t-[z^{-1}])}{\tilde{\tau}(0,t-[z^{-1}])} = \frac{\tilde{\tau}(x,t)}{\tilde{\tau}(0,t)}.$$

From this equation it follows that $\tilde{\tau}(x,t)/\tilde{\tau}(0,t) = f(x)$ does not depend on $t_1, t_2, \ldots$. Thus, we finally obtain

$$\tau_q(x,t) = f(x)\tilde{\tau}(0,t+[x]_q),$$

which finishes the proof of the theorem. □

Let us take a classical tau-function $\tau(t)$ which does not depend on t_N, t_{2N}, $t_{3N}, \ldots$. This is a tau-function for a solution of the N-th Gelfand-Dickey KdV hierarchy, that is, the N-th power of the corresponding pseudo-differential operator $L_1 = \partial + \sum_{j\ge 1} b_j \partial^{-j}$ is a "purely" differential operator. From the relation (2.4.1), it follows that $\tau_q(x;t)$ will not depend on t_N, t_{2N}, t_{3N}, ... Let us denote by L the q-pseudo-difference operator, corresponding to $\tau_q(x;t)$. Formula (2.2.4) shows that L^N is a pure q-difference operator which is a solution of a q-deformation of the N-th KdV hierarchy

$$\frac{\partial L^N}{\partial t_k} = [L_+^k, L^N]. \tag{2.4.7}$$

If we rewrite L^N in D notation

$$L^N = \frac{1}{q^{N(N-1)/2}(q-1)^N x^N} D^N + v_1 D^{N-1} + \cdots + v_0, \tag{2.4.8}$$

it follows from (2.4.7) that the "free term" v_0 depends only on x. We shall prove that in fact

$$v_0 = \frac{1}{x^N(1-q)^N}, \tag{2.4.9}$$

which means that L^N is an operator of the form used in [40]

$$L^N = D_{q,x}^N + (q-1)x\left(\sum_{j=0}^{N-2}(1-q)^j x^j u_{j+2}\right) D_{q,x}^{N-1} + u_2 D_{q,x}^{N-2} + \ldots + u_{N-1}D_{q,x} + u_N, \tag{2.4.10}$$

for some functions $u_2, u_3, \ldots, u_N$. The proof of (2.4.9) is based on an identity among the coefficients ψ_i of the wave function

$$\Psi(t) = \left(\sum_{i=0}^{\infty}\frac{\psi_i(t)}{z^i}\right)\exp\left(\sum_{i=1}^{\infty}t_i z^i\right), \qquad \psi_0 = 1$$

and the coefficients ψ_j^* of the adjoint wave function

$$\Psi^*(t) = \left(\sum_{j=0}^{\infty}\frac{\psi_j^*(t)}{z^j}\right)\exp\left(-\sum_{i=1}^{\infty}t_i z^i\right), \qquad \psi_0^* = 1$$

for a classical solution L_1^N of the N-th KdV hierarchy.

Lemma 2.4.2. *Let*

$$T_l = \sum_{\substack{i+j=l\\ i,j\geq 0}}\psi_i(t+[\alpha])\psi_j^*(t), \tag{2.4.11}$$

where α is a free parameter and $[\alpha]$ is defined as in (2.3.4)*. Then*

$$\sum_{l=1}^{N}\alpha^{l-1}T_l = 0. \tag{2.4.12}$$

Proof. From the bilinear identity (for the classical KP hierarchy) we have

$$\mathrm{res}_z\left(\Psi(t+[\alpha])\ \Psi^*(t)\right) = 0, \tag{2.4.13}$$
$$\mathrm{res}_z\left(\partial_N\Psi(t+[\alpha])\ \Psi^*(t)\right) = 0. \tag{2.4.14}$$

But

$$\Psi(t+[\alpha]) = \left(1+\sum_{s=1}^{\infty}\alpha^s z^s\right)\left(\sum_{k=0}^{\infty}\frac{\psi_k(t+[\alpha])}{z^k}\right)\exp\left(\sum_{i=1}^{\infty}t_i z^i\right),$$

so (2.4.13) gives

$$\mathrm{res}_z\left[\left(1+\sum_{s=1}^{\infty}\alpha^s z^s\right)\sum_{l=0}^{\infty}\frac{T_l}{z^l}\right] = \sum_{l=1}^{\infty}\alpha^{l-1}T_l = 0. \tag{2.4.15}$$

Making the same computation, but using this time (2.4.14) we get

$$\sum_{k=0}^{\infty}\alpha^k T_{k+N+1} = 0,$$

or equivalently

$$\sum_{l=N+1}^{\infty} \alpha^{l-1} T_l = 0. \tag{2.4.16}$$

Subtracting (2.4.15) and (2.4.16) we obtain (2.4.12). □

Now we can formulate the following theorem.

Theorem 2.4.3. *Let $\tau(t)$ be a tau function for a solution of the N-th KdV hierarchy, and let L be the q-pseudo-difference operator constructed in Theorem 2.4.1. Then L^N is a q-difference operator of the form* (2.4.10) *which solves the N-th q-deformed KdV hierarchy* (2.4.7)*.*

Proof. We have already proved everything, except that (2.4.9) holds. Let us denote by S the wave operator corresponding to L. Then from Theorem 2.4.1 we have

$$S = \sum_{i=0}^{\infty} \psi_i(t + [x]_q) D_{q,x}^{-i} \tag{2.4.17}$$

and

$$(S^{-1})^*|_{x/q} = \sum_{j=0}^{\infty} \psi_j^*(t + [x]_q)(-1)^j D_{1/q,x}^{-j}, \tag{2.4.18}$$

where ψ_i and ψ_j^* are the coefficients of the wave and adjoint wave function for L_1. From (2.4.18) it follows that

$$S^{-1} = \sum_{j=0}^{\infty} D_{q,x}^{-j} \cdot \psi_j^*(t + [xq]_q)$$

and we obtain that

$$L^N = \sum_{i,j=0}^{\infty} \psi_i(t + [x]_q) D_{q,x}^{N-i-j} \cdot \psi_j^*(t + [xq]_q).$$

Using the fact that L^N is a pure q-difference operator we can write

$$L^N = \sum_{0 \le i+j \le N} \psi_i(t + [x]_q) D_{q,x}^{N-i-j} \cdot \psi_j^*(t + [xq]_q).$$

From the last formula and from

$$D_{q,x}^s = \frac{1}{q^{s(s-1)/2}(q-1)^s x^s} D^s + \cdots + \frac{1}{x^s(1-q)^s},$$

we obtain that

$$v_0 = \frac{1}{x^N(1-q)^N} + \sum_{1 \le i+j \le N} \psi_i(t + [x]_q) \psi_j^*(t + [xq]_q) \frac{1}{(x(1-q))^{N-i-j}}.$$

If we denote $\alpha = x(1-q)$ and $t' = t + [xq]_q$, then

$$[x]_q - [xq]_q = [\alpha]$$

and

$$v_0 = \frac{1}{x^N(1-q)^N} + \frac{1}{\alpha^{N-1}} \sum_{l=1}^{N} \alpha^{l-1} \sum_{i+j=l} \psi_i(t' + [\alpha])\psi_j^*(t').$$

The second term in the above expression is zero by Lemma 2.4.2 which concludes the proof. □

Example 2.4.4 (The q-KdV hierarchy). Let $L = D_{q,x}^2 + (q-1)xuD_{q,x} + u$ be a solution of the q-KdV hierarchy. The well known formula expressing $u(x;t)$ in terms of the tau-function (see (1.2.17)) admits the following q-analogue

$$u(x;t) = D_{q,x} \frac{\partial}{\partial t_1} \log \tau_q(x;t)\tau_q(xq;t) \tag{2.4.19}$$

Indeed, from (2.3.5) we get

$$S = 1 - \frac{\partial}{\partial t_1} \log \tau_q(x;t) D_{q,x}^{-1} + \cdots$$

and

$$S^{-1} = 1 + \frac{\partial}{\partial t_1} \log \tau_q(x;t) D_{q,x}^{-1} + \cdots .$$

Equating the coefficients of $D_{q,x}$ in $L = SD_{q,x}^2 S^{-1}$ we obtain

$$x(q-1)u(x;t) = \frac{\partial}{\partial t_1} \log \tau(xq^2;t) - \frac{\partial}{\partial t_1} \log \tau(x;t),$$

which gives (2.4.19).

Example 2.4.5 (Soliton solutions). Let α_k, β_k, a_k, $k = 1, 2, \ldots, n$ be distinct complex numbers (n is the number of solitons) and

$$y_k = \exp_q(x\alpha_k) \exp\left(\sum_{s=1}^{\infty} t_s \alpha_k^s\right) + a_k \exp_q(x\beta_k) \exp\left(\sum_{s=1}^{\infty} t_s \beta_k^s\right).$$

We define a wave operator via the formula

$$S = \frac{1}{\Delta} \begin{vmatrix} y_1 & \cdots & y_n & D_{q,x}^{-n} \\ D_{q,x} y_1 & \cdots & D_{q,x} y_n & D_{q,x}^{-n+1} \\ \vdots & & \vdots & \vdots \\ D_{q,x}^n y_1 & \cdots & D_{q,x}^n y_n & 1 \end{vmatrix},$$

where Δ is the q-Wronskian of $y_1, y_2, \ldots, y_n$ and the above expression should be understood as follows: in the expansion of the determinant, $D_{q,x}^i$ must be written to the right. Then $L = SD_{q,x}S^{-1}$ is a solution of the q-KP hierarchy.

If $\alpha_k = \epsilon\beta_k$, where $\epsilon^N = 1$ then L^N is pure q-difference operator and satisfies the N-th q-KdV-hierarchy.

2.5 Solutions to FKLR hierarchy

In this section we show how one can adapt the previous ideas to obtain some solutions to the q-deformation of the KP hierarchy proposed by Frenkel [25] and Khesin et al. [51]. The corresponding shift was obtained in [44]. We refer the reader to [3] for another set of solutions of this hierarchy, based on a certain isomorphism between $R\{D_{q,x}\}$ and $R\{\Delta_q\}$.

We shall denote by Δ_q, the q-difference operator, acting on functions of x by

$$\Delta_q f(x) = f(xq) - f(x). \tag{2.5.1}$$

Using the fact that for $n \in \mathbb{Z}_+$

$$\Delta_q^n(fg) = \sum_{k=0}^{n} \binom{n}{k} (D^{n-k}\Delta_q^k f)\Delta_q^{n-k} g,$$

one can define $\Delta_q^n \cdot f$ for any $n \in \mathbb{Z}$ as the formal pseudo-difference operator

$$\Delta_q^n \cdot f = \sum_{k=0}^{\infty} \binom{n}{k} (D^{n-k}\Delta_q^k f)\Delta_q^{n-k}. \tag{2.5.2}$$

As in the classical case it is not difficult to see that the formal q-difference operators form an associative ring $R\{\Delta_q\}$.

We define the formal adjoint to the operator $P = \sum a_i \Delta_q^i$ to be $P^* = \sum \Delta_{1/q}^i \cdot a_i$. One can easily deduce the usual identity

$$(PQ)^* = Q^* P^* \tag{2.5.3}$$

for any pseudo-difference operators P and Q. Consider the formal q-pseudo-difference operator

$$L = \Delta_q + a_0 + \sum_{i=1}^{\infty} a_i \Delta_q^{-i}. \tag{2.5.4}$$

The FKLR hierarchy is defined by the Lax equations

$$\frac{\partial L}{\partial t_j} = [L_+^j, L], \tag{2.5.5}$$

where as usual L_+^j stands for the positive q-difference part of the operator L^j.
Let us denote by $S = 1 + \sum_{k=1}^{\infty} w_k \Delta_q^{-k}$ the wave operator, i.e. the operator which conjugates L to Δ_q

$$L = S\Delta_q S^{-1}. \tag{2.5.6}$$

The vector fields can be extended by

$$\frac{\partial S}{\partial t_j} = -L_-^j S. \tag{2.5.7}$$

Denoting by $E_q(x,z)$ the "q-exponential" function

$$E_q(x,z) = \exp\left(\frac{\log x \;\log(1+z)}{\log q}\right). \tag{2.5.8}$$

we have

$$\Delta_q E_q = zE_q \qquad \text{and} \qquad \Delta_q^* E_{1/q} = zE_{1/q}.$$

For a q-pseudo-difference operator $P = \sum p_i \Delta_q^i$ we introduce the following notation

$$P|_{x/t} = \sum p_i(x/t)\Delta_q^i,$$

which corresponds to the linear change of variable $y = x/t$ in the operator P. If we denote

$$\operatorname{res}_{\Delta_q}(\sum b_j \Delta_q^j) = b_{-1},$$

one can check that for any two operators P and Q we have

$$\operatorname{res}_z\left(PE_q(x,z)\, Q^*|_{x/q}\, E_{1/q}(x,z)\right) = \operatorname{res}_{\Delta_q}(PQ).$$

Let us define the q-wave function $\Psi_q(x,t,z)$ and the q-adjoint wave function $\Psi_q^*(x,t,z)$ for the FKLR hierarchy by

$$\Psi_q(x,t,z) = SE_q(x,z)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right)$$

and

$$\Psi_q^*(x,t,z) = (S^*)^{-1}|_{x/q}\, E_{1/q}(x,z)\exp\left(-\sum_{i=1}^{\infty} t_i z^i\right).$$

Now we can formulate the appropriate "bilinear identity" which characterizes the solutions of the deformed hierarchy.

Proposition 2.5.1. *If L is a solution of the FKLR hierarchy* (2.5.5)*, then for any $n \in \mathbb{Z}_+$ and $\alpha = (\alpha_1, \alpha_2, \dots)$ multi-index with $\alpha_i \in \mathbb{Z}_+$ we have*

$$\operatorname{res}_z(\Delta_q^n \partial^\alpha \Psi_q\; \Psi_q^*) = 0. \tag{2.5.9}$$

Conversely, if

$$\Psi_q(x,t,z) = \left(1 + \sum_{i=1}^{\infty} \psi_i z^{-i}\right) E_q(x,z)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right)$$

and

$$\Psi_q^*(x,t,z) = \left(1 + \sum_{i=1}^{\infty} \psi_i^* z^{-i}\right) E_{1/q}(x,z)\exp\left(-\sum_{i=1}^{\infty} t_i z^i\right)$$

are formal series, with ψ_i and ψ_i^* functions of $(x, t_1, t_2, \dots)$ and if (2.5.9) *holds for any $n \in \mathbb{Z}_+$ and for any multi-index α with nonnegative components α_i, then the operator $L = S\Delta_q S^{-1}$, where $S = 1 + \sum_{i=1}^{\infty} w_i \Delta_q^{-i}$ is a solution to the FKLR hierarchy.*

Using this proposition one can express the wave and the adjoint wave functions in terms of a q-tau function $\tau_q(x;t)$

$$\Psi_q(x,t,z) = \frac{\tau_q(x;t-[z^{-1}])}{\tau_q(x;t)} E_q(x;z) \exp\left(\sum_{i=1}^{\infty} t_i z^i\right)$$
$$\Psi_q^*(x,t,z) = \frac{\tau_q(x;t+[z^{-1}])}{\tau_q(x;t)} E_{1/q}(x,z) \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right).$$

Expanding (2.5.8) with respect to z one can state the following theorem.

Theorem 2.5.2. *Let $\tau(t)$ be a tau-function of the classical KP hierarchy. Then*

$$\tau_q(x;t) = \tau(t + \{x\}_q), \tag{2.5.10}$$

where

$$\{x\}_q = \left(\frac{\log x}{\log q}, -\frac{1}{2}\frac{\log x}{\log q}, \frac{1}{3}\frac{\log x}{\log q}, \dots\right) \tag{2.5.11}$$

is a tau-function of the FKLR hierarchy.

Remark 2.5.3. When $\tau(t)$ is a tau-function for a solution of the of the N-th Gelfand-Dickey hierarchy, then L^N is a q-difference operator of the form

$$L = D^N - u_1 D^{N-1} + u_2 D^{N-2} + \cdots + (-1)^N \tag{2.5.12}$$

which solves the N-th deformed hierarchy [25]. For example, the second order operator

$$L = D^2 - (2-u)D + 1$$

is a solution of the q-KdV hierarchy, where

$$u = \Delta_q \frac{\partial}{\partial t_1} \log \tau_q(x;t)\tau_q(xq;t)$$

and $\tau(t)$ is a tau-function for the classical KdV equation.

Chapter 3

Bispectral commutative rings of q-difference operators

In this chapter we show that for any plane $W \in Gr$, corresponding to an algebro-geometrical solution, one can associate a commutative ring of q-difference operators. Deforming appropriately the adelic Grassmannian Gr^{ad} we obtain another sub-Grassmannian Gr_q^{ad} which parametrizes bispectral commutative rings of q-difference operators. The main difference with the $q = 1$ case is that some special soliton solutions are contained in Gr_q^{ad}. The proof of the bispectral property is based on a q-version of a lemma due to Reach [70], which was used in [40] to prove the bispectral property of the q-deformed Schur polynomials. In the $q = 1$ case, this lemma was first explored by Zubelli [81], who showed the bispectral property of the classical Schur polynomials, and later by Liberati [60] who extended the construction to the adelic Grassmannian.

3.1 The deformed adelic Grassmannian

Let W be a plane in Sato Grassmannian. The q-wave function $\Psi_W^q(x,t,z) = \Psi_W(t+[x]_q, z)$ constructed in Theorem 2.4.1 can be characterized as the unique function $\Psi_W^q(x,t,z) \in W$ of the form

$$\Psi_W^q(x,t,z) = \left(1 + \sum_{i=1}^{\infty} \alpha_i(x,t) z^{-i}\right) e_q^{xz} e^{\sum_{i=1}^{\infty} t_i z^i}.$$

Consider the ring A_W of meromorphic functions $f(z)$, with poles only at $z = \infty$, that leave W invariant:

$$A_W = \{f(z) : f(z)W \subset W\}.$$

From the above characterization of the q-wave function $\Psi_W^q(x,t,z)$ and the definition of A_W, one can easily show that for any $f(z) \in A_W$, there exists a q-difference operator $L_f(x,t,D_{q,x})$ such that

$$L_f(x,t,D_{q,x}) \Psi_W^q(x,t,z) = f(z) \Psi_W^q(x,t,z). \tag{3.1.1}$$

If L_W denotes the solution of the q-KP hierarchy, corresponding to the plane W, from (2.2.9) we can write the following "explicit" formula for $L_f(x,t,D_{q,x})$

$$L_f(x,t,D_{q,x}) = f(L_W). \tag{3.1.2}$$

Now, if we define

$$\mathcal{A}_W^q = \{L_f(x,0,D_{q,x}) : f(z) \in A_W\},$$

we obtain a commutative ring of q-difference operators isomorphic to A_W with common eigenfunction $\bar{\Psi}^q(x,z) = \Psi^q(x,0,z)$. $\mathcal{A}_W^q$ can be looked up as a deformation of the commutative ring $\mathcal{A}_W$ of ordinary differential operators, defined in Section 1.4. Below, we shall use the above construction for planes W from the sub-Grassmannian Gr^{rat}. In this case $A_W \subset \mathbb{C}[z]$.

Inspired by Wilson [78], we consider the linear functionals (*q-conditions*) $e_q(m,\lambda)$ on $\mathbb{C}[z]$, defined by

$$\langle e_q(m,\lambda), g\rangle = (D_{q,z}^m g)(\lambda),$$

for $m \geq 0$ and $\lambda \in \mathbb{C}$. We denote by $\mathcal{C}_\lambda^q$ the infinite dimensional space over $\mathbb{C}$, generated by $e_q(m,\lambda)$ for $m \geq 0$, and by $\mathcal{C}^q$ the infinite dimensional space over $\mathbb{C}$, generated by all q-conditions. In contrast to the classical case, $e_q(m,\lambda)$ are no longer linearly independent. It is obvious from the definition that, for $\lambda \neq 0$,

$$\cdots \subset \mathcal{C}_{\lambda q^2}^q \subset \mathcal{C}_{\lambda q}^q \subset \mathcal{C}_\lambda^q \subset \mathcal{C}_{\lambda q^{-1}}^q \subset \mathcal{C}_{\lambda q^{-2}}^q \subset \cdots$$

A functional c is called a *one point q-condition* if it is a finite linear combination of q-conditions supported at single point λ, i.e. $c \in \mathcal{C}_\lambda^q$. For each finite dimensional subspace $C \subset \mathcal{C}$, we set

$$V_C = \{g(z) \in \mathbb{C}[z] : \langle c, g\rangle = 0, \text{ for } c \in C\}.$$

Now we are ready to give the definition of the q-deformed adelic Grassmannian Gr_q^{ad}.

Definition 3.1.1. A plane $W \in Gr$ belongs to Gr_q^{ad} if W has the form $W = r^{-1}(z)V_C$, for some finite dimensional subspace $C \subset \mathcal{C}$, which possesses a basis of one-point q-conditions, and $r(z)$ is the unique polynomial in z of degree $\deg r(z) = \dim C$, such that

$$\lim_{x\to\infty} \psi_W^q \mid_{t=0} = 1.$$

Remark 3.1.2. From the definition it follows that Gr_q^{ad} is contained in the Grassmannian Gr^{rat}, which corresponds to the algebro-geometric solutions of the KP hierarchy, arising from rational algebraic curves. In particular, for any $W \in Gr_q^{ad}$, SpecA_W is a rational curve. The intersection $Gr^{ad} \cap Gr_q^{ad}$ is the sub-Grassmannian Gr_0, corresponding to planes $W \in Gr$, with a tau function $\tau_W(t)$ polynomial in finite many time variables $t_1, t_2, \ldots$.

Remark 3.1.3. The group Γ_- of rational functions $\gamma(z)$ with $\gamma(\infty) = 1$ acts on Gr^{rat} by scalar multiplication and the q-wave function of $\gamma(z)W$ is just $\gamma(z)\Psi_W^q$. Thus, the ring $\mathcal{A}_W^q$ constructed from W depends only on the Γ_--orbit in Gr^{rat}, which gives us some freedom in choosing $r(z)$. The special choice of $r(z)$ above is made to fix the plane in each orbit of Γ_-, whose tau function is, up to unessential factor, a polynomial in x with constant leading coefficient (i.e. it can be taken to be a monic polynomial). This normalization is used for the extension of the bispectral involution in the next chapter. The explicit formula for $r(z)$ will be computed later (see (3.1.9)).

Let us fix a plane $W = r^{-1}(z)V_C \in Gr_q^{ad}$ with $C = \{c_1, c_2, \ldots, c_N\}$ as in the definition. Since $\{c_i\}$ are one point q-conditions we can write

$$c_k = \sum_{i=1}^{s_k} \gamma_{ki} e_q(i, \lambda_k),$$

where s_k is the order[1] of the condition c_k. From the characterization of the q-wave function in the previous section and the definition of W, one obtains the following explicit formula for $\Psi_W^q(x,t,z)$:

$$\Psi_W^q(x,t,z) = \frac{1}{r(z)} \frac{\mathrm{Wr}_q(f_1, f_2, \ldots, f_N, e_q^{xz})}{\mathrm{Wr}_q(f_1, f_2, \ldots, f_N)} e^{\sum_{k=1}^\infty t_k z^k}, \tag{3.1.3}$$

where $f_k(x,t) = \langle c_k, e_q^{xz} e^{\sum t_i z^i} \rangle$, and $\mathrm{Wr}_q(f_1, \ldots, f_N)$ denotes the q-Wronskian determinant $\det(D_{q,x}^{i-1} f_j)$. From the defining relation of $\{f_k\}$, it is not difficult to check that they satisfy

$$f_k(x, t - [z^{-1}]) = f_k(x,t) - \frac{1}{z} D_{q,x} f_k(x,t). \tag{3.1.4}$$

Using (3.1.3), (3.1.4) and the elementary properties of determinants we can rewrite $\Psi_W^q(x,t,z)$ in the form

$$\Psi_W^q(x,t,z) = \frac{z^N}{r(z)} \frac{\mathrm{Wr}_q(f_1(x, t-[z^{-1}]), \ldots, f_N(x, t-[z^{-1}]))}{\mathrm{Wr}_q(f_1(x,t), \ldots, f_N(x,t))} e_q^{xz} e^{\sum_{k=1}^\infty t_k z^k}. \tag{3.1.5}$$

From the last equality, it follows that $\tau_W^q(x,t)$ is a polynomial in x, given by

$$\tau_W^q(x,t) = \mathrm{Wr}_q(f_1, f_2, \ldots, f_N) \left(e_q^{\lambda_1 x} \ldots e_q^{\lambda_N x} \right)^{-1} e^{\sum_{i=1}^\infty \beta_i t_i}, \tag{3.1.6}$$

where $\{\beta_i\}$ are constants determined by the equality

$$\frac{r(z)}{z^N} = \exp\left(\sum_{i=1}^\infty \frac{\beta_i}{i z^i} \right).$$

Substituting $t_1 = t_2 = \cdots = 0$ in (3.1.6) and (3.1.3) we obtain

$$\bar{\tau}_W^q(x) := \tau_W^q(x,0) = \mathrm{Wr}_q(p_1(x) e_q^{\lambda_1 x}, \ldots, p_N(x) e_q^{\lambda_N x}) \left(e_q^{\lambda_1 x} \ldots e_q^{\lambda_N x} \right)^{-1} \tag{3.1.7}$$

$$\bar{\Psi}_W^q(x,z) = \Psi_W^q(x,0,z) = \frac{\mathrm{Wr}_q(p_1(x) e_q^{\lambda_1 x}, \ldots, p_N(x) e_q^{\lambda_N x}, e_q^{xz})}{r(z) \tau_W^q(x,0) e_q^{\lambda_1 x} \ldots e_q^{\lambda_N x}}, \tag{3.1.8}$$

[1] One should be careful here because the same condition can be written as a condition at the point $\lambda_k q^{-m}$ and then the order will be $s_k + m$.

where $p_k(x) = f_k(x,0)(e_q^{\lambda_k x})^{-1} = \sum_{i=1}^{s_k} \gamma_{ki} x^i$, for $k = 1, 2, \ldots, N$ and $\bar{\tau}_W^q(x)$ are polynomials in x.

To write an explicit formula for $r(z)$ we shall suppose that $\lambda_i q^{s_i} \neq \lambda_j q^{s_j}$ for $i \neq j$. This inequality can always be achieved by picking an appropriate basis of C. Indeed, $\lambda_i q^{s_i} = \lambda_j q^{s_j}$ means that c_i and c_j can be looked up as conditions of the same order supported at the point $\lambda_i q^{-s}$ for some s big enough; taking appropriate linear combinations we may assume that this never happens. Now, in the limit $x \to \infty$, from (3.1.7) and (3.1.8) one can deduce that

$$r(z) = \prod_{k=1}^{N} (z - \lambda_k q^{s_k}). \tag{3.1.9}$$

3.2 The bispectral property

In this section we prove that the rings $\mathcal{A}_W^q$, for $W \in Gr_q^{ad}$, are bispectral. First we formulate a q-analogue of a lemma due to Reach [70].

Lemma 3.2.1. *Let $g_0, g_1, \ldots, g_{N+1}$ be functions of x. Define*

$$G(x) = \sum_{k=1}^{N+1} (-1)^{N+1+k} g_k(x) \int g_0(x) \mathrm{Wr}_q(g_1, \ldots, \hat{g}_k, \ldots, g_{N+1})\, d_q x. \tag{3.2.1}$$

Then

$$\mathrm{Wr}_q(g_1, g_2, \ldots, g_N, G) = \theta(x) \mathrm{Wr}_q(g_1, g_2, \ldots, g_{N+1}) \tag{3.2.2}$$

with

$$\theta(x) = \left(\int g_0(x) \mathrm{Wr}_q(g_1, g_2, \ldots, g_N)\, d_q x \right) |_{xq}, \tag{3.2.3}$$

where $\int d_q x$ denotes the standard q-integral.

<u>*Proof*</u>. Expanding with respect to the last row the identity

$$\begin{vmatrix} g_1 & g_2 & \cdots & g_{N+1} \\ D_{q,x} g_1 & D_{q,x} g_2 & \cdots & D_{q,x} g_{N+1} \\ \vdots & \vdots & & \vdots \\ D_{q,x}^{N-1} g_1 & D_{q,x}^{N-1} g_2 & \cdots & D_{q,x}^{N-1} g_{N+1} \\ g_1(q^s x) & g_2(q^s x) & \cdots & g_{N+1}(q^s x) \end{vmatrix} = 0,$$

for $s = 0, 1, \ldots, N-1$, gives

$$\sum_{k=1}^{N+1} (-1)^{N+1+k} g_k(q^s x) \mathrm{Wr}_q(g_1, \ldots, \hat{g}_k, \ldots, g_{N+1}) = 0. \tag{3.2.4}$$

We now compute $D_{q,x}G, D_{q,x}^2G, \dots$. We have

$$D_{q,x}G = \sum_{k=1}^{N+1}(-1)^{N+1+k}g_k(qx)g_0(x)\mathrm{Wr}_q(g_1,\dots,\hat{g}_k,\dots,g_{N+1}) + \sum_{k=1}^{N+1}(-1)^{N+1+k}D_{q,x}g_k\int g_0(x)\mathrm{Wr}_q(g_1,\dots,\hat{g}_k,\dots,g_{N+1})\,d_qx.$$

The first term is zero by (3.2.4) with $s=1$. Continuing the process inductively and taking appropriate linear combinations of the identities (3.2.4), we get that

$$D_{q,x}^jG = \sum_{k=1}^{N+1}(-1)^{N+1+k}D_{q,x}^jg_k\int g_0(x)\mathrm{Wr}_q(g_1,\dots,\hat{g}_k,\dots,g_{N+1})d_qx, \quad \text{for } j=1,2,\dots,N-1. \tag{3.2.5}$$

This gives then that

$$D_{q,x}^NG = \sum_{k=1}^{N+1}(-1)^{N+1+k}(D_{q,x}^{N-1}g_k)(qx)g_0(x)\mathrm{Wr}_q(g_1,\dots,\hat{g}_k,\dots,g_{N+1}) + \sum_{k=1}^{N+1}(-1)^{N+1+k}D_{q,x}^Ng_k\int g_0(x)\mathrm{Wr}_q(g_1,\dots,\hat{g}_k,\dots,g_{N+1})d_qx. \tag{3.2.6}$$

Note that now (and this is the main difference with the case $q=1$) the first term is non-zero, but it would be zero if instead of $(D_{q,x}^{N-1}g_k)(qx)$ we had $(D_{q,x}^{N-1}g_k)(x)$. Thus we can rewrite $D_{q,x}^NG$ as

$$\begin{aligned} D_{q,x}^NG =&(q-1)x\sum_{k=1}^{N+1}(-1)^{N+1+k}(D_{q,x}^Ng_k)(x)g_0(x)\mathrm{Wr}_q(g_1,\dots,\hat{g}_k,\dots,g_{N+1}) \\ &+ \text{same second term as in (3.2.6)} \\ =&(q-1)xg_0(x)\mathrm{Wr}_q(g_1,\dots,g_{N+1}) \\ &+\sum_{k=1}^{N+1}(-1)^{N+1+k}D_{q,x}^Ng_k\int g_0(x)\mathrm{Wr}_q(g_1,\dots,\hat{g}_k,\dots,g_{N+1})d_qx. \end{aligned} \tag{3.2.7}$$

Putting (3.2.1), (3.2.5) and (3.2.7) into $\mathrm{Wr}_q(g_1,\dots,g_N,G)$, most of the terms disappear by column elimination and we obtain

$$\begin{aligned} \mathrm{Wr}_q(g_1,\dots,g_N,G) =&\mathrm{Wr}_q(g_1,\dots,g_{N+1})\Big[\int g_0(x)\mathrm{Wr}_q(g_1,\dots,g_N)\,dx \\ &+(q-1)xg_0(x)\mathrm{Wr}_q(g_1,\dots,g_N)\Big] \\ =&\theta(x)\mathrm{Wr}_q(g_1,\dots,g_{N+1}) \end{aligned}$$

with $\theta(x)$ as in (3.2.3), which proves the lemma. □

Now we can state the main result of this chapter.

Theorem 3.2.2. *For any plane $W \in Gr_q^{ad}$ the commutative ring of q-difference operators $\mathcal{A}_W^q$ is bispectral. Precisely, the function $\bar{\Psi}_W^q(x,z)$ satisfies*

$$L_f(x, D_{q,x})\bar{\Psi}_W^q(x,z) = f(z)\bar{\Psi}_W^q(x,z) \tag{3.2.8}$$

for any $f(z) \in A_W$, and for any polynomial $\theta(x)$ such that $D_{q,x}\theta(x)$ is divisible by $\bar{\tau}_W^q(xq)$, there exists a q-difference operator in z, $B_\theta(z, D_{q,z})$ independent of x such that

$$B_\theta(z, D_{q,z})\bar{\Psi}_W^q(x,z) = \theta(x)\bar{\Psi}_W^q(x,z). \tag{3.2.9}$$

Proof. By q-integration by parts, for any polynomial $h(x)$, we have

$$\int h(x)e_q^{xz}(e_q^{\lambda x})^{-1}\, d_qx =$$
$$-\sum_{k=0}^{\infty} \frac{(\lambda(q-1)x+q)(\lambda(q-1)x+q^2)\dots(\lambda(q-1)x+q^{k+1})}{q^{k(k+1)/2}(\lambda-qz)(\lambda-q^2z)\dots(\lambda-q^{k+1}z)}(D_{q,x}^k h)(x/q^{k+1})e_q^{xz}(e_q^{\lambda x})^{-1}. \tag{3.2.10}$$

Now we apply Lemma 3.2.1 with $g_0(x) = p(x)\prod_{i=1}^N (e_q^{\lambda_i x})^{-1}$, where $p(x)$ is a polynomial in x, $g_i(x) = p_i(x)e_q^{\lambda_i x}$ for $i = 1, 2, \dots, N$, and $g_{N+1} = e_q^{xz}/r(z)$. Using (3.2.10) we see that G can be written as

$$G = P(x,z)e_q^{xz},$$

where $P(x,z)$ is polynomial in x with rational in z coefficients. Thus, replacing xe_q^{xz} by $D_{q,z}e_q^{xz}$ we get

$$G = :P(D_{q,z}, z): e_q^{xz} = B(z, D_{q,z})\frac{e_q^{xz}}{r(z)}. \tag{3.2.11}$$

Putting (3.2.11) into (3.2.2) and using (3.1.7) and (3.1.8), we obtain

$$B(z, D_{q,z})\bar{\Psi}_W^q(x,z) = \theta(x)\bar{\Psi}_W^q(x,z)$$

with

$$\theta(x) = \left(\int p(x)\bar{\tau}_W^q(x)\, d_qx\right)|_{xq},$$

from which it follows that $\theta(x)$ can be any polynomial in x such that $D_{q,x}\theta(x)$ is divisible by $\bar{\tau}_W^q(xq)$. □

We now illustrate all steps of the above construction in the next example.

Example 3.2.3. Let $C = \mathbb{C}e_q(1,1)$ be the space generated by the single condition $e_q(1,1)$ at the point $z = 1$. For $q \neq 1$ this is the simplest example of soliton solution of the KP hierarchy, consisting of a solitary wave. We have that $r(z) = z - q$ and, by (3.1.6), the q-tau function for the corresponding plane $W = r^{-1}V_C$ is

$$\tau_W^q(x,t) = x + \frac{e^{\sum_{i=1}^\infty t_i q^i} - e^{\sum_{i=1}^\infty t_i}}{q-1}.$$

Thus

$$\bar{\tau}^q_W(x) = x.$$

The wave function, computed at $t_1 = t_2 = t_3 = \ldots = 0$, is given by formula (3.1.8)

$$\bar{\Psi}^q_W(x,z) = \left(1 - \frac{1}{x(z-q)}\right) e_q^{xz}.$$

The ring A_W is generated by

$$f(z) = z^2 - (q+1)z + q$$

and

$$h(z) = z^3 - \frac{3}{2}(q+1)z^2 + \frac{q^2+4q+1}{2}z - \frac{q^2+q}{2},$$

where

$$L_f = D^2_{q,x} - \frac{(q+1)(q^2x+q-1)}{q^2x} D_{q,x} + \frac{(q^3x^2-q-1)}{q^2x^2}.$$

If we choose for example $\theta = x^2$, the bispectral operator $B(z, D_{q,z})$ becomes

$$B(z, D_{q,z}) = D^2_{q,z} + \frac{(1-q^2)z}{q(z-q)(zq-1)} D_{q,z} - \frac{q+1}{q(z-q)(zq-1)}.$$

Let us take $\xi = f(z)$ and $\eta = h(z)$ as generators of the coordinate ring. The corresponding curve is

$$\eta^2 = \xi^3 + \left(\frac{q-1}{2}\right)^2 \xi^2.$$

The sole singularity is a double point at the origin which becomes a cusp in the limit $q \to 1$, in agreement with Wilson's result. □

Remark 3.2.4. From (3.1.2) it follows that $L_f = Sf(D_{q,x})S^{-1}$, i.e. the operator L_f can be obtained by a Darboux transformation (in the sense of [9, 10]) or by N (or fewer) successive Darboux transformations in the usual sense [60], from the constant coefficient operator $f(D_{q,x})$. The operator L_f in the above example is a Darboux transform of

$$f(D_{q,x}) = (D_{q,x} - 1)(D_{q,x} - q).$$

Note also that the operator $(D_{q,x} - 1)^2$ cannot be "rationally" factorized in a different way. Thus, Theorem 3.2.2 provides us with constant coefficient q-difference operators, for which there exist "rational" factorizations on which, Darboux transformations preserving the bispectrality, can be performed.

Chapter 4

Rational solutions to q-KP hierarchy and Calogero-Moser type hierarchy

In this chapter we study the dynamics of the poles of the rational solutions (in x) to the q-KP hierarchy. In the first section we consider the "generic" case of planes $W \in Gr_q^{ad}$, generated by first order conditions supported at different points. In the case $q = 1$, this is the clue to the connection with the Calogero-Moser system (see [48, 79]). We show that such a plane W can be determined by a pair of matrices (X, Y) satisfying

$$\operatorname{rank}\left([X, Y]_q + E\right) = 1.$$

In particular, this gives a very simple explanation of the bispectral involution: on pair of matrices it corresponds to the map

$$\beta : (X, Y) \to (-Y^t, -X^t).$$

The tau function is expressed in terms of X and Y by the following formula (see Shiota [74] and G. Wilson [79] for the $q = 1$ case)

$$\tau_W^q(x, t) = \det\left(xE - X_0 e^{\sum t_i (1-q^i)(-Y)^i} + \frac{e^{\sum t_i (1-q^i)(-Y)^i} - E}{1-q} (-Y)^{-1} \right).$$

In the second section, we examine the dynamics of the poles of the rational solutions (in x) to the q-KP hierarchy and show that the motion is governed by a hierarchy of Hamiltonian systems. The n-th Hamiltonian, corresponding to the n-th KP flow, is of the form

$$H_n = (-1)^n \frac{[n]_q}{n} \operatorname{tr}(Y^n),$$

where Y is a deformation of Calogero-Moser matrix, see Theorem 4.2.1. This result can be looked up as a q-analogue of the mysterious connection between the KP and Calogero-Moser hierarchy [7, 57, 74]. The derivation of the system (4.2.6) is obtained by a suitable adaptation of the approach of Shiota [74] for the classical case, within the context of the

q-KP hierarchy. The main difficulty here, compared to the $q = 1$ case, comes from the non triviality of the first q-KP flow. The key new ingredient is that $\partial/\partial t_1$ can be rewritten in a Lax form (see Lemma 4.2.2), which allows us to write the system in the Hamiltonian form above.

4.1 Calogero-Moser pair of matrices and the bispectral problem

Let us take first order conditions $\{c_1, \dots, c_N\}$, supported at different points $\{\lambda_1, \dots, \lambda_N\}$, which satisfy also[1] $\lambda_i \neq q\lambda_j$ for $i \neq j$, i.e. $c_i = e_q(1, \lambda_i) + \alpha_i e_q(0, \lambda_i)$ and consider the plane

$$W = \prod_{j=1}^{N} (z - q\lambda_j)^{-1} V_C. \tag{4.1.1}$$

Let us denote by Λ and α the diagonal matrices $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_N)$ and $\alpha = \mathrm{diag}(\alpha_1, \alpha_2, \dots, \alpha_N)$, and by $\mathrm{Van}(\lambda) = (\lambda_i^{j-1})_{1\le i,j\le N}$ - the Vandermonde matrix. Now, using (3.1.6) one can write the tau function in the form

$$\tau_W^q(x,t) = \det \left[x\mathrm{Van}(q\lambda) + V + \mathrm{Van}(\lambda) \left(\alpha e^{\sum t_i(1-q^i)\Lambda^i} + \frac{e^{\sum t_i(1-q^i)\Lambda^i} - E}{1-q} \Lambda^{-1} \right) \right], \tag{4.1.2}$$

where $\mathrm{Van}(q\lambda) = \mathrm{Van}(q\lambda_1, q\lambda_2, \dots, q\lambda_N)$, $E = E_N$ is the identity $N \times N$ matrix and

$$V = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ [2]_q \lambda_1 & [2]_q \lambda_2 & \dots & [2]_q \lambda_N \\ \vdots & \vdots & & \vdots \\ [N-1]_q \lambda_1^{N-2} & [N-1]_q \lambda_2^{N-2} & \dots & [N-1]_q \lambda_N^{N-2} \end{pmatrix},$$

with

$$[n]_q = \frac{1-q^n}{1-q}.$$

We shall need also two diagonal matrices $A(\lambda) = \mathrm{diag}(A_1, A_2, \dots, A_N)$ and $A'(\lambda) = \mathrm{diag}(A'_1, A'_2, \dots, A'_N)$ defined by

$$A_i = \prod_{j\neq i} \frac{q\lambda_i - \lambda_j}{\lambda_i - \lambda_j} \qquad \text{and} \qquad A'_i = \prod_{j\neq i} (q\lambda_i - \lambda_j).$$

With these notations one can check that V can be written as

$$V = -\mathrm{Van}(\lambda)\, A'^{-1}\, \tilde{\Lambda}\, A', \tag{4.1.3}$$

[1] In the rest of the chapter we shall briefly call $\{\lambda_i\}$ "q-different" if they are different and $\lambda_i \neq q\lambda_j$ for $i \neq j$.

where $\tilde{\Lambda}$ is a matrix of Calogero-Moser type

$$\tilde{\Lambda}_{ij} = \frac{A_i(\lambda)}{\lambda_i - q\lambda_j} \qquad \text{for } i \neq j \tag{4.1.4}$$

$$\tilde{\Lambda}_{ii} = \frac{1 - A_i(\lambda)}{\lambda_i(q-1)}. \tag{4.1.5}$$

From (4.1.3) it easily follows that

$$\mathrm{Van}(q\lambda) = \mathrm{Van}(\lambda)A'^{-1}(E + (1-q)\tilde{\Lambda}\Lambda)A'. \tag{4.1.6}$$

Now, using (4.1.2), (4.1.3), (4.1.6) and the fact that A', Λ and α are diagonal matrices (and so, in particular, they commute) we get the following formula for τ_W^q

$$\tau_W^q(x,t) = \det(\mathrm{Van}(\lambda))\times \det\left(x(E + (1-q)\tilde{\Lambda}\Lambda) - \tilde{\Lambda} + \alpha e^{\sum t_i(1-q^i)\Lambda^i} + \frac{e^{\sum t_i(1-q^i)\Lambda^i} - E}{1-q}\Lambda^{-1}\right). \tag{4.1.7}$$

From (4.1.6) it follows that

$$\det(E + (1-q)\tilde{\Lambda}\Lambda) = q^{N(N-1)/2} \neq 0,$$

thus we can define

$$X_t = (E + (1-q)\tilde{\Lambda}\Lambda)^{-1}\left(\tilde{\Lambda} - \alpha e^{\sum t_i(1-q^i)\Lambda^i} + \frac{E - e^{\sum t_i(1-q^i)\Lambda^i}}{1-q}\Lambda^{-1}\right). \tag{4.1.8}$$

In particular, for $t = 0$, we have

$$X_0 = (E + (1-q)\tilde{\Lambda}\Lambda)^{-1}(\tilde{\Lambda} - \alpha). \tag{4.1.9}$$

X_t and X_0 are connected by

$$X_t = X_0 e^{\sum t_i(1-q^i)\Lambda^i} + \frac{E - e^{\sum t_i(1-q^i)\Lambda^i}}{1-q}\Lambda^{-1}. \tag{4.1.10}$$

From the last equality we get

$$X_{t-[z^{-1}]} = (X_t(zE - \Lambda) + E)\,(zE - q\Lambda)^{-1},$$

and thus we can finally write explicit formulae (cf. [74, 79]) for the tau function and the wave function

$$\tau_W^q(x,t) = \det(\mathrm{Van}(q\lambda))\det\left(xE - X_0 e^{\sum t_i(1-q^i)\Lambda^i} + \frac{e^{\sum t_i(1-q^i)\Lambda^i} - E}{1-q}\Lambda^{-1}\right) \tag{4.1.11}$$

$$\Psi_W^q(x,t,z) = \frac{\det(xzE - xq\Lambda - zX_t + X_t\Lambda - E)}{\det(xE - X_t)\det(zE - q\Lambda)} e_q^{xz} e^{\sum_{k=1}^{\infty} t_k z^k}. \tag{4.1.12}$$

One can check, that the matrices $\Lambda, \tilde{\Lambda}$ and X_t satisfy the following relations

$$[\Lambda, \tilde{\Lambda}]_q + E = AT;$$
$$[X_t, \Lambda]_q - E = ([X_0, \Lambda] - E)e^{\sum t_i(1-q^i)\Lambda^i};$$
$$[X_0, \Lambda]_q - E = -(E + (1-q)\tilde{\Lambda}\Lambda)^{-1}AT(E + (q-1)\Lambda X_0),$$

where $[P, Q]_q = PQ - qQP$ denotes the q-commutator and

$$T = T_N = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}.$$

In particular, we have

$$\operatorname{rank}([X_t, \Lambda]_q - E) = 1.$$

Our next goal will be to extend the symmetry in the adelic Grassmannian to the $q \neq 1$ case for a generic plane $W \in Gr_q^{ad}$ of the form (4.1.1). Before that, we shall formulate a technical lemma.

Lemma 4.1.1. *In the notations above, the identity*

$$(E + (1-q)\tilde{\Lambda}\Lambda)AT = q^{N-1}AT \tag{4.1.13}$$

holds.

Proof. The equality (4.1.13) is equivalent to

$$\sum_{k=1}^{N} \frac{\lambda_k}{\lambda_s - q\lambda_k} \prod_{i \neq k} \frac{q\lambda_k - \lambda_i}{\lambda_k - \lambda_i} = \frac{q^{N-1}}{1-q}. \tag{4.1.14}$$

The left-hand side of (4.1.14) can be re-written as

$$\frac{1}{\det(\mathrm{Van}(\lambda))} \left[\sum_{k=1}^{N} \frac{\lambda_k}{\lambda_s - q\lambda_k} \det(\mathrm{Van}(\lambda_1, \lambda_2, \dots, q\lambda_k, \dots, \lambda_N)) \right] = \frac{F(\lambda)}{\det(\mathrm{Van}(\lambda))}.$$

$F(\lambda)$ is a polynomial in $\{\lambda_i\}$ which is zero for $\lambda_i = \lambda_j$, hence $\det(\mathrm{Van}(\lambda))/F(\lambda)$. But since $\det(\mathrm{Van}(\lambda))$ and $F(\lambda)$ have the same degree, it follows that the left-hand side of (4.1.14) is a constant, which depends only on q and N. Taking $\lambda_s \to 0$, (4.1.14) reduces to

$$\sum_{k \neq s} \prod_{i \neq k, s} \frac{q\lambda_k - \lambda_i}{\lambda_k - \lambda_i} = [N-1]_q.$$

Remembering that the left-hand side does not depend on $\{\lambda_i\}$, one can easily prove the last equality by induction. □

Now, we are ready to characterize the planes of the form (4.1.1) by the next proposition.

Proposition 4.1.2. *Let X and Y be two $n\times n$ matrices, such that the eigenvalues of Y are q-distinct and*

$$\operatorname{rank}([X,Y]_q+E_n)=1.$$

Then, there exists $N\le n$ and a matrix $U\in GL(n,\mathbb{C})$ such that

$$Y=-U\operatorname{diag}(\underbrace{\lambda_1,\dots,\lambda_N}_{\Lambda},\underbrace{\lambda_{N+1},\dots,\lambda_n}_{\Lambda'})U^{-1}=-U\begin{pmatrix}\Lambda & 0\\ 0 & \Lambda'\end{pmatrix}U^{-1},$$

and X can be written in the block form

$$X=U\begin{pmatrix}X_0 & *\\ 0 & \Lambda''\end{pmatrix}U^{-1},$$

with $X_0=(E_N+(1-q)\tilde\Lambda\Lambda)^{-1}(\tilde\Lambda-\alpha)$, the $N\times N$ matrix given by (4.1.9) for some diagonal matrix α, and $\Lambda''=((1-q)\Lambda')^{-1}$. Defining a plane $W=W(X,Y)\in Gr_q^{ad}$ by (4.1.1), its wave function at $t=0$ is given by

$$\bar\Psi_W^q(x,z)=\bar\Psi^q(x,z,X,Y)=\frac{\det(xzE_n+xqY-zX-XY-E_n)}{\det(xE_n-X)\det(zE_n+qY)}e_q^{xz}.$$

Proof. We can diagonalize the matrix Y by a matrix U_1 and write

$$[U_1^{-1}XU_1,\operatorname{diag}(\lambda_1,\dots,\lambda_n)]_q-E_n=S'T_nS'',$$

where $Y=-U_1\operatorname{diag}(\lambda_1,\dots,\lambda_n)U_1^{-1}$, and S' and S'' are diagonal matrices (we still have the freedom to conjugate by diagonal matrices). We may assume that $S'_i\neq 0$ for $i=1,2,\dots,N$ and $S'_i=0$ for $i=N+1,\dots,n$. Then X is of the form

$$X=U_1\begin{pmatrix}X' & *\\ 0 & \Lambda''\end{pmatrix}U_1^{-1},$$

and conjugating by diagonal matrices we can make S'_i (for $i=1,\dots,N$) arbitrary nonzero numbers. Let us fix $S'_i=-A_i(\lambda)$. Thus we get

$$[X',\Lambda]_q-E_N=-AT_NS, \tag{4.1.15}$$

for some $N\times N$ matrix S. If we put

$$\alpha=\tilde\Lambda-(E_N+(1-q)\tilde\Lambda\Lambda)X',$$

and multiply (4.1.15) to the left by $(E_N+(1-q)\tilde\Lambda\Lambda)$, using Lemma 4.1.1, we obtain

$$[\Lambda,\alpha]=AT_N(E_N+(q-1)\Lambda X'-q^{N-1}S).$$

From the last equality it follows that α is a diagonal matrix. Since

$$\bar\Psi^q(x,z,X,Y)=\bar\Psi^q(x,z,X',-\Lambda),$$

the rest of the argument is clear from (4.1.12). ☐

As an immediate corollary of Proposition 4.1.2, we can state the main result of this section.

Theorem 4.1.3. *Let X and Y be $n \times n$ matrices which have q-different eigenvalues and satisfy*

$$\operatorname{rank}([X, Y]_q + E_n) = 1.$$

Let $W = W(X, Y)$ and $W' = W'(-qY^t, -q^{-1}X^t)$ denote the planes constructed in Proposition 4.1.2. Then we have

$$\bar{\Psi}^q_W(x, z) = \bar{\Psi}^q_{W'}(z, x),$$

that is, on pair of matrices, the bispectral involution corresponds to the map

$$\beta : (X, Y) \to (-qY^t, -q^{-1}X^t).$$

Remark 4.1.4. Following Wilson [79], let us denote by V_n the complex vector space of pairs (X, Y), where X and Y are $n \times n$ matrices, and by $\tilde{C}^q_n$ the sub-variety of V_n, consisting of all (X, Y) satisfying the equation

$$\operatorname{rank}([X, Y]_q + E_n) = 1. \tag{4.1.16}$$

The group $GL(n, \mathbb{C})$ acts on V_n by simultaneous conjugation of X and Y. Clearly this action preserves (4.1.16). Let C^q_n stand for the quotient space $\tilde{C}^q_n/GL(n, \mathbb{C})$. Formula (4.1.10) suggests to introduce q-analogues of the Calogero-Moser flows on C^q_n, induced by the $GL(n, \mathbb{C})$ invariant flows on $\tilde{C}^q_n$

$$(X, Y) \to \left(X e^{\sum t_i (1-q^i)(-Y)^i} + \frac{E - e^{\sum t_i (1-q^i)(-Y)^i}}{1-q}(-Y)^{-1}, Y \right). \tag{4.1.17}$$

One can check that the above formula defines properly commutative flows on V_n, which preserve the condition (4.1.16). However, in the case $q \neq 1$ these flows are not Hamiltonian (in the standard coordinates), and thus the reduction procedure of [50] cannot be easily applied. In the next section, we shall write the corresponding dynamical system on the reduced phase space in a Hamiltonian form, using the approach of Shiota [74].

4.2 Calogero-Moser type hierarchy

The aim of the present section is to find the system of equations for the poles of the rational solutions to the q-KP hierarchy. Since the poles come from zeros of the tau function, let

$$\tau^q(x, t) = (x - x_1(t))(x - x_2(t)) \dots (x - x_N(t)) \tag{4.2.1}$$

be a q-tau function of the KP hierarchy, which is a polynomial in x. We may assume that $\partial x_j/\partial t_1 \neq 0$ for $j = 1, 2, \dots, N$. Indeed, if $\partial x_j/\partial t_1 = 0$ for some j, then it is not difficult to see that $\partial x_j/\partial t_n = 0$ for any n (cf. (4.2.9), (4.2.10) and (4.2.12) below), which means that $(x - x_j)$ is just an unessential factor in the tau function. We shall suppose also that $x_i \neq q^k x_j$ for $i \neq j$ and $k = 0, 1$. This is a natural restriction since in the limit $q \to 1$ it reduces to $x_i \neq x_j$ for $i \neq j$, which can always be achieved by picking an appropriate neighbourhood of the $\{t_i\}$'s, see [73, 74]. Let us denote by $A_i = A_i(x_1, x_2, \ldots, x_N)$ the expression from the previous section

$$A_i = \prod_{j\neq i} \frac{qx_i - x_j}{x_i - x_j} \tag{4.2.2}$$

and introduce $y_1, y_2, \ldots, y_N$ by

$$-x_i' = A_i e^{(1-q)x_i y_i}, \tag{4.2.3}$$

where for simplicity we have posed $' = \partial/\partial t_1$. We define the q-deformed Calogero-Moser matrix Y by

$$Y_{ij} = -\frac{x_i'}{x_i - qx_j} = \frac{A_i e^{(1-q)x_i y_i}}{x_i - qx_j}, \qquad \text{for } i \neq j \tag{4.2.4}$$

$$Y_{ii} = \frac{1 + x_i'}{x_i(q-1)} = \frac{1 - A_i e^{(1-q)x_i y_i}}{x_i(q-1)}. \tag{4.2.5}$$

With these notations we can state the main result of this section.

Theorem 4.2.1. *Let $\tau^q(x,t) = \prod_{i=1}^N (x - x_i(t))$ be a tau function of the q-KP hierarchy, which is a monic polynomial in x. Then the motion of the zeros of τ^q is governed by a hierarchy of Hamiltonian systems, which is a q-deformation of the Calogero-Moser hierarchy. Precisely, if we define*

$$H_n = (-1)^n \frac{[n]_q}{n} \mathrm{tr}(Y^n),$$

we have

$$\frac{\partial}{\partial t_n} \begin{pmatrix} x_i \\ y_i \end{pmatrix} = \begin{pmatrix} \partial H_n/\partial y_i \\ -\partial H_n/\partial x_i \end{pmatrix}, \qquad n = 1, 2, \ldots \tag{4.2.6}$$

Proof. Consider the wave function

$$\Psi^q(x,t) = \left(\sum_{k=0}^{\infty} \psi_k z^{-k}\right) e_q^{xz} e^{\sum_{k=1}^{\infty} t_k z^k}, \tag{4.2.7}$$

where $\psi_0 = 1$ and ψ_k, $k > 0$, is given by

$$\psi_k = \frac{p_k(-\tilde{\partial})\tau^q}{\tau^q}. \tag{4.2.8}$$

For our special choice (4.2.1) of the tau function we can write

$$\psi_k = \sum_{i=1}^N \frac{w_{k,i}}{x - x_i}, \qquad k \geq 1 \tag{4.2.9}$$

and in particular, for $k = 1$ formula (4.2.8) gives

$$w_{1,i} = -x_i'. \tag{4.2.10}$$

Putting (4.2.1) and (4.2.7) in (2.3.8) and comparing the coefficients of z^{-k} we obtain

$$\psi_{k+1} + \frac{\partial \psi_k}{\partial t_1} = \psi_{k+1}(xq) + D_{q,x}\psi_k + x(q-1)\sum_{i=1}^{N}\frac{x_i'}{(x-x_i)(xq-x_i)}\psi_k. \tag{4.2.11}$$

For $k>0$, the coefficient of $(x-x_i/q)^{-1}$ in (4.2.11) gives the reccurence relation

$$-\frac{w_{k+1,i}}{q} = \frac{1+x_i'}{x_i(q-1)}w_{k,i} - \sum_{j\neq i}\frac{x_i'}{x_i-qx_j}w_{k,j}, \tag{4.2.12}$$

and the coefficient of $(x-x_i)^{-1}$ in (4.2.11) gives

$$w_{k+1,i}+\frac{\partial w_{k,i}}{\partial t_1} = \left(-\frac{1+x_i'}{x_i(q-1)} + \sum_{j\neq i}\frac{(q-1)x_ix_j'}{(x_i-x_j)(x_iq-x_j)}\right)w_{k,i}+\sum_{j\neq i}\frac{x_i'}{x_i-x_j}w_{k,j}. \tag{4.2.13}$$

Denoting $X=\mathrm{diag}(x_1,x_2,\ldots,x_N)$, $w_k=(w_{k,1},\ldots,w_{k,N})^t$, $\vec{e}=(1,1,\ldots,1)^t$ we have from (4.2.10) and (4.2.12)

$$w_k = (-qY)^{k-1}X'\vec{e}, \tag{4.2.14}$$

where Y is the q-deformed Calogero-Moser matrix defined by (4.2.4) and (4.2.5). Eliminating $w_{k+1,i}$ from equations (4.2.12) and (4.2.13) we obtain

$$\frac{\partial w_{k,i}}{\partial t_1} = \left(\frac{1+x_i'}{x_i} + \sum_{j\neq i}\frac{(q-1)x_ix_j'}{(x_i-x_j)(x_iq-x_j)}\right)w_{k,i}-\sum_{j\neq i}\frac{(q-1)x_ix_i'}{(x_i-x_j)(x_i-qx_j)}w_{k,j}. \tag{4.2.15}$$

Let

$$\Psi^{q*}(x,t,z) = \left(1+\sum_{k=1}^{\infty}\psi_k^* z^{-k}\right)e_{1/q}^{-xz}e^{-\sum_{k=1}^{\infty}t_kz^k},$$

be the q-adjoint wave function. Writing ψ_k^* as

$$\psi_k^* = \sum_{i=1}^{N}\frac{w_{k,i}^*}{x-x_i}$$

and comparing the coefficients of $(x-qx_i)^{-1}$ in

$$\partial_1\Psi^{q*} = -(L\mid_{x/q})_+^*\Psi^{q*} = (D_{1/q,x}-a_0(x/q))\Psi^{q*},$$

we obtain as above

$$w_k^* = -X'(-Y^t)^{k-1}\vec{e}, \tag{4.2.16}$$

where $w_k^*=(w_{k,1}^*,w_{k,2}^*,\ldots,w_{k,N}^*)^t$. Now, we compute the coefficients of $D_{q,x}^{-1}$ in the equation

$$\partial_n S = -(SD_{q,x}^nS^{-1})_-S. \tag{4.2.17}$$

The left-hand side is

$$\begin{aligned}\partial_n S &= \sum_{k=1}^{\infty}\sum_{i=1}^{N}\left(\frac{(\partial_n x_i)w_{k,i}}{(x-x_i)^2}+\frac{\partial_n w_{k,i}}{x-x_i}\right)D_{q,x}^{-k}\\ &= \left(\frac{x_i'(\partial_n x_i)}{(x-x_i)^2}+\frac{\partial_n x_i'}{x-x_i}\right)D_{q,x}^{-1}+O(D_{q,x}^{-2}).\end{aligned} \tag{4.2.18}$$

On the other hand, from the definition of the q-adjoint wave function we have

$$S^{-1}=\sum_{j=0}^{\infty}D_{q,x}^{-j}\cdot\psi_j^*(xq),$$

so the right-hand side of (4.2.17) becomes

$$\begin{aligned}-(SD_{q,x}^{n}S^{-1})_{-}S &= -\left(\sum_{k+l\geq n+1}\psi_k D_{q,x}^{n-k-l}\cdot\psi_l^*(xq)\right)\left(1+\sum_{k\geq 1}\psi_k D_{q,x}^{-k}\right)\\ &= -\sum_{k+l=n+1}\psi_k\psi_l^* D_{q,x}^{-1}+O(D_{q,x}^{-2}),\end{aligned}$$

and thus

$$\frac{x_i'\partial_n x_i}{(x-x_i)^2}+\frac{\partial_n x_i'}{x-x_i}=-\sum_{k+l=n+1}\psi_l^*\psi_k.$$

Comparing the coefficients of $(x-x_i)^{-2}$ in the above identity, and using (4.2.14) and (4.2.16), we get

$$\begin{aligned}x_i'(\partial_n x_i) &= -\sum_{k=1}^{n}w_{n+1-k,i}^*w_{k,i}=\sum_{k=1}^{n}w_{n+1-k}^{*\,t}E_{ii}w_k\\ &= (-1)^{n+1}\sum_{k=1}^{n}q^{k-1}\vec{e}^{\,t}Y^{n-k}X'E_{ii}Y^{k-1}X'\vec{e}.\end{aligned}$$

Here, as usual, E_{ij} denotes the matrix with 1 at the (i,j)th entry, with all other entries zero. Since $X'E_{ii}=x_i'E_{ii}$, we can cancel x_i' and rewrite the above equality as

$$\partial_n x_i=(-1)^{n+1}\mathrm{tr}\left(\sum_{k=1}^{n}q^{k-1}Y^{n-k}E_{ii}Y^{k-1}X'\begin{pmatrix}1&\dots&1\\ \vdots&&\vdots\\ 1&\dots&1\end{pmatrix}\right).$$

Finally, replacing $X'\begin{pmatrix}1&\dots&1\\ \vdots&&\vdots\\ 1&\dots&1\end{pmatrix}$ by $-(E+XY-qYX)$, and using the elementary properties of the trace operator we get

$$\begin{aligned}\partial_n x_i =&(-1)^n \mathrm{tr}\left(\sum_{k=1}^{n} q^{k-1} Y^{n-k} E_{ii} Y^{k-1}(E+XY-qYX)\right)\\ =&(-1)^n \left(\sum_{k=1}^{n} q^{k-1}\mathrm{tr}(E_{ii}Y^{n-1}) + \sum_{k=1}^{n} q^{k-1}\mathrm{tr}(Y^{n+1-k}E_{ii}Y^{k-1}X)\right.\\ &\left.-\sum_{k=1}^{n} q^{k}\mathrm{tr}(Y^{n-k}E_{ii}Y^{k}X)\right)\\ =&(-1)^n [n]_q \mathrm{tr}\left((E_{ii}+(1-q)E_{ii}XY)Y^{n-1}\right).\end{aligned} \tag{4.2.19}$$

But since

$$\frac{\partial Y}{\partial y_i} = E_{ii} + (1-q)E_{ii}XY,$$

(4.2.19) gives the first equation in (4.2.6). To get the second equation, we need to represent "nicely" the first flow $\partial/\partial t_1$ on the matrix Y, which is the content of the next lemma.

Lemma 4.2.2. *The first flow $\partial/\partial t_1$ can be written in the Lax form*

$$\frac{\partial Y}{\partial t_1} = [Y, M], \tag{4.2.20}$$

where M is another deformation of the Calogero-Moser matrix given by

$$\begin{aligned} M_{ij} &= -\frac{x_i'}{x_i - x_j} \qquad \text{for } i \neq j \\ M_{ii} &= \frac{1+x_i'}{x_i(q-1)} + \sum_{k\neq i}\left(\frac{x_k'}{x_i q - x_k} - \frac{x_k'}{x_i - x_k}\right). \end{aligned}$$

Proof of Lemma 4.2.2. The equality (4.2.20) can be checked directly, using (4.2.10) and (4.2.15) for $k=1$, and the definition (4.2.4) and (4.2.5) of Y. ☐

We can now finish the proof of Theorem 4.2.1. From (4.2.3), (4.2.19) and Lemma 4.2.2 one can easily deduce

$$\frac{\partial y_i}{\partial t_n} = (-1)^n [n]_q \mathrm{tr}(BY^{n-1}), \tag{4.2.21}$$

where

$$\begin{aligned} B =& \frac{1}{(q-1)x_i x_i'}[E_{ii}, \hat{M}] + \frac{1}{x_i}E_{ii}Y + \frac{1}{x_i'}(\hat{M}E_{ii}Y - YE_{ii}\hat{M}) + \\ & \frac{1}{(q-1)x_i}\sum_j \frac{\partial \log A_i}{\partial x_j}E_{jj} - \left(\sum_j \frac{x_j}{x_i}\frac{\partial \log A_i}{\partial x_j}E_{jj}\right)Y - \frac{y_i}{x_i}(E_{ii} + (1-q)x_i E_{ii} Y), \end{aligned}$$

with $\hat{M}_{jk} = (1-\delta_{jk})M_{jk}$. On the other hand, a direct computation shows that

$$B + \frac{\partial Y}{\partial x_i} = \left[\sum_{j\neq i}\frac{\partial \log A_j}{\partial x_i}E_{jj} + \frac{1}{(1-q)x_i}E_{ii}, Y\right], \tag{4.2.22}$$

which combined with (4.2.21) gives the second equation in (4.2.6). ☐

Remark 4.2.3 (The limiting case $q = 1$). We should note that our choice of "dual" variables $\{y_i\}$ does not reduce exactly to the standard one $\{\xi_i\}$ with $\xi_i = \partial x_i/\partial t_2$ in the classical case $q = 1$. Indeed, in the limit $q \to 1$, from (4.2.4) and (4.2.5), it follows that

$$\lim_{q\to 1} Y_{ij} = \frac{1}{x_i - x_j} \qquad \text{for } i \neq j$$

$$\lim_{q\to 1} Y_{ii} = \frac{\partial x_i}{\partial t_2} = y_i + \sum_{j\neq i} \frac{1}{x_j - x_i},$$

i.e.

$$\xi_i = y_i + \sum_{j\neq i} \frac{1}{x_j - x_i}.$$

From these relations, it is not difficult to see that the system of equations (4.2.6) is equivalent to equation (1.5.3).

Chapter 5

Commutative rings of difference operators and an adelic flag manifold

This Chapter represents our joint paper [41] with Professor L. Haine. We show that maximal rank 1 commutative rings of difference operators can be systematically constructed from their differential analogues by making an appropriate shift in the variables of the tau function of the KP hierarchy. When the spectrum of the ring is a unicursal curve, the operators in the ring enjoy a bispectral property reminiscent of the familiar bispectral property satisfied by the classical orthogonal polynomials. As an illustration, the tau functions leading to the rings with spectral curves $y^2 = r^{2K+1}(r+1)$, $K = 1, 2, 3, \dots$ are explicitly characterized.

5.1 The discrete KP hierarchy

We denote by Λ and Δ respectively, the customary shift and difference operators acting on the ring R of functions of a discrete variable $n \in \mathbb{Z}$ by

$$\Lambda f(n) = f(n+1),$$
$$\Delta f(n) = f(n+1) - f(n) = (\Lambda - I) f(n),$$

where I is the identity operator. Defining for any $j \in \mathbb{Z}$

$$\Delta^j \cdot f = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^i f)(n+j-i)\Delta^{j-i}, \qquad \binom{j}{i} = \frac{j(j-1)\dots(j-i+1)}{i!},$$

we obtain an associative ring of formal pseudo-difference operators

$$R\{\Delta\} = \left\{ X = \sum_{j=-\infty}^{d} a_j(n)\Delta^j, \quad a_j(n) \in R \right\}.$$

We shall denote by $X_+ = \sum_{j=0}^{d} a_j(n)\Delta^j$ the positive difference part of X and by $X_- = \sum_{j=-\infty}^{-1} a_j(n)\Delta^j$, the Volterra part of X. If we think of X as an infinite matrix with finitely many bands above the main diagonal, X_+ is the positive part (including the diagonal) of the matrix, and X_- is the strictly lower part of the matrix. The ring $R\{\Delta\}$ enjoys the same properties as the usual ring of formal pseudo-differential operators. One exception however is that the formal adjoint (i.e. the transpose, if we think in terms of matrices) to $X \in R\{\Delta\}$ belongs to another ring $R\{\Delta^*\}$ with

$$\Delta^* f(n) = \left(\Lambda^{-1} - I\right) f(n) = f(n-1) - f(n)$$

and

$$\Delta^{*j} \cdot f = \sum_{i=0}^{\infty} \binom{j}{i} (\Delta^{*i} f)(n+i-j) \Delta^{*j-i}.$$

The formal adjoint to $X \in R\{\Delta\}$ is defined to be $X^* = \sum_{j=-\infty}^{d} \Delta^{*j} \cdot a_j(n)$ and satisfies $(X \cdot Y)^* = Y^* \cdot X^*$.

Let L be a general first order formal pseudo-difference operator

$$L = \Delta + \sum_{j=0}^{\infty} a_j(n) \Delta^{-j} = \Lambda + (a_0(n) - 1)I + \sum_{j=1}^{\infty} b_j(n) \Lambda^{-j},$$

with

$$b_j(n) = \sum_{k=1}^{j} \binom{j-1}{k-1} a_k(n).$$

The discrete KP hierarchy is the family of evolution equations in infinitely many time variables $t = (t_1, t_2, t_3, \dots)$

$$\frac{\partial L}{\partial t_i} = [(L^i)_+, L]. \tag{5.1.1}$$

There are two ways to develop a "Sato type" theory for this hierarchy. One can either conjugate L to the shift Λ as in [6, 19, 39, 75], or to the difference operator Δ as in [44]. Although equivalent, these two approaches lead to a different definition of the tau function, as will be clear in a moment. For us, it will be convenient to follow the approach of [44], which makes the parallelism with the classical KP theory more transparent, just replacing d/dx by Δ. Let

$$S(n;t) = 1 + \sum_{j=1}^{\infty} \psi_j(n;t) \Delta^{-j}, \tag{5.1.2}$$

be the wave operator which conjugates L to Δ, that is

$$L = S \Delta S^{-1}. \tag{5.1.3}$$

As in the continuous case, S is determined up to multiplication to the right by a constant coefficient (independent of n) pseudo-difference operators $1 + \sum_{j=1}^{\infty} c_j \Delta^{-j}$. The vector fields (5.1.1) can be extended by

$$\frac{\partial S}{\partial t_i} = -(L^i)_- S. \tag{5.1.4}$$

We shall denote by $\mathrm{Exp}(n;t,z)$ the exponential function

$$\begin{aligned}\mathrm{Exp}(n;t,z) &\equiv (1+z)^n \exp\left(\sum_{i=1}^{\infty} t_i z^i\right)\\ &= \exp\left(\sum_{i=1}^{\infty}\left(t_i + n\frac{(-1)^{i-1}}{i}\right) z^i\right)\\ &= \sum_{j\in\mathbb{Z}} S_j(t_1+n, t_2-n/2, t_3+n/3, \dots) z^j\\ &\equiv \sum_{j\in\mathbb{Z}} S_j(n;t) z^j,\end{aligned} \tag{5.1.5}$$

where $S_j(t_1,t_2,t_3,\dots)$ are the standard elementary Schur polynomials defined by the generating function $\exp\left(\sum_{i=1}^{\infty} t_i z^i\right) = \sum_{j\in\mathbb{Z}} S_j(t_1,t_2,t_3,\dots)z^j$, with the understanding that $S_j(t_1,t_2,t_3,\dots)=0$ for $j<0$. One checks easily that

$$\Delta\mathrm{Exp}(n;t,z) = z\mathrm{Exp}(n;t,z) \qquad \text{and} \qquad \Delta^*\mathrm{Exp}^{-1}(n;t,z) = z\mathrm{Exp}^{-1}(n;t,z). \tag{5.1.6}$$

We define the wave function $\Psi(n;t,z)$ and the adjoint wave function $\Psi^*(n;t,z)$ of the discrete KP hierarchy by

$$\begin{aligned}\Psi(n;t,z) &= S(n;t)\mathrm{Exp}(n;t,z)\\ &= \left(1+\frac{\psi_1(n;t)}{z}+\frac{\psi_2(n;t)}{z^2}+\cdots\right)\mathrm{Exp}(n;t,z)\end{aligned} \tag{5.1.7}$$

and

$$\begin{aligned}\Psi^*(n;t,z) &= \left(S^{-1}(n-1;t)\right)^* \mathrm{Exp}^{-1}(n;t,z)\\ &= \left(1+\frac{\psi_1^*(n;t)}{z}+\frac{\psi_2^*(n;t)}{z^2}+\cdots\right)\mathrm{Exp}^{-1}(n;t,z).\end{aligned} \tag{5.1.8}$$

We introduce the ring

$$\mathbb{C}\{z\} = \left\{\sum_{j=-\infty}^{d} c_j z^j\right\}, \tag{5.1.9}$$

of formal power series "meromorphic at infinity" and define as before the residue by

$$\mathrm{res}_z \sum_{j=-\infty}^{d} c_j z^j = c_{-1}. \tag{5.1.10}$$

With these definitions, we have

Proposition 5.1.1 (Discrete KP bilinear identities). *For any $j\in\mathbb{Z}_+$ and for any multi-index $\alpha=(\alpha_1,\alpha_2,\dots,\alpha_k)$ with $\alpha_s\geq 0$, we have*

$$\mathrm{res}_z\left(\Delta^j\partial^\alpha\Psi(n;t,z)\right)\Psi^*(n;t,z)=0, \tag{5.1.11}$$

with $\partial^\alpha = \partial_{t_1}^{\alpha_1}\partial_{t_2}^{\alpha_2}\cdots\partial_{t_k}^{\alpha_k}$.

Proof. The proof is an adaptation of the classical argument as presented for instance in [18, pp. 82-83]. From (5.1.3)-(5.1.7) we have

$$\frac{\partial \Psi(n;t,z)}{\partial t_i} = (L^i)_+ \Psi(n;t,z), \tag{5.1.12}$$

and thus it is enough to establish (5.1.11) with $(\alpha_1, \alpha_2, \ldots, \alpha_k) = (0, 0, \ldots, 0)$. Defining $\operatorname{res}_\Delta \sum_{j=-\infty}^{d} a_j(n)\Delta^j = a_{-1}(n)$, one checks easily that for $X, Y \in R\{\Delta\}$

$$\operatorname{res}_z \left(X(n)\operatorname{Exp}(n;t,z)\right)\left(Y^*(n-1)\operatorname{Exp}^{-1}(n;t,z)\right) = \operatorname{res}_\Delta XY.$$

Thus,

$$\begin{aligned}
&\operatorname{res}_z \left(\Delta^j \Psi(n;t,z)\right) \Psi^*(n;t,z) \\
&\quad = \operatorname{res}_z \left(\Delta^j S(n;t)\operatorname{Exp}(n;t,z)\right)\left(S^{-1}(n-1;t)\right)^* \operatorname{Exp}^{-1}(n;t,z) \\
&\quad = \operatorname{res}_\Delta \Delta^j = 0, \qquad \text{for } j \geq 0,
\end{aligned}$$

which establishes the proposition. □

If we fix n, from (5.1.11) applied with $j = 0$, $\Psi(n;t,z)(1+z)^{-n}$ and $\Psi^*(n;t,z)(1+z)^n$ are formal power series of the type $(1+O(z^{-1}))\exp\left(\sum t_i z^i\right)$ and $(1+O(z^{-1}))\exp\left(-\sum t_i z^i\right)$, which satisfy the bilinear relations of the continuous KP hierarchy. Thus, by the classical theory (see Section 1.2), there exists a tau function $\tau(n;t)$ (in the variables $t = (t_1, t_2, \ldots)$), such that

$$\Psi(n;t,z) = \frac{\tau(n;t-[z^{-1}])}{\tau(n;t)}\operatorname{Exp}(n;t,z), \tag{5.1.13}$$

and

$$\Psi^*(n;t,z) = \frac{\tau(n;t+[z^{-1}])}{\tau(n;t)}\operatorname{Exp}^{-1}(n;t,z), \tag{5.1.14}$$

where $[z] = (z, z^2/2, z^3/3, \ldots)$.

Remark 5.1.2. Constructing the theory by conjugating L to Λ instead of Δ, as it is done in [6, 19, 39, 75], amounts to write the wave and adjoint wave functions in (5.1.13) and (5.1.14) as

$$z^{\pm n}\frac{\left(1+z^{-1}\right)^{\pm n}\tau(n;t \mp [z^{-1}])}{\tau(n;t)}\exp\left(\pm\sum_{i=1}^{\infty} t_i z^i\right).$$

This shows that the tau functions constructed in this way are obtained from ours by multiplication with a factor $\exp\left(n\sum_{i=1}^{\infty}(-1)^i t_i\right)$.

Remark 5.1.3. As in the differential and q-difference case, it is possible to prove a converse of Proposition 5.1.1. Namely, given formal series

$$\begin{aligned}
\Psi(n;t,z) &= z^n\left(1+z^{-1}\right)^n\left(1+\frac{\psi_1(n;t)}{z}+\frac{\psi_2(n;t)}{z^2}\cdots\right)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right) \\
&= z^n\left(1+O(z^{-1})\right)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right),
\end{aligned}$$

and

$$\Psi^*(n;t,z) = z^{-n}\left(1+O(z^{-1})\right)\exp\left(-\sum_{i=1}^{\infty} t_i z^i\right),$$

satisfying the bilinear identities (5.1.11), $\Psi(n;t,z)$ and $\Psi^*(n;t,z)$ are necessarily wave and adjoint wave functions of the discrete KP hierarchy. The pseudo-difference operator $S(n;t) = 1 + \sum_{j=1}^{\infty} \psi_j(n)\Delta^{-j}$ satisfies (5.1.4) and $L = S\Delta S^{-1}$ solves the discrete KP hierarchy.

The next theorem gives a procedure to construct a large class of tau functions of the discrete KP hierarchy starting from tau functions of the continuous KP hierarchy. It will play a crucial rôle in the rest of the Chapter.

Theorem 5.1.4. *Let $\tau(t)$ be a tau function for the continuous KP hierarchy. Then,*

$$\tau(n;t) = \tau(t_1+n, t_2-n/2, t_3+n/3, \dots) \tag{5.1.15}$$

is a tau function for the discrete KP hierarchy.

Proof. The key point of the proof is to observe that the following *formal* identity holds, as long as j is a positive integer:

$$(1+z)^j = \exp\left(j\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i}z^i\right), \qquad j \geq 0. \tag{5.1.16}$$

We define $\Psi(n;t,z)$ and $\Psi^*(n;t,z)$ by formulae (5.1.13) and (5.1.14) with $\tau(n;t)$ as in (5.1.15). According to Remark 5.1.3, in order to establish the theorem, it is enough to show that $\Psi(n;t,z)$ and $\Psi^*(n;t,z)$ so defined, satisfy the bilinear identities (5.1.11). Let $\Psi(t,z)$ and $\Psi^*(t,z)$ denote the wave and dual wave functions of the continuous KP hierarchy defined by $\tau(t)$. They satisfy

$$\mathrm{res}_z \Psi(t',z)\Psi(t,z) = 0, \qquad \forall\, t, t', \tag{5.1.17}$$

where $\Psi(t',z)$ is understood as the formal Taylor series $\sum_\alpha (1/\alpha!)\partial^\alpha\Psi(t,z)(t'-t)^\alpha$. Making the change of variables

$$s_i = t_i + n\frac{(-1)^{i-1}}{i},$$

as long as $j \geq 0$, one shows easily using (5.1.16) that the following identity holds in $\mathbb{C}\{z\}$:

$$\begin{aligned}(\partial^\alpha\Psi(n+j;t,z))\,\Psi^*(n;t,z) &= (\partial^\alpha\Psi(s',z))\Psi(s,z),\\ &\text{with } s' = (s_1+j, s_2-j/2, s_3+j/3, \dots).\end{aligned}$$

But then, from (5.1.17), we deduce that

$$\mathrm{res}_z\,(\partial^\alpha\Psi(n+j;t,z))\,\Psi^*(n;t,z) = 0,$$

for all $j \geq 0$ and any multi-index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$, $\alpha_s \geq 0$, which is equivalent to (5.1.11). This proves the theorem. □

Sometimes it will be useful to interpret our constructions in the framework of the (formal) infinite dimensional Sato Grassmannian. We remind the reader that to a wave function of the KP hierarchy $\Psi(t,z) = S(t)\exp\left(\sum_{i=1}^{\infty} t_i z^i\right)$, with $S = 1 + \psi_1(t)\partial^{-1} + \psi_2(t)\partial^{-2} + \cdots$, $\partial = d/dx$, $x = t_1$, one associates a plane V in the Sato Grassmannian

$$W = \mathrm{span}\{w_0(z), w_1(z), w_2(z), \dots\}, \tag{5.1.18}$$

via the formal expansion

$$S(x,0,0,\dots)\exp(xz) = \sum_{i=0}^{\infty} w_i(z)\frac{x^i}{i!}. \tag{5.1.19}$$

This expansion assumes that the wave function does not blow up at $x = 0$, meaning that the plane W belongs to the big cell of the Grassmannian. In this case the $w_i(z)$ admit a formal expansion in z, $w_i(z) = z^i\left(1 + O(z^{-1})\right)$. We shall sometimes denote by $\tau_W(t)$, $\Psi_W(t,z)$ and $\Psi_W^*(t,z)$ the tau function, the wave function and the adjoint wave function corresponding to the plane W. The functions $\Psi_W^*(-t,z)$ and $\Psi_W(-t,z)$ are respectively the wave and the adjoint wave functions of another plane W^* in Sato Grassmannian, called the dual plane of W, that is

$$\Psi_W^*(-t,z) = \Psi_{W^*}(t,z), \qquad \Psi_W(-t,z) = \Psi_{W^*}^*(t,z), \tag{5.1.20}$$

and

$$\tau_{W^*}(t) = \tau_W(-t). \tag{5.1.21}$$

In the case of the discrete KP hierarchy, instead of a plane, we have a flag

$$\mathcal{W}: \quad \cdots \subset W_{n+1} \subset W_n \subset W_{n-1} \subset \cdots \tag{5.1.22}$$

with

$$W_n = \text{span of } \{g_n, g_{n+1}, g_{n+2}, \dots\}, \quad g_n = \Psi(n;0,z), \tag{5.1.23}$$

as well as a dual flag

$$\mathcal{W}^*: \quad \cdots \supset W_{n+1}^* \supset W_n^* \supset W_{n-1}^* \supset \cdots \tag{5.1.24}$$

with

$$W_n^* = \text{span of } \{g_n^*, g_{n-1}^*, g_{n-2}^*, \dots\}, \quad g_n^* = \Psi^*(n;0,z). \tag{5.1.25}$$

The discrete KP tau functions $\tau(n;t)$ can then be seen as the tau functions of the plane W_n, see [6, 19, 39]. The bilinear relations (5.1.11) can be interpreted as expressing the orthogonality of W_n and W_n^* with respect to the residue pairing (5.1.10).

In the rest of the Chapter, we shall be concerned only with non-formal tau functions, whose corresponding planes belong to the Segal-Wilson Grassmannian. In this case (5.1.19) converges around $x = 0$, and the expansion of $w_i(z)$ around $z = \infty$ is also convergent. The functions $w_i(z)$ form a so-called algebraic basis of a suitable L^2 closure $\bar{W}$ of W, see [72].

5.2 Tau functions of rank 1 commutative rings of difference operators

In the previous section, only the formal aspects of the theory of the discrete KP hierarchy have been discussed and the reader may wonder if the shift considered in Theorem 5.1.4 may have any analytical meaning, when we are dealing with a continuous KP tau function depending on infinitely many time variables. In this section, we answer this question when the tau function is built from the theta function of an arbitrary non-singular complete irreducible complex algebraic curve. We show that we construct in this way “all” (except for those coming from singular curves) rank 1 commutative rings of difference operators. We are confident that the result should extend to singular complete irreducible complex curves as well. We shall treat the case of rational curves, most important for us, in the next section.

We start by reminding the reader Krichever's construction [56] of rank 1 commutative rings of differential operators, following closely the account in [73, pp. 343-354]. Let X be a compact Riemann surface, P_∞ a distinguished point on it thought of as the point at infinity, z^{-1} a local coordinate near P_∞ extending up to $|z| = 1$, and let $D = \sum_{i=1}^{g} P_i$ be a non-special divisor of degree g on $X \setminus \{P_\infty\}$. For each f in the coordinate ring of the affine curve $X \setminus \{P_\infty\}$

$$A = \{\text{meromorphic functions } f \text{ on } X, \text{ with poles only at } P_\infty\}, \tag{5.2.1}$$

one can construct an ordinary differential operator L_f in the variable $x = t_1$ (depending on the parameters $t_2, t_3, \dots$), of order equal to the order of the pole of f at P_∞, such that

$$L_f\Psi(t,P) = f(P)\Psi(t,P), \qquad t = (x = t_1, t_2, t_3, \dots), \quad P \in X, \tag{5.2.2}$$

where $\Psi(t, P)$ is the Baker-Akhiezer function. It is the unique (up to a constant) common eigenfunction of the commuting operators L_f. An explicit expression for this function is given by

$$\begin{aligned}\Psi(t,P) = &\exp\left(\sum_{j=1}^{\infty} t_j \int^P \eta^{(j)}\right) \times \\ &\frac{\theta(A(P) + \sum t_j U_j - A^{(g)}(D) - \mathcal{K})}{\theta(A(P_\infty) + \sum t_j U_j - A^{(g)}(D) - \mathcal{K})} \frac{\theta(A(P_\infty) - A^{(g)}(D) - \mathcal{K})}{\theta(A(P) - A^{(g)}(D) - \mathcal{K})}.\end{aligned} \tag{5.2.3}$$

We briefly recall the meaning of the various quantities involved in the formula (see Section 1.3): $\{\alpha_j, \beta_j\}$ is a canonical homology basis on X, and $\omega = (\omega_1, \omega_2, \dots, \omega_g)$ is a normalized basis of holomorphic differentials such that $\int_{\alpha_j} \omega_n = \delta_{jn}$. $\theta(w, B)$ (in short $\theta(w)$), $w = (w_1, w_2, \dots, w_g) \in \mathbb{C}^g$, $B_{jn} = \int_{\beta_j} \omega_n$, denotes Riemann's theta function. $\eta^{(n)}$, $n \geq 1$, denotes the meromorphic 1-form on X, with a unique pole of order $n+1$ at P_∞ of the form $\eta^{(n)} = dz^n + O(1)$ and normalized by $\int_{\alpha_j} \eta^{(n)} = 0$, $1 \leq j \leq g$. The integral of $\eta^{(n)}$ is so defined that

$$\int^P \eta^{(n)} = z^n + \sum_{j=1}^{\infty} \frac{c_{jn}}{j z^j}, \qquad \text{for } P \text{ near } P_\infty. \tag{5.2.4}$$

$2\pi i U_n$ denotes the vector of the β-periods of $\eta^{(n)}$, i.e.

$$(U_n)_j = \frac{1}{2\pi i} \oint_{\beta_j} \eta^{(n)}. \tag{5.2.5}$$

Finally, $A(P) = \int^P \omega$ is the Abel map, and $\mathcal{K}$ is the vector of Riemann constants.

By an identity of Riemann, the c_{ji} in (5.2.4) satisfy $c_{ji} = c_{ij}$. Introducing the quadratic form

$$\Omega(t) = -\frac{1}{2} \sum_{i,j \geq 1} c_{ij} t_i t_j, \tag{5.2.6}$$

one shows that the function

$$\tau(t) = \exp(\Omega(t))\, \theta(A(P_\infty) + \sum_{j\geq 1} t_j U_j - A^{(g)}(D) - \mathcal{K}), \tag{5.2.7}$$

is a tau function for the continuous KP hierarchy. In fact

$$\Psi(t,P) = \frac{\tau(t-[z^{-1}])}{\tau(t)} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right), \qquad \text{for } P \text{ near } P_\infty, \tag{5.2.8}$$

up to multiplication by a series $1 + O(z^{-1})$, with constant coefficients (independent of t), i.e. the Baker-Akhiezer function coincides with the wave function of the KP hierarchy.

The theory of rank 1 commutative rings of difference operators has been worked out in [58, 77, 66]. The analogous formula to (5.2.3) for the Baker-Akhiezer function can be found in the appendix of [2]. The spectrum of the ring completes now into an irreducible complex curve by adding *two* non-singular points P_∞ and Q_∞, instead of *one* non-singular point P_∞ in the case of differential operators. Our next theorem shows that this is precisely the geometric meaning of the shift (5.1.15) performed with the (continuous) KP tau function (5.2.7). More precisely, we have

Theorem 5.2.1. *Let $\tau(t)$ be as in* (5.2.7) *and let $\tau(n;t)$ be the discrete KP tau function defined in Theorem 5.1.4 according to* (5.1.15). *Then, the corresponding wave function of the discrete KP hierarchy defined via* (5.1.13) *coincides precisely with the Baker-Akhiezer function of the rank* 1 *commutative ring of difference operators associated with the affine curve $X \setminus \{P_\infty, Q_\infty\}$, with P_∞ and Q_∞ the points with respective coordinates $z=\infty$ and $z=-1$. That is, up to multiplication by a series $1+O(z^{-1})$ with constant coefficients, we have*

$$\begin{aligned}\Psi(n;t,z) = &\exp\left(\sum_{j=1}^{\infty} t_i \int^P \eta^{(j)}\right) \left[\frac{E(Q_\infty,P)}{E(P_\infty,P)}\right]^n \frac{1}{E(Q_\infty,P_\infty)^n} \times \\ &\frac{\theta(A(P) + \sum t_j U_j + n\int_{P_\infty}^{Q_\infty}\omega - A^{(g)}(D) - \mathcal{K})}{\theta(A(P_\infty) + \sum t_j U_j + n\int_{P_\infty}^{Q_\infty}\omega - A^{(g)}(D) - \mathcal{K})} \frac{\theta(A(P_\infty) - A^{(g)}(D) - \mathcal{K})}{\theta(A(P) - A^{(g)}(D) - \mathcal{K})}\end{aligned} \tag{5.2.9}$$

with P the point near P_∞ with coordinate z and $E(P,Q)$ the prime form of the curve X.

Proof. We refer the reader to [67, pp. 207-213] for the definition of the prime form $E(P,Q)$. Denoting by $s(P) = z^{-1}(P)$ the local coordinate of a point P near P_∞, we shall use the following formula from [24, Section 1, formula (2)]:

$$\log \frac{E(P_\infty,Q)}{s(Q)} \frac{E(P_\infty,P)}{s(P)} \frac{s(P)-s(Q)}{E(Q,P)} = \sum_{i,j\geq 1} \sigma_{ij} \frac{s(P)^i}{i} \frac{s(Q)^j}{j}, \tag{5.2.10}$$

for any P, Q near P_∞, with c_{ij} as in (5.2.4). Applying this formula to $Q = Q_\infty$ with coordinate $s(Q_\infty) = -1$ and P an arbitrary point near P_∞ with coordinate $s(P) = z^{-1}$, gives

$$\log(1+z) + \sum_{i,j\geq 1} \frac{(-1)^{j-1}\sigma_{ij}}{ijz^i} = -\log E(Q_\infty,P_\infty) + \log \frac{E(Q_\infty,P)}{E(P_\infty,P)}. \tag{5.2.11}$$

From this equality and (5.2.4) it follows that making the shift in the exponential part $\exp\left(\sum_{j=1}^\infty t_i \int^P \eta^{(j)}\right)$ of the Baker-Akhiezer function (5.2.3), gives an extra factor

$$\left[\frac{E(Q_\infty,P)}{E(P_\infty,P)}\right]^n \frac{1}{E(Q_\infty,P_\infty)^n}. \tag{5.2.12}$$

On the other hand, from (1.3.18), for P near P_∞ with local coordinate $s(P) = z^{-1}$ we have

$$A(P) - A(P_\infty) = \int_{P_\infty}^P \omega_j = -\sum_{j=1}^\infty U_j \frac{1}{jz^j}, \tag{5.2.13}$$

and thus, in particular,

$$\int_{P_\infty}^{Q_\infty} \omega_j = \sum_{j=1}^\infty U_j \frac{(-1)^{j-1}}{j}, \tag{5.2.14}$$

since $s(Q_\infty) = -1$, which finishes the proof. □

Following [2], it is possible to develop the theory of rank 1 commutative rings of difference operators in the manner of Segal and Wilson [72], for the ordinary differential operators. Given the data $\{X, P_\infty, z^{-1}, D, Q_\infty\}$, we introduce the flag

$$\mathcal{W}: \qquad \cdots \subset W_{n+1} \subset W_n \subset W_{n-1} \subset \cdots, \tag{5.2.15}$$

with

$$W_n = \{\text{meromorphic functions } g \text{ on } X \text{ such that } (g) + D - nQ_\infty \geq 0 \text{ on } X \setminus \{P_\infty\}\}, \tag{5.2.16}$$

$\dim W_n/W_{n+1} = 1$. In words: W_n is the space of meromorphic functions on X, with at worst simple poles at the points of the divisor D, a zero of order n or bigger at Q_∞ if $n \geq 0$ (or a pole of order at most $-n$ at Q_∞ if $n < 0$), and an arbitrary behaviour at P_∞. Let A be the coordinate ring of the affine curve $X \setminus \{P_\infty, Q_\infty\}$:

$$A = \{\text{meromorphic functions on } X, \text{ with poles only at } P_\infty \text{ and } Q_\infty\}. \tag{5.2.17}$$

Obviously, for a meromorphic function f on X, we have

$$f \in A \Leftrightarrow \exists k \in \mathbb{Z} \text{ such that } fW_n \subset W_{n+k},\ \forall n. \tag{5.2.18}$$

With these definitions, one shows in the standard manner that for each $f \in A$, there is a finite band difference operator L_f such that

$$L_f \Psi(n; t, z) = f(z)\Psi(n; t, z), \tag{5.2.19}$$

with $\Psi(n; t, z)$ as in (5.2.9). The size of the operator L_f is determined by the order of the poles of f at P_∞ and Q_∞. Namely if f has a pole of order i at P_∞ and a pole of order j at Q_∞, the operator L_f, thought of as a finite band matrix, has i diagonals above the main diagonal, and j diagonals below it.

It follows from the Riemann-Roch theorem that, for j big enough, there always exists a meromorphic function on X with a simple pole at P_∞ and a pole of order j at Q_∞. The corresponding operator L_f will have one diagonal above the main diagonal and j diagonals below it, thus providing a finite band operator of the type needed to solve the discrete KP hierarchy (5.1.1). We shall now show that, modulo a change of time variables, such an operator does indeed solve the discrete KP hierarchy. This result will be used in Section 5.5. Expanding f around P_∞ as

$$f(z) = z + c_0 + c_1 z^{-1} + \cdots, \qquad \text{near } P_\infty,$$

it follows from (5.2.19) and the definition of the wave function (5.1.7) that

$$L_f = SL_0S^{-1}, \tag{5.2.20}$$

with

$$L_0 = \Delta + c_0 I + c_1 \Delta^{-1} + \cdots . \tag{5.2.21}$$

The constant coefficient pseudo-difference operator L_0 is not a finite band operator in general, but it will be of that type when the curve X is rational, as will be discussed in the next section.

For an integer $j \geq 1$, we define coefficients c_{ij} by expanding L_0^j as

$$L_0^j = \sum_{i=1}^{j} c_{ij}\Delta^i + (\text{constant})I + (L_0^j)_-, \tag{5.2.22}$$

with $c_{jj} = 1$. We introduce the new time variables $s = (s_1, s_2, s_3, \dots)$ via the triangular change of variables

$$t_i = s_i + \sum_{j=i+1}^{\infty} c_{ij} s_j. \tag{5.2.23}$$

With these notations we have

Proposition 5.2.2. *Viewed as a function of the time variables* $s_1, s_2, s_3, \dots$ *defined in* (5.2.23)*, the operator* L_f *defined in* (5.2.20) *is a finite band solution of the discrete KP hierarchy, that is*

$$\frac{\partial L_f}{\partial s_j} = [(L_f^j)_+, L_f].$$

Proof. Conjugating equation (5.2.22) by S and taking the Volterra part, we obtain

$$(L_f^j)_- = \sum_{i=1}^{j} c_{ij}(L^i)_- + S(L_0^j)_- S^{-1},$$

with $L = S\Delta S^{-1}$ as in (5.1.3). Multiplying this equation to the right by S and using (5.1.4) and (5.2.23) we get

$$-(L_f^j)_- S + S(L_0^j)_- = \sum_{i=1}^{j} c_{ij}\frac{\partial S}{\partial t_i} = \frac{\partial S}{\partial s_j}.$$

Thus:

$$\begin{aligned}
\frac{\partial L_f}{\partial s_j} =& \frac{\partial S}{\partial s_j} L_0 S^{-1} - S L_0 S^{-1}\frac{\partial S}{\partial s_j} S^{-1} \\
=& -(L_f^j)_- S L_0 S^{-1} + S(L_0^j)_- L_0 S^{-1} \\
&+ S L_0 S^{-1}(L_f^j)_- S S^{-1} - S L_0 S^{-1} S (L_0^j)_- S^{-1} \\
=& [-(L_f^j)_-, L_f] + S[(L_0^j)_-, L_0]S^{-1} \\
=& [(L_f^j)_+, L_f],
\end{aligned}$$

since constant coefficient pseudo-difference operators commute between themselves. This concludes the proof of the proposition. □

5.3 A rational flag manifold

The Grassmannian Gr^{rat} parametrizes the maximal rank 1 commutative rings of differential operators A such that the curve $X = \operatorname{Spec}(A) \cup \{P_\infty\}$ is rational, that is the map $\mathbb{C} \cup \{\infty\} \to X$ is an isomorphism outside of the inverse image of the singular locus of X. We call z the rational parameter, $z \in \mathbb{C}$, with $z(P_\infty) = \infty$. In this section, we apply Theorem 5.1.4 to a tau function $\tau_W(t)$ of the continuous KP hierarchy with $W \in Gr^{rat}$. We obtain in this way a rational flag $\mathcal{W}$ associated with a maximal rank 1 commutative ring of difference operators $A_{\mathcal{W}}$, with $X = \operatorname{Spec}(A_{\mathcal{W}}) \cup \{P_\infty, Q_\infty\}$, $z(Q_\infty) = -1$.

We denote by $\mathcal{P} = \mathbb{C}[z]$ the space of polynomials in the variable z, and by $\mathcal{R} = \mathbb{C}(z)$ the space of rational functions in z. The following beautiful and efficient characterization of Gr^{rat} was obtained in [72] (cf. Section 1.6).

Definition 5.3.1. A subspace $W \subset \mathcal{R}$ belongs to Gr^{rat} if and only if there are polynomials $p(z)$ and $q(z)$ such that $p\mathcal{P} \subset W \subset q^{-1}\mathcal{P}$, and the codimension of W in $q^{-1}\mathcal{P}$ is equal to the degree of q.

Equivalently, following [78], one can introduce the subspace $\mathcal{C}$ of the algebraic dual of $\mathcal{P}$ spanned by the linear functionals

$$\langle e(r,\lambda), g\rangle = g^{(r)}(\lambda), \qquad \lambda \in \mathbb{C}, \quad r \geq 0. \tag{5.3.1}$$

Then $W \in Gr^{rat}$ is equivalent to say that there is a finite dimensional subspace of conditions $C \subset \mathcal{C}$, of dimension equal to the degree of q, such that

$$W = q^{-1}V_C = q^{-1}\{g \in \mathcal{P} : \langle c, g\rangle = 0 \text{ for all } c \in C\}. \tag{5.3.2}$$

Multiplying eventually W by q/z^K, with K =degree of q, one can always assume that $q = z^K$. This amounts to multiply the wave function by a series $1+O(z^{-1})$ with constant coefficients, or equivalently, to multiply the tau function by a factor $\exp\left(\sum_{i=1}^{\infty} k_i t_i\right)$, for some appropriate constants k_i. In this case, there is a nice Wronskian formula for the tau function $\tau_W(t)$ associated with the plane W, generalizing the classical Wronskian formula describing the K solitons solutions of the KP hierarchy

$$\tau_W(t) = \mathrm{Wr}_{t_1}(\phi_1(t), \phi_2(t), \ldots, \phi_K(t)) \tag{5.3.3}$$

with

$$\phi_j(t) = \left\langle c_j, \exp\left(\sum_{i=1}^{\infty} t_i z^i\right)\right\rangle, \tag{5.3.4}$$

c_j, $1 \le j \le K$, a basis of the space of conditions C. The notation Wr_{t_1} means the Wronskian with respect to the variable t_1.

It will be useful to consider the non-degenerate bilinear form on $\mathcal{R}$

$$B(f,g) = \mathrm{res}_{z=\infty} f(z)g(z), \qquad f, g \in \mathcal{R}. \tag{5.3.5}$$

Notice that, expanding $f \in \mathcal{R}$ around $z = \infty$, $f(z) \in \mathbb{C}\{z\}$ as defined in (5.1.9) and the formal residue in (5.1.10) is exactly the residue at infinity up to a sign, that is $\mathrm{res}_z f(z) = -\mathrm{res}_{z=\infty} f(z)$. Remembering (5.1.20), the bilinear identities (5.1.17) of the KP theory imply that

$$\mathrm{res}_{z=\infty} \Psi_W(x, 0, 0, \ldots, z)\Psi_{W^*}(y, 0, 0, \ldots, z) = 0, \qquad \forall x, y, \tag{5.3.6}$$

meaning that the plane W and the dual plane W^* defined as in (5.1.18), (5.1.19) are orthogonal complements of each other relative to the bilinear pairing (5.3.5).

Modulo some rescaling, one can always assume that the points λ at which the conditions (5.3.1) are evaluated satisfy $|\lambda| < 1$, see [78]. Applying then Theorem 5.1.4 to the tau function (5.3.3), that is shifting t_i to $t_i + n(-1)^{i-1}/i$, one computes easily that

$$\tau(n;t) = \mathrm{Wr}_\Delta(\phi_1(n;t), \phi_2(n;t), \ldots, \phi_K(n;t)) \tag{5.3.7}$$

with

$$\phi_j(n;t) = \langle c_j, \mathrm{Exp}(n;t,z)\rangle, \tag{5.3.8}$$

and $\mathrm{Exp}(n;t,z)$ as in (5.1.5). The notation Wr_Δ means the discrete Wronskian with respect to the variable n, and is often called a Casorati determinant:

$$\mathrm{Wr}_\Delta(\phi_1(n), \phi_2(n), \ldots, \phi_K(n)) = \det(\Delta^{i-1}\phi_j(n))_{1\le i,j\le K}. \tag{5.3.9}$$

Lemma 5.3.2. *The wave function of the discrete KP hierarchy corresponding to the plane $W = q^{-1}V_C \in Gr^{rat}$ is given by*

$$\Psi(n;t,z) = q(z)^{-1}\frac{\mathrm{Wr}_\Delta(\phi_1(n;t), \phi_2(n;t), \ldots, \phi_K(n;t), \mathrm{Exp}(n;t,z))}{\mathrm{Wr}_\Delta(\phi_1(n;t), \phi_2(n;t), \ldots, \phi_K(n;t))}, \tag{5.3.10}$$

with $\phi_j(n;t)$ as in (5.3.8).

Proof. As explained above, one can always assume that $q = z^K$ and then multiply the final result by z^K/q. We mimic the standard trick used in the case of the continuous KP hierarchy, based now on the identity

$$\phi_j(n; t - [z^{-1}]) = \phi_j(n; t) - \frac{1}{z}\Delta\phi_j(n; t), \tag{5.3.11}$$

which follows easily from (5.3.8). Using this identity, one computes that

$$\mathrm{Wr}_\Delta(\phi_1(n; t), \phi_2(n; t), \dots, \phi_K(n; t), \mathrm{Exp}(n; t, z)) = z^K \tau(n; t - [z^{-1}])\mathrm{Exp}(n; t, z),$$

by subtracting $1/z\times$ row $(i+1)$ from row i, $1 \le i \le K$, in the determinant on the left-hand side, and expanding it along the last column, all of whose entries are zero, except the last one which is $z^K\mathrm{Exp}(n; t, z)$. The result follows then from formula (5.1.13) for the discrete KP wave function, concluding the proof of the lemma. □

Corollary 5.3.3. *The wave and dual wave functions of the discrete KP hierarchy built from a plane $W \in Gr^{rat}$, according to the recipe of Theorem 5.1.4, can be written as*

$$\Psi(n; t, z) = Q(n; t)q(\Delta)^{-1}\mathrm{Exp}(n; t, z), \tag{5.3.12}$$
$$\Psi^*(n; t, z) = P^*(n - 1; t)p(\Delta^*)^{-1}\mathrm{Exp}^{-1}(n; t, z), \tag{5.3.13}$$

with $q(z)$ and $p(z)$ the polynomials appearing in Definition 5.3.1, and $Q(n; t) = \Delta^K + \sum_{i=1}^K a_i(n; t)\Delta^{K-i}$ and $P(n; t) = \Delta^{K'} + \sum_{i=1}^{K'} \Delta^{K'-i} \cdot b_i(n + 1; t)$, positive difference operators with $K =$ degree of $q(z)$ and $K' =$ degree of $p(z)$.

Proof. Formula (5.3.12) follows immediately from (5.3.10) by expanding the discrete Wronskian in the numerator of this formula along the last column, and using that

$$q(\Delta)^{-1}\mathrm{Exp}(n; t, z) = q(z)^{-1}\mathrm{Exp}(n; t, z).$$

Passing to the orthogonal complements in Definition 5.3.1, with respect to the bilinear form B (5.3.5), the dual plane W^* satisfies $q(z)\mathcal{P} \subset W^* \subset p(z)^{-1}\mathcal{P}$. Thus, applying again Lemma 5.3.2, the wave function $\Psi_{W^*}(n; t, z)$ built via Theorem 5.1.4 from the tau function $\tau_{W^*}(t)$, can be written as

$$\Psi_{W^*}(n; t, z) = \left(\Delta^{K'} + \sum_{i=1}^{K'} b_i^*(n; t)\Delta^{K'-i}\right) p(\Delta)^{-1}\mathrm{Exp}(n; t, z),$$

with $K' =$ degree of $p(z)$. By (5.1.20), the dual wave function $\Psi^*(n; t, z)$ (built from the original plane W) satisfies

$$\Psi^*(n; t, z) = \Psi_{W^*}(-n; -t, z) = \left(\Delta^{*K'} + \sum_{i=1}^{K'} b_i(n; t)\Delta^{*K'-i}\right) p(\Delta^*)^{-1}\mathrm{Exp}^{-1}(n; t, z),$$

with $b_i(n; t) = b_i^*(-n; -t)$. This establishes (5.3.13) and concludes the proof of the corollary. □

For $W \in Gr^{rat}$, A_W denotes the ring of polynomials that leave W invariant

$$A_W = \{f(z) \in \mathcal{P} : f(z)W \subset W\}. \tag{5.3.14}$$

As in the case of non-singular curves that we reviewed in Section 5.2, for each $f \in A_W$, there is a unique differential operator L_f in the variable t_1, of order equal to the degree of f, such that $L_f\Psi(t,z) = f(z)\Psi(t,z)$, where $\Psi(t,z)$ is the corresponding wave function of the KP hierarchy. The affine curve $\mathrm{Spec}(A_W)$ completes into an irreducible rational curve $X = \mathrm{Spec}(A_W) \cup \{P_\infty\}$, with $z(P) = \infty$, see [72]. From (5.3.10), the flag

$$\mathcal{W}: \qquad \cdots \subset W_{n+1} \subset W_n \subset W_{n-1} \subset \cdots, \tag{5.3.15}$$

with

$$W_n = \mathrm{span}\{\Psi(n;0,z), \Psi(n+1;0,z), \dots\}, \tag{5.3.16}$$

defined in terms of the wave function $\Psi(n;t,z)$ as in (5.1.22) and (5.1.23), is a flag of subspaces of the set of rational functions $\mathcal{R}$. We introduce the ring $A_{\mathcal{W}}$ of rational functions which preserve the flag $\mathcal{W}$:

$$\begin{aligned} A_{\mathcal{W}} = \{f(z) \in \mathcal{R} \text{ with poles only at } z = -1 \text{ and } z = \infty, \\ \text{such that } \exists k \in \mathbb{Z} \text{ for which } f(z)W_n \subset W_{n+k},\ \forall n\}. \end{aligned} \tag{5.3.17}$$

This is the analogue of the ring A defined in (5.2.17), (5.2.18) in our analysis of non-singular curves.

Theorem 5.3.4. *For each $f \in A_{\mathcal{W}}$, there is a finite band operator L_f with i diagonals above the main diagonal and j diagonals below it, with i and j denoting respectively the order of the poles of f at $z = \infty$ and $z = -1$, such that*

$$L_f\Psi(n;t,z) = f(z)\Psi(n;t,z). \tag{5.3.18}$$

Moreover,

$$\mathrm{Spec}(A_{\mathcal{W}}) = \mathrm{Spec}(A_W) \setminus \{Q_\infty\}, \tag{5.3.19}$$

where Q_∞ is the point with coordinate $z = -1$ on the complete irreducible rational curve $X = \mathrm{Spec}(A_W) \cup \{P_\infty\}$, $z(P_\infty) = \infty$.

Proof. The first assertion (5.3.18) is proved exactly like in the non-singular case, following the method of [72]. Thus, we concentrate on (5.3.19). For each $f(z) \in A_W$, there exists an ordinary differential operator L_f in t_1 such that $L_f\Psi(t,z) = f(z)\Psi(t,z)$, with $\Psi(t,z)$ the wave function of the (continuous) KP hierarchy. By performing the shift $t_i \to t_i + n(-1)^{i-1}/i$ in the coefficients of this operator (which makes sense, because the shift can be performed in the wave function), we obtain an ordinary differential operator in t_1 which satisfies the same equation, with $\Psi(t,z)$ replaced by $\Psi(n;t,z)$. By (5.1.12)

$$\frac{\partial\Psi(n;t,z)}{\partial t_1} = (\Delta + a_0(n;t))\Psi(n;t,z),$$

with $\Delta + a_0(n;t) = L_+$, and thus we can re-express all the t_1 derivatives in terms of difference derivatives, leading to a *positive difference* operator L_f satisfying

$$L_f\Psi(n;t,z) = f(z)\Psi(n;t,z).$$

Since L_f is positive, remembering the definitions (5.3.15)-(5.3.17), this equation means that $f(z)W_n \subset W_n$, establishing thus that $A_W \subset A_{\mathcal{W}}$ or equivalently $\mathrm{Spec}(A_{\mathcal{W}}) \subset \mathrm{Spec}(A_W)$.

We now show that A_W is strictly included in $A_{\mathcal{W}}$. Let us define

$$L_0 = \Lambda^j q(\Delta)p(\Delta), \qquad j \in \mathbb{Z}, \tag{5.3.20}$$

with $q(z)$ and $p(z)$ as in Definition 5.3.1. Remembering the general definition of the wave function (5.1.7) and the adjoint wave function (5.1.8), from (5.3.12) and (5.3.13) in Corollary 5.3.3, we deduce that

$$S(n;t) = Q(n;t)q(\Delta)^{-1} \qquad \text{and} \qquad S^{-1}(n;t) = p(\Delta)^{-1}P(n;t). \tag{5.3.21}$$

Thus, using that constant coefficient pseudo-difference operators commute between themselves, we get that

$$\begin{aligned} SL_0S^{-1} &= Q(n;t)q(\Delta)^{-1}\Lambda^j q(\Delta)p(\Delta)p(\Delta)^{-1}P(n;t) \\ &= Q(n;t)\Lambda^j P(n;t). \end{aligned} \tag{5.3.22}$$

Since both Q and P are positive difference operators, this shows that the operator SL_0S^{-1} is a *finite band* operator. Moreover, since $\Lambda\mathrm{Exp}(n;t,z) = (z+1)\mathrm{Exp}(n;t,z)$, from the definition of L_0, we also have that

$$(SL_0S^{-1})S\mathrm{Exp}(n;t,z) = (z+1)^j p(z)q(z)S\mathrm{Exp}(n;t,z). \tag{5.3.23}$$

The upshot is that the function $f(z) = (z+1)^j p(z)q(z) \in A_{\mathcal{W}}$ for any $j \in \mathbb{Z}$. Indeed $L_f = SL_0S^{-1}$ satisfies $L_f\Psi(n;t,z) = f(z)\Psi(n;t,z)$, meaning $f(z)W_n \subset W_{n+j}$. As soon as j is a negative integer, $f(z)$ is not a polynomial and thus $f(z) \notin A_W$. Since $f(z)$ has a pole at $z = -1$, the point $Q_\infty \in \mathrm{Spec}(A_W)$ with coordinate $z(Q_\infty) = -1$ does not belong to $\mathrm{Spec}(A_{\mathcal{W}})$. By Theorem I of [66], the spectrum of any maximal rank 1 commutative ring of difference operators is a complete irreducible complex curve *minus exactly two* non-singular points. Thus there can be no other point than Q_∞ in $\mathrm{Spec}(A_W)$ missing from $\mathrm{Spec}(A_{\mathcal{W}})$. This establishes (5.3.19) and concludes the proof of the theorem. □

Example 5.3.5. We illustrate Theorem 5.3.4 above with the simple example of the plane $W \in Gr^{rat}$ leading to the 1-soliton solution of the KP hierarchy:

$$W = \frac{1}{z}\{g \in \mathcal{P} : \langle c, g\rangle \equiv g(\lambda) - \alpha g(\mu) = 0\}.$$

One computes easily that the algebra A_W is generated by the functions $f = (z-\lambda)(z-\mu)$ and $g = z(z-\lambda)(z-\mu)$, while the bigger algebra $A_{\mathcal{W}}$ is generated by $h = (z-\lambda)(z-\mu)/(z+1)$ and f as above. One checks that $g = fh + (\lambda+\mu+1)f - (\lambda+1)(\mu+1)h$. The corresponding affine curves have respective equations

$$\mathrm{Spec}(A_W): \qquad 2g - (\lambda+\mu)f = \pm f\sqrt{4f + (\lambda-\mu)^2}, \tag{5.3.24}$$

and

$$\mathrm{Spec}(A_{\mathcal{W}}): \qquad 2f - (\lambda+\mu)h - h(h+2) = \pm h\sqrt{h^2 + 2(\lambda+\mu+2)h + (\lambda-\mu)^2}. \tag{5.3.25}$$

Notice that $\mathrm{Spec}(A_W)$ is completed by adding *one* non-singular point at infinity, while $\mathrm{Spec}(A_{\mathcal{W}})$ is completed by adding *two* non-singular points at infinity, both curves being rational with one singular ordinary double point. The case $\mu = -\lambda$ corresponds to the 1-soliton solution of the Korteweg-de Vries hierarchy.

Remark 5.3.6. From the equation $L_f S = SL_0$, $L_0 = f(\Delta)$, using (5.3.21) and (5.3.22), one checks easily that L_0 in (5.3.20) can also be written as

$$L_0 = \left(\Lambda^j P(n;t)\right) Q(n;t). \tag{5.3.26}$$

Comparing this equation with (5.3.22) shows that we can think of L_f as being obtained by factorizing the finite band constant coefficients operator L_0 as a product of two factors and then exchanging the order of the factors. This is precisely what is usually called a *Darboux transformation*.

Remark 5.3.7. If we take L_0 in (5.3.20) to be

$$L_0 = \Lambda^{1-K-K'} q(\Delta)p(\Delta), \text{ with } K = \text{degree of } q(z), \ K' = \text{degree of } p(z), \tag{5.3.27}$$

L_0 has one diagonal above the main diagonal and Proposition 5.2.2 applies to the corresponding L_f.

5.4 An adelic flag manifold and its bispectral property

The adelic Grassmannian $Gr^{ad} \subset Gr^{rat}$, introduced by Wilson [78], parametrizes the maximal rank 1 commutative rings of ordinary differential operators A, such that the curve $X = \mathrm{Spec}(A) \cup \{P_\infty\}$ is unicursal, i.e. the map $\mathbb{C} \cup \{\infty\} \to X$ from the projective line to the curve is bijective. Such rational curves can be characterized as having cusps as only singularities. As shown in [78], those rings are the only maximal rank 1 commutative rings of ordinary differential operators enjoying the bispectral property alluded to in the introduction. In this section we show that if we start from a plane W in the smaller (compared with Gr^{rat}) Grassmannian Gr^{ad}, the maximal rank 1 commutative ring of difference operators constructed in Theorem 5.3.4 enjoys a bispectral property reminiscent of the familiar bispectral property satisfied by the classical orthogonal polynomials.

Following [78], the planes $W = q^{-1}V_C \in Gr^{ad}$ can be characterized as being defined by a subspace $C \subset \mathcal{C}$ of *homogeneous* conditions. This means that C has a basis consisting of conditions c_j, $1 \le j \le K$, each one evaluated at *one point* λ_j:

$$\langle c_j, g\rangle = \sum_{k,\text{ finite}} \alpha_{jk} g^{(k)}(\lambda_j), \tag{5.4.1}$$

eventually some of the points λ_j may coincide. In this case, there is a global factor $\prod_{j=1}^K \exp\left(\sum_{i=1}^\infty t_i\lambda_j^i\right)$, which factors out of the Wronskian determinant (5.3.3) defining $\tau_W(t)$; the remaining factor is a monic polynomial in t_1 with variable coefficients (which are also polynomials in any finite number of the t_i's, $i = 2, 3, \ldots$). One checks easily that by choosing $q(z) = \prod_{j=1}^K (z-\lambda_j)$ as a normalization factor in the definition of W in (5.3.2), instead of $q(z) = z^K$, one gets rid of the exponential factor in front of the tau function, so that it is a monic polynomial in t_1. As shown in [78], this normalization has the effect that the wave function $\Psi(t,z)$ of the KP hierarchy satisfies $\Psi(t,z)\exp\left(-\sum_{i=1}^{\infty}t_iz^i\right)\to 1$ if either z or $t_1\to\infty$. This is crucial in establishing the bispectral property satisfied by the wave function. In the sequel, when we speak of a plane $W\in Gr^{ad}$, it will always be assumed that it is so normalized.

With this definition of Gr^{ad}, it is proved in [79] that Gr^{ad} can be parametrized by the space of all pairs (X,Z) of $N\times N$ complex matrices satisfying the condition

$$[X,Z]+I \text{ has rank } 1,\ I=\text{identity matrix}, \tag{5.4.2}$$

modulo simultaneous conjugation by a matrix of $GL(N,\mathbb{C})$. The expression of the tau function in terms of the "Calogero-Moser" matrices X and Z, due to Shiota [74], will be most useful to us

$$\tau(t)=\det\left\{X+\sum_{i=1}^{\infty}it_i(-Z)^{i-1}\right\}. \tag{5.4.3}$$

The assumption that all points λ_j involved in (5.4.1) satisfy $|\lambda_j|<1$, amounts to say that all the eigenvalues of the matrix Z are in the interior of the unit circle. Thus $(I-Z)^{-1}=\sum_{i=1}^{\infty}Z^{i-1}$ and applying Theorem 5.1.4 to this tau function, we get that the corresponding discrete KP tau function is given by

$$\tau(n;t)=\det\left\{\tilde{X}+n(I-Z)^{-1}\right\}, \tag{5.4.4}$$

with

$$\tilde{X}=X+\sum_{i=1}^{\infty}it_i(-Z)^{i-1}. \tag{5.4.5}$$

This shows that $\tau(n;t)$ is a polynomial in the discrete variable n (and a quasi-polynomial in the variables $t_1,t_2,\dots$).

Lemma 5.4.1. *Let $\tau(n;t)$ be as in* (5.4.4) *and let $\Psi(n;t,z)$ and $\Psi^*(n;t,z)$ be the corresponding wave and adjoint wave functions of the discrete KP hierarchy. Then, $\Psi(n;t,z)\times$ $\mathrm{Exp}^{-1}(n;t,z)$ and $\Psi^*(n;t,z)\mathrm{Exp}(n;t,z)$ are rational functions of z and n which tend to* 1 *either as z or $n\to\infty$.*

Proof. The assertion in z is always true, whichever normalization is taken for the tau function, but this is not the case for the variable n, for which the normalization (5.4.4) is crucial. A direct computation, using formulae (5.1.13) and (5.1.14), shows that

$$\Psi(n;t,z)\mathrm{Exp}^{-1}(n;t,z)=\det\left\{I-\left(\tilde{X}+n(I-Z)^{-1}\right)^{-1}(zI+Z)^{-1}\right\}, \tag{5.4.6}$$

$$\Psi^*(n;t,z)\mathrm{Exp}(n;t,z)=\det\left\{I+\left(\tilde{X}+n(I-Z)^{-1}\right)^{-1}(zI+Z)^{-1}\right\}, \tag{5.4.7}$$

which makes the assertion of the lemma obviously true, finishing the proof. □

We define

$$\tilde{\Psi}(n,z)=\Psi(n;t,z)\exp\left(-\sum_{i=1}^{\infty}t_iz^i\right), \tag{5.4.8}$$

$$\tilde{\Psi}^*(n,z)=\Psi^*(n;t,z)\exp\left(\sum_{i=1}^{\infty}t_iz^i\right), \tag{5.4.9}$$

omitting to write the explicit t dependence, which is irrelevant for what follows (we just think of $t_1, t_2, \dots$ as parameters). Putting

$$z = e^x - 1 \qquad \text{and} \qquad n = z, \tag{5.4.10}$$

in (5.4.8), the factor $(1+z)^n$ that remains on the right-hand side becomes e^{xz}, and from (5.4.6) we have

$$\tilde{\Psi}(z, e^x - 1) = \det \left\{ I - \left(\tilde{X} + z(I - Z)^{-1} \right)^{-1} ((e^x - 1)I + Z)^{-1} \right\} e^{xz}. \tag{5.4.11}$$

Notice that the first factor on the right-hand side of (5.4.11) is a rational function in z and e^x, and tends to 1 as $z \to \infty$. This makes it plausible for $\tilde{\Psi}(z, e^x - 1)$ to be a wave function for the *continuous* KP hierarchy corresponding to some plane $W' \in Gr^{rat}$, i.e. a wave function of "solitonic type". That this is in fact true is the content of our next theorem. In the statement of the theorem $\Psi_{W'}(x, z)$ should be interpreted as a wave function of the continuous KP hierarchy evaluated at $t_1 = x$, $t_2 = t_3 = \cdots = 0$; the "original t dependence" in $\tilde{\Psi}(n; z)$ (see (5.4.8)) should be thought of as hidden in W'.

Theorem 5.4.2. *Let $W \in Gr^{ad}$ and let $\tilde{\Psi}(n, z)$ be defined in terms of the corresponding discrete KP wave function $\Psi(n; t, z)$ according to* (5.4.8). *There exists a plane $W' \in Gr^{rat}$ whose wave function $\Psi_{W'}(x, z)$ satisfies*

$$\Psi_{W'}(x, z) = \tilde{\Psi}(z, e^x - 1). \tag{5.4.12}$$

As a consequence, the common eigenfunction $\tilde{\Psi}(n, z)$ of the maximal rank 1 *commutative ring of difference operators $A_{\mathcal{W}}$ constructed in Theorem 5.3.4, satisfying*

$$L_f \tilde{\Psi}(n, z) = f(z) \tilde{\Psi}(n, z), \qquad \forall f \in A_{\mathcal{W}}, \tag{5.4.13}$$

is also the common eigenfunction of a maximal rank 1 *commutative ring of differential operators $\mathcal{A}_{W'}$ (in the variable z), that is*

$$B_\theta(z, d/dz) \tilde{\Psi}(n, z) = \theta(n) \tilde{\Psi}(n, z), \qquad \forall \theta \in A_{W'}. \tag{5.4.14}$$

Before giving the proof of the theorem (which is an adaptation of Wilson's original argument [78]), we need to give a few definitions. We introduce the following multiplicative groups of formal operators:

$$\mathcal{V} = \Big\{ 1 + \sum_{i,j=1}^{\infty} v_{ij} n^{-j} \Lambda^{-i} \Big\}, \qquad \mathcal{V}^* = \Big\{ 1 + \sum_{i,j=1}^{\infty} v_{ij} n^{-j} \Lambda^{i} \Big\},$$
$$\mathcal{V}' = \Big\{ 1 + \sum_{i,j=1}^{\infty} v_{ij} e^{-jx} \partial^{-i} \Big\}, \tag{5.4.15}$$

with $v_{ij} \in \mathbb{C}$ and $\partial = d/dx$. We denote by a_Λ and a_∂ the adjoint isomorphism from $\mathcal{V}$ to $\mathcal{V}^*$ and the adjoint involution from $\mathcal{V}'$ to $\mathcal{V}'$ defined respectively by

$$a_\Lambda(S(n)) = (S(n)^*)^{-1}, \; S(n) \in \mathcal{V}; \; a_\partial(S) = (S^*)^{-1}, \; S \in \mathcal{V}', \; \partial^* = -\partial. \tag{5.4.16}$$

We define the anti-isomorphism $b : \mathcal{V} \to \mathcal{V}'$ by

$$b(n) = \partial,\ b(\Lambda) = e^x, \text{ i.e. } b\left(1 + \sum_{i,j=1}^{\infty} v_{ij} n^{-j} \Lambda^{-i}\right) = 1 + \sum_{i,j=1}^{\infty} v_{ij} e^{-ix} \partial^{-j}. \tag{5.4.17}$$

Notice, that thinking of n as a continuous variable x, since $\Lambda = e^{\partial}$, this is the natural extension of the bispectral anti-isomorphism b defined in [78] by $b(x) = \partial$ and $b(\partial) = x$. Finally, we define an anti-isomorphism $b_s : \mathcal{V}^* \to \mathcal{V}'$ in a similar manner, except for a change of sign

$$b_s(n) = -\partial \qquad \text{and} \qquad b_s(\Lambda) = e^{-x}. \tag{5.4.18}$$

With these definitions, it is straightforward to check that we have the relation

$$a_{\partial} b = b_s a_{\Lambda}. \tag{5.4.19}$$

We shall need the following lemma.

Lemma 5.4.3. *Let $S = 1 + \psi_1(x)\partial^{-1} + \psi_2(x)\partial^{-2} + \cdots$. Assume that the functions*

$$\Psi(x,z) \equiv Se^{xz} = \sum_{i=0}^{\infty} w_i(z) \frac{x^i}{i!} \text{ and } \Psi^*(x,z) \equiv (S^*)^{-1} e^{-xz} = \sum_{i=0}^{\infty} w_i^*(z) \frac{x^i}{i!}, \tag{5.4.20}$$

are analytic functions in x in the neighbourhood of $x = 0$ and rational functions in z (for all x). Then, the subspaces W and W^* of $\mathcal{R}$ spanned respectively by the rational functions $w_i(z)$ and $w_i^*(z)$, $i = 0, 1, 2, \ldots$ are orthogonal complements of each other with respect to the bilinear form B defined in (5.3.5)*.*

Proof. The proof follows from the standard proof establishing the formal bilinear identities satisfied by any wave and dual wave functions of the continuous KP hierarchy, see formula (5.3.6) above and Proposition 7.6 in [78]. □

We can now give the

Proof of Theorem 5.4.2. From formula (5.4.11) the function $\Psi'(x,z)$ defined by

$$\Psi'(x,z) \equiv \tilde{\Psi}(z, e^x - 1), \tag{5.4.21}$$

is of the type requested in Lemma 5.4.3. Denoting by W' the subspace of $\mathcal{R}$ spanned by its Taylor coefficients $w_i'(z)$, $i = 0, 1, 2, \ldots$, around $x = 0$, we shall show that $W' \in Gr^{rat}$. Once this fact is established *and only then,* it implies that a suitable L^2 closure $\bar{W}'$ of W' belongs to the Segal-Wilson Grassmannian (see the "warning" in [78] at the top of p. 199). But then $\Psi'(x,z) \in \bar{W}'$ and it must agree with the Baker-Akhiezer function of the plane $\bar{W}'$, since there is only one element in that plane of the form $\left(1 + O(z^{-1})\right) e^{xz}$. This implies that $\Psi'(x,z)$ is the common eigenfunction of a maximal rank 1 commutative ring of differential operators in x. Making the change of variable $x = \log(1+z)$ in these operators, because of the definition (5.4.21), this statement amounts to (5.4.14).

We now show that indeed $W' \in Gr^{rat}$, according to Definition 5.3.1. From (5.4.11), it is clear that $\Psi'(x,z)$ can be written as $f(x,z)e^{xz}/\tau(z;t)$, with $\tau(z;t) = \det\left\{\tilde{X} + z(I - Z)^{-1}\right\}$ as in (5.4.4) polynomial in z, and $f(x,z)$ polynomial in z too. Thus $W' \subset q(z)^{-1}\mathcal{P}$, with $q(z) = \tau(z;t)$. Expanding the rational functions $w_i'(z)$ around $z = \infty$, one has $w_i'(z) = z^i\left(1 + O(z^{-1})\right)$, showing that the projection of W' onto $\mathcal{P}$ is an isomorphism; from that it follows easily that the codimension of W' in $q(z)^{-1}\mathcal{P}$ is equal to the degree of $q(z)$.

To show that there is a polynomial $p(z)$ such that $p(z)\mathcal{P} \subset W'$ amounts to show that $(W')^{\perp} \subset p(z)^{-1}\mathcal{P}$, for some polynomial $p(z)$, with $(W')^{\perp}$ the orthogonal complement to W' with respect to the bilinear form B defined in (5.3.5). In order to identify $(W')^{\perp}$, we introduce the function $\Psi'^*(x,z)$ defined by

$$\Psi'^*(x,z) \equiv \tilde{\Psi}^*(z+1, e^x - 1)e^x, \tag{5.4.22}$$

with $\tilde{\Psi}^*(n,z)$ as in (5.4.9). From (5.4.7) we have

$$\Psi'^*(x,z) = \det\left\{I + \left(\tilde{X} + (z+1)(I-Z)^{-1}\right)^{-1}((e^x-1)I + Z)^{-1}\right\}e^{-xz}, \tag{5.4.23}$$

showing that $\Psi'^*(x,z)$ admits an expansion $\sum_{i=0}^{\infty} w_i'^*(z)x^i/i!$ around $x = 0$ with $w_i'^*(z)$ rational functions of z. Let $(W')^*$ be the subspace of $\mathcal{R}$ spanned by the functions $w_i'^*(z)$. We claim that $(W')^* = (W')^{\perp}$. From Lemma 5.4.1, the coefficients $\psi_j(n;t)$ of the wave operator $S(n;t)$ in (5.1.7) (in short $S(n)$), defining the wave function $\Psi(n;t,z)$ in (5.4.6), are rational functions of n, vanishing as $n \to \infty$. Expanding these coefficients around $n = \infty$, shows that $S(n) \in \mathcal{V}$, with $\mathcal{V}$ as in (5.4.15). Also, it follows readily from the definition of the anti-isomorphism b in (5.4.17) that

$$b\left(S(n)\right)e^{xz} = \Psi'(x,z), \tag{5.4.24}$$

with $\Psi'(x,z)$ as in (5.4.21). From (5.1.8) and (5.4.9), we have that

$$\left(S(n)^*\right)^{-1}(1+z)^{-(n+1)} = \tilde{\Psi}^*(n+1,z),$$

from which one checks easily that

$$\Psi'^*(x,z) = \left(b_s a_\Lambda(S(n))\right)e^{-xz},$$

with $\Psi'^*(x,z)$ as in (5.4.22) and a_Λ, b_s as in (5.4.16), (5.4.18). Using (5.4.19), this gives that

$$\Psi'^*(x,z) = \left(a_\partial b(S(n))\right)e^{-xz},$$

which combined with (5.4.24), shows that $\Psi'(x,z)$ and $\Psi'^*(x,z)$ satisfy the hypothesis of Lemma 5.4.3, with S there given here by $b(S(n))$, and thus $(W')^* = (W')^{\perp}$. From formula (5.4.23), it is clear that $(W')^* \subset p(z)^{-1}\mathcal{P}$, with $p(z) = \det\left\{\tilde{X} + (z+1)(I-Z)^{-1}\right\} = \tau(z+1;t)$, thus showing that $p(z)\mathcal{P} \subset W'$. This concludes the proof of the theorem. □

Example 5.4.4. Let $W \in Gr^{ad}$ be defined by

$$W = \frac{1}{z-\lambda}\{g \in \mathcal{P} : g'(\lambda) - \alpha g(\lambda) = 0\}, \tag{5.4.25}$$

for some $\lambda, \alpha \in \mathbb{C}$, $|\lambda| < 1$. The ring A_W is generated by $u = (z-\lambda)^2$ and $v = (z-\lambda)^3$; thus $\mathrm{Spec}(A_W)$ is the affine curve

$$\mathrm{Spec}(A_W): \qquad v^2 = u^3, \tag{5.4.26}$$

with a cusp at the origin $u = v = 0$.

One computes easily from (5.3.3), remembering that the normalization (5.4.25) kills the exponential factor in front of $\tau(t)$, that the corresponding discrete KP tau function is given by

$$\tau(n;t) = \sum_{i=1}^{\infty} it_i\lambda^{i-1} - \alpha + n(1+\lambda)^{-1},$$

leading to the following formula for $\tilde{\Psi}(n;z)$ in (5.4.8):

$$\tilde{\Psi}(n;z) = \left(1 - \frac{1+\lambda}{(z-\lambda)(n-\beta(1+\lambda))}\right)(1+z)^n, \text{ with } \beta = \alpha - \sum_{i=1}^{\infty} it_i\lambda^{i-1}. \quad (5.4.27)$$

Using that $\tau(n;t) = n(1+\lambda)^{-1} - \beta$, one checks that

$$\frac{(z-\lambda)^2}{z+1}\tilde{\Psi}(n,z) = \tilde{\Psi}(n+1,z) + b(n)\tilde{\Psi}(n,z) + a(n)\tilde{\Psi}(n-1,z), \quad (5.4.28)$$

with

$$a(n) = \frac{(1+\lambda)^2\tau(n-1;t)\tau(n+1;t)}{\tau(n;t)^2}, \; b(n) = -2(1+\lambda) + \frac{1}{\tau(n+1;t)} - \frac{1}{\tau(n;t)},$$

which shows that the function $r = (z-\lambda)^2/(z+1)$ belongs to the ring $A_{\mathcal{W}}$ as defined in (5.3.17). It is easy to see that $A_{\mathcal{W}}$ is generated by r and u as above. In fact, $v = ur + (1+\lambda)u - (1+\lambda)^2 r$. The affine curve $\mathrm{Spec}(A_{\mathcal{W}})$ has the following equation

$$\begin{aligned} \mathrm{Spec}(A_{\mathcal{W}}): \quad & (u-(1+\lambda)r)^2 = ur^2, \\ \Leftrightarrow \quad & y^2 = r^3(r+4(1+\lambda)), \text{ with } y = u - r(r+2(1+\lambda)). \end{aligned} \quad (5.4.29)$$

The only singular point of $\mathrm{Spec}(A_{\mathcal{W}})$ is a cusp at the origin $y = r = 0$. Notice that, as expected from Theorem 5.3.4, this curve completes by adding *two* non-singular points at infinity, instead of *one* non-singular point in the case of $\mathrm{Spec}(A_W)$ above.

From (5.4.12) and (5.4.27) we obtain

$$\Psi_{W'}(x,z) = \left(1 - \frac{1+\lambda}{(e^x-1-\lambda)(z-\beta(1+\lambda))}\right)e^{xz},$$

which is the wave function of the following plane $W' \in Gr^{rat}$

$$W' = \frac{1}{z-\beta(1+\lambda)}\{g \in \mathcal{P} : g(\beta(1+\lambda)) - (1+\lambda)g(\beta(1+\lambda)-1) = 0\}.$$

This plane corresponds to a 1-soliton solution of the continuous KP hierarchy (see Example 5.3.5). The curve $\mathrm{Spec}(A_{W'})$ is given by (5.3.24) where λ and μ in that formula must be replaced by $\beta(1+\lambda)$ and $\beta(1+\lambda)-1$ respectively, and it has one singular ordinary double point.

Remark 5.4.5. Since the wave and adjoint wave functions of a plane $W \in Gr^{ad}$ are given respectively by $\det\left\{I \mp (xI+X)^{-1}(zI+Z)^{-1}\right\}e^{\pm xz}$, it means that one can always take for the polynomials $p(z)$ and $q(z)$ in Definition 5.3.1, $p(z) = q(z) = \det(zI+Z)$. According to Remark 5.3.7, this shows that the operator $L = SL_0S^{-1}$, $L_0 = \Lambda^{1-2N}\det(\Delta I + Z)^2$, with N the size of the matrices X and Z, can always be chosen as an operator L_f solving the discrete KP hierarchy as in Proposition 5.2.2, with $f(z) = (z+1)^{1-2N}\det(zI+Z)^2$.

5.5 Bispectral orthogonal functions on the circle and the Toda lattice hierarchy

In this last section, we exhibit a family of maximal rank 1 bispectral commutative rings of difference operators $A_{\mathcal{W}}$, with $\mathrm{Spec}(A_{\mathcal{W}})$ the affine curve with equation $y^2 = r^{2K+1}(r+1)$, $K = 1, 2, \ldots$. These rings have the peculiarity that they contain a tridiagonal operator with one diagonal above and one diagonal below the main diagonal. Thus the common eigenfunction $\tilde{\Psi}(n, z)$ in (5.4.13) satisfies a three-term recursion relation, reminiscent of the familiar recursion relation satisfied by any family of orthogonal polynomials. We shall show that the functions $\tilde{\Psi}(n, z)$ (thought of as a family of functions of z) are in fact orthogonal on the circle and that, with an appropriate choice of "time" variables, the tridiagonal operator mentioned above satisfies the Toda lattice hierarchy.

The simplest example, corresponding to $K = 1$, was already discussed in the previous section, see formula (5.4.28) in Example 5.4.4. In this case the ring $A_{\mathcal{W}}$ was obtained from the commutative ring of differential operators A_W, with $\mathrm{Spec}(A_W)$ defined by $v^2 = u^3$, see (5.4.26). However, as emphasized in the introduction, it would be misleading to believe that for $K \geq 2$, the rings $A_{\mathcal{W}}$ are obtained by applying Theorem 5.4.2 to the rank 1 commutative rings of differential operators A_W, with $\mathrm{Spec}(A_W)$ given by $v^2 = u^{2K+1}$.

In [33] Grünbaum and Haine computed the first few iterations of the Darboux transformation from the discrete second derivative operator

$$L_0 = \Lambda - 2I + \Lambda^{-1}, \tag{5.5.1}$$

and checked that by choosing appropriately the Darboux free parameters, one obtains in this way rational solutions of the Toda lattice hierarchy and that the resulting operators provide solutions to the discrete-continuous version of the bispectral problem. Here we shall show that the resulting tridiagonal matrices can be obtained from Theorem 5.4.2 applied to suitable plane $W \in Gr^{ad}$. The tau function of this plane $\tau_W(t)$ turns out to be a *polynomial* tau function of the KP hierarchy but, contrary to the case of the Korteweg-de Vries hierarchy, it is *not* a Schur polynomial.

The lattice version of the elementary Darboux transformation, as originally developed in [62], amounts to perform a lower-upper factorization of the doubly infinite matrix L_0 and to produce a new matrix by exchanging the order of the factors. The factorization involves a free parameter which is present in the new operator. Iterating the Darboux process K times from L_0 is then equivalent to perform a lower-upper factorization of the operator

$$L_0^K = \left(\Lambda^{-K} P\right) Q, \tag{5.5.2}$$

with P and Q monic positive operators of order K. The kernel of Q is specified by K functions $\phi_1, \phi_2, \ldots, \phi_K$ satisfying

$$L_0 \phi_j = \phi_{j-1}, \qquad 1 \leq j \leq K, \tag{5.5.3}$$

with the convention that $\phi_0 = 0$. Indeed from (5.5.3) we have that $\mathrm{Ker}Q \subset \mathrm{Ker}QL_0$, which means that QL_0 can be factorized to the right by Q, that is

$$QL_0 = LQ, \tag{5.5.4}$$

for some new tridiagonal matrix L. In other words, the Darboux transform $Q(\Lambda^{-K}P)$ of $L_0^K = (\Lambda^{-K}P)Q$ is the K-th power of the tridiagonal matrix L. Indeed, from (5.5.2) and (5.5.4) we have

$$L^K = QL_0^K Q^{-1} = Q\Lambda^{-K}PQQ^{-1} = Q\left(\Lambda^{-K}P\right).$$

In [40], we have solved a factorization problem similar to (5.5.2), with the discrete derivative replaced by a q-derivative. We discovered that the problem can be solved in terms of a shifted version of the Schur polynomials. In the present context, the appropriate shifted elementary Schur polynomials are the polynomials $S_j(n;t)$ introduced in (5.1.5), as shown in the next lemma.

Lemma 5.5.1. *Let us define*

$$\phi_j(n;t) = S_{2j-1}(n+j-1;t). \tag{5.5.5}$$

Then

$$L_0\phi_j = \phi_{j-1}. \tag{5.5.6}$$

Proof. From the definition (5.1.5), we deduce that

$$\Delta S_j(n;t) = S_{j-1}(n;t). \tag{5.5.7}$$

By repeated use of this identity, we find

$$\begin{aligned}\phi_j(n+1;t) - 2\phi_j(n;t) + \phi_j(n-1;t) &= \Delta S_{2j-1}(n+j-1;t) - \Delta S_{2j-1}(n+j-2;t)\\ &= S_{2j-2}(n+j-1;t) - S_{2j-2}(n+j-2;t)\\ &= \Delta S_{2j-2}(n+j-2;t)\\ &= S_{2j-3}(n+j-2;t)\\ &= \phi_{j-1}(n;t).\end{aligned}$$

This establishes the lemma. □

At this point, we can make contact with the theory which was developed in the previous sections. The monic difference operator of order K with kernel given by the functions $\phi_j(n;t)$ as in (5.5.5) is given by

$$Q(n;t)f(n) = \frac{\mathrm{Wr}_\Delta(\phi_1(n;t),\dots,\phi_K(n;t),f(n))}{\mathrm{Wr}_\Delta(\phi_1(n;t),\dots,\phi_K(n;t))}. \tag{5.5.8}$$

We define

$$\tau(n;t) = \mathrm{Wr}_\Delta(\phi_1(n;t),\dots,\phi_K(n;t)). \tag{5.5.9}$$

We can rewrite (5.5.7) as

$$S_j(n+1;t) = S_j(n;t) + S_{j-1}(n;t),$$

from which we deduce easily by induction that

$$\begin{aligned}\phi_j(n;t) \equiv S_{2j-1}(n+j-1;t) &= \sum_{i=0}^{j-1}\binom{j-1}{i}S_{2j-i-1}(n;t)\\ &= \langle c_j, \mathrm{Exp}(n;t,z)\rangle,\end{aligned} \tag{5.5.10}$$

with the linear functionals c_j (acting on functions of z) defined by

$$\langle c_j, g(z)\rangle = \sum_{i=0}^{j-1} \binom{j-1}{i} \frac{1}{(2j-i-1)!} g^{(2j-i-1)}(0), \qquad 1 \le j \le K. \tag{5.5.11}$$

Remembering (5.3.7) and (5.3.8), this shows that $\tau(n;t)$ in (5.5.9) is a (polynomial) tau function of the discrete KP hierarchy, defined by the K one point conditions (5.5.11) all at the point zero, $\lambda_1 = \lambda_2 = \ldots = \lambda_K = 0$.

The next sequence of lemmas, Lemma 5.5.2, Lemma 5.5.3 and Lemma 5.5.4, are peculiar to the situation we are dealing with. Lemma 5.5.2 gives an explicit relation between the wave and the dual wave operators as described in Corollary 5.3.3. It will play a crucial rôle in establishing the orthogonality relations satisfied by the wave functions in Theorem 5.5.5 below. Lemma 5.5.3 gives explicit formulae for the entries of the tridiagonal operator L in (5.5.4), in terms of the tau function $\tau(n;t)$ introduced in (5.5.9). Finally, Lemma 5.5.4 specifies the constants c_{ij} involved in formula (5.2.22).

Lemma 5.5.2. *The wave and adjoint wave functions of the discrete KP hierarchy defined by the one point conditions* (5.5.11) *are given by*

$$\Psi(n;t,z) = Q(n;t)\Delta^{-K}\mathrm{Exp}(n;t,z) \tag{5.5.12}$$

and

$$\Psi^*(n;t,z) = P^*(n-1;t)(\Delta^*)^{-K}\mathrm{Exp}^{-1}(n;t,z), \tag{5.5.13}$$

with Q and P the positive monic difference operators of order K defining the Darboux factorization (5.5.2)*. Moreover, we have an explicit Wronskian formula for the adjoint operator P^* in terms of the functions $\phi_j(n;t)$ defining the operator Q in* (5.5.8)*, namely*

$$P^*(n;t)f(n) = \frac{\mathrm{Wr}_{\Delta^*}(\psi_1(n;t), \ldots, \psi_K(n;t), f(n))}{\mathrm{Wr}_{\Delta^*}(\psi_1(n;t), \ldots, \psi_K(n;t))}, \tag{5.5.14}$$

with

$$\psi_j(n;t) = \phi_j(n+K;t), \qquad 1 \le j \le K, \tag{5.5.15}$$

and Wr_{Δ^*} *defined as in* (5.3.9) *with Δ replaced by Δ^*.*

Proof. From (5.3.7), (5.3.8), (5.5.10) and (5.5.11), the function $\tau(n;t)$ defined in (5.5.9) is a tau function of the discrete KP hierarchy constructed from a (polynomial) tau function of the KP hierarchy, according to the recipe described in Theorem 5.1.4. By the definition of Q in (5.5.8), formula (5.5.12) coincides then with the Wronskian formula for the wave function given in (5.3.10).

In order to establish (5.5.13), we must show that $\Psi^*(n;t,z)$ as defined by (5.5.13), (5.5.14) and (5.5.15) satisfies (5.1.14), that is we have to prove that

$$P^*(n-1;t)(\Delta^*)^{-K}\mathrm{Exp}^{-1}(n;t,z) = \frac{\tau(n;t+[z^{-1}])}{\tau(n;t)}\mathrm{Exp}^{-1}(n;t,z), \tag{5.5.16}$$

with $\tau(n;t)$ as in (5.5.9). In order to establish (5.5.16), we shall need the following formula

$$\phi_j(n;t+[z^{-1}]) - \frac{1+z}{z^2}\phi_{j-1}(n;t+[z^{-1}]) = \phi_j(n;t) - \frac{1}{z}\Delta^*\phi_j(n;t). \tag{5.5.17}$$

We invite the reader to compare this formula with (5.3.11) and like to stress that, contrary to (5.3.11), this formula depends on the special form (5.5.5) of the functions $\phi_j(n;t)$. Assuming for a moment (5.5.17), the trick to establish (5.5.16) is similar to the one used to establish the Wronskian formula (5.3.10) for the wave function. Indeed:

$$\begin{aligned}
&\mathrm{Wr}_{\Delta^*}(\psi_1(n-1;t),\dots,\psi_K(n-1;t),\mathrm{Exp}^{-1}(n;t,z))\\
&\quad = z^K\mathrm{Exp}^{-1}(n;t,z)\det\Big((\Delta^*)^{i-1}\Big\{\phi_j(n-1+K;t+[z^{-1}])\\
&\qquad\qquad - \frac{1+z}{z^2}\phi_{j-1}(n-1+K;t+[z^{-1}])\Big\}\Big)_{1\le i,j\le K}\\
&\quad = z^K\mathrm{Exp}^{-1}(n;t,z)\det\left((\Delta^*)^{i-1}\phi_j(n-1+K;t+[z^{-1}])\right)_{1\le i,j\le K}.
\end{aligned}$$

The first equality is obtained after replacing row i by row $i-(1/z)\times$ row $(i+1)$, $1\le i\le K$, using (5.5.17) at each step, and then expanding the determinant along the last column, all of whose entries are zero, except the last one which is $z^K\mathrm{Exp}^{-1}(n;t,z)$. The second equality is obtained after replacing column j by column $j+(1+z)/z^2\times$ column $(j-1)$, $2\le j\le K$, remembering that $\phi_0(n;t)=0$. Combining the last formula with

$$\begin{aligned}
\det\left((\Delta^*)^{i-1}\phi_j(n-1+K;t)\right)_{1\le i,j\le K} &= \det\left(\phi_j(n-i+K;t)\right)_{1\le i,j\le K}\\
&= (-1)^{K(K-1)/2}\det\left(\phi_j(n+i-1;t)\right)_{1\le i,j\le K}\\
&= (-1)^{K(K-1)/2}\mathrm{Wr}_{\Delta}\left(\phi_1(n;t),\dots,\phi_K(n;t)\right)\\
&= (-1)^{K(K-1)/2}\tau(n;t).
\end{aligned}$$

we obtain (5.5.16).

It remains to establish (5.5.17). Applying Δ^* to (5.3.11), and using that

$$L_0 = \Delta+\Delta^* = -\Delta^*\Delta,$$

we get

$$\begin{aligned}
\Delta^*\phi_j(n;t-[z^{-1}]) &= \Delta^*\phi_j(n;t) - \frac{1}{z}\Delta^*\Delta\phi_j(n;t)\\
&= -\Delta\phi_j(n;t) + L_0\phi_j(n;t) + \frac{1}{z}L_0\phi_j(n;t)\\
&= z\phi_j(n;t-[z^{-1}]) - z\phi_j(n;t) + \frac{1+z}{z}\phi_{j-1}(n;t),
\end{aligned}$$

using (5.3.11) and (5.5.6) to deduce the last equality. Replacing t by $t+[z^{-1}]$ in the above formula and dividing by z, we get (5.5.17). This completes the proof of Lemma 5.5.2. □

Lemma 5.5.3. *The operator L in* (5.5.4) *can be expressed in terms of the tau function $\tau(n;t)$ in* (5.5.9) *via the formula*

$$Lf(n) = f(n+1) + \left(-2+\frac{\partial}{\partial t_1}\log\frac{\tau(n+1;t)}{\tau(n;t)}\right)f(n) + \frac{\tau(n-1;t)\tau(n+1;t)}{\tau(n;t)^2}f(n-1). \tag{5.5.18}$$

Proof. From (5.1.5), (5.5.5) and (5.5.7) one checks that

$$\frac{\partial}{\partial t_1}\phi_j(n;t) = \Delta\phi_j(n;t),$$

from which one computes that the operator Q in (5.5.8) can be expanded as

$$Q = \Lambda^K - \left(\frac{\partial}{\partial t_1}\log\tau(n;t) + KI\right)\Lambda^{K-1} + \cdots + (-1)^K\frac{\tau(n+1;t)}{\tau(n;t)}I.$$

Writing the operator L as

$$L = \Lambda + b(n)I + a(n)\Lambda^{-1},$$

and comparing the coefficients of Λ^K and Λ^{-1} in equation (5.5.4), one obtains respectively

$$-2 - \left(\frac{\partial}{\partial t_1}\log\tau(n;t) + K\right) = b(n) - \left(\frac{\partial}{\partial t_1}\log\tau(n+1;t) + K\right)$$

and

$$\frac{\tau(n+1;t)}{\tau(n;t)} = a(n)\frac{\tau(n;t)}{\tau(n-1;t)},$$

which is the desired result and establishes Lemma 5.5.3. □

Lemma 5.5.4. *With L_0 as in* (5.5.1)*, the coefficients c_{ij} entering* (5.2.22) *are given by*

$$c_{ij} = (-1)^{i+j}\binom{2j-i-1}{j-1}, \qquad i \le j. \tag{5.5.19}$$

Proof. Writing L_0 in (5.5.1) as

$$L_0 = \Delta - I + \Delta^{-1} - \Delta^{-2} + \Delta^{-3} - \Delta^{-4} + \cdots,$$

one shows easily by induction on j that

$$L_0^j = \sum_{i\le j} c_{ij}\Delta^i,$$

using the combinatorial identity

$$\sum_{s=i-1}^{j}\binom{2j-s-1}{j-1} = \binom{2j-i+1}{j}, \qquad i \le j+1,$$

which proves Lemma 5.5.4. □

Our final theorem is an application of Proposition 5.2.2 and Theorem 5.4.2 above. Notice that the proof of Proposition 5.2.2 just requires L to be conjugated to a first order constant coefficient pseudo-difference operator L_0 as in (5.2.21), via a wave operator of the discrete KP hierarchy. It does not depend on the assumption that the curves we were dealing with in Section 5.2 were non-singular. The only new feature in the theorem below concerns the orthogonality relations (5.5.24) satisfied by the wave functions, viewed as functions of $x \equiv z+1$, thus extending all the familiar properties of the classical orthogonal polynomials.

Theorem 5.5.5. *After the change of variables*

$$t_i = s_i + \sum_{j=i+1}^{\infty} c_{ij} s_j, \tag{5.5.20}$$

with c_{ij} defined as in (5.5.19)*, the operator L in* (5.5.18)*, which is obtained by performing K elementary successive Darboux transformations from the discrete second derivative operator L_0 in* (5.5.1)*, provides a rational solution (in n) to the Toda lattice hierarchy*

$$\frac{\partial L}{\partial s_i} = [(L^i)_+, L]. \tag{5.5.21}$$

The operator L belongs to a maximal rank 1 *commutative ring of difference operators $A_{\mathcal{W}}$ with the affine curve* $\mathrm{Spec}(A_{\mathcal{W}})$ *defined by the equation*

$$\mathrm{Spec}(A_{\mathcal{W}}): \qquad y^2 = r^{2K+1}(r+1). \tag{5.5.22}$$

The ring $A_{\mathcal{W}}$ can be constructed from an appropriate plane $W \in Gr^{ad}$, following the general procedure explained in Theorem 5.4.2. As a consequence, the function

$$p(n,x) \equiv \Psi(n;t,x-1)\exp\left(-\sum_{i=1}^{\infty} t_i (x-1)^i\right), \tag{5.5.23}$$

which is the common eigenfunction of the operators belonging to the ring $A_{\mathcal{W}}$, is also the common eigenfunction of a maximal rank 1 *commutative ring of differential operators in the variable x, corresponding to some plane $W' \in Gr^{rat}$. Moreover, viewed as a family of functions of x, the functions $p(n;x)$ satisfy the orthogonality relations*

$$\frac{1}{2\pi i}\oint p(n,x)p(m,x^{-1})\frac{dx}{x} = \frac{\tau(n+1;t)}{\tau(n;t)}\delta_{nm}, \text{ for all } n,m \in \mathbb{Z}, \tag{5.5.24}$$

where the integral is taken along any circle in the complex plane of center $x=0$ and radius $R \neq 1$, and $\tau(n;t)$ is the tau function defined in (5.5.9).

Proof. Only (5.5.22) and (5.5.24) remain to be established. We start with (5.5.22). Since $L = SL_0S^{-1}$ (with $S = Q\Delta^{-K}$ as in (5.5.12)) and $L_0(1+z)^n = \left(z^2/(1+z)\right)(1+z)^n$, the function $r = z^2/(1+z)$ belongs to $A_{\mathcal{W}}$. This function has a simple pole at $z=-1$ and $z=\infty$. Thus, we can assume that all other generators of $A_{\mathcal{W}}$ can be taken to be polynomials, that is they must belong to the ring A_W, where W is the plane in Gr^{ad} defined by the K one point conditions (5.5.11). One checks that the ring A_W is generated by one polynomial of degree j for each $j \geq K+1$, and does not contain any lower degree polynomial. This corresponds in fact to the generic situation for a curve of (arithmetic) genus K, when the point at infinity P_∞ is not a Weierstrass point. Now, it follows from (5.5.12) and (5.5.13) that the polynomials $q(z)$ and $p(z)$ in Definition 5.3.1 can be taken to be $q(z) = p(z) = z^K$. Going back to the proof of Theorem 5.3.4, specifically to formula (5.3.23), this shows that the functions $(1+z)^j z^{2K} \in A_{\mathcal{W}}$, for all $j \in \mathbb{Z}$. Using the function r above, one can kill the pole of these functions at $z=-1$, and produce in this way polynomials of degree $K+1, K+2, K+3, \ldots$ belonging to the ring. Thus, together with

r, these functions generate the ring $A_{\mathcal{W}}$. From this one checks that two functions are enough to generate $A_{\mathcal{W}}$. The choice

$$r = \frac{z^2}{4(z+1)}, \qquad y = \frac{z^{2K+1}(z+2)}{4^{K+1}(z+1)^{K+1}},$$

leads to equation (5.5.22).

In order to establish the orthogonality relations (5.5.24), we need to introduce the function

$$p^*(n,x) = \Psi^*(n;t,x-1)\exp\left(\sum_{i=1}^{\infty} t_i(x-1)^i\right). \tag{5.5.25}$$

Notice that the functions $p(n,x)$ and $p^*(n,x)$ coincide with the functions $\tilde{\Psi}(n,z)$ and $\tilde{\Psi}^*(n,z)$ introduced in (5.4.8) and (5.4.9), after the change of variable $z = x-1$. The proof of (5.5.24) will follow from a combination of the discrete KP bilinear identities (see Proposition 5.1.1) together with the relation

$$p(n,x^{-1}) = \frac{\tau(n+1;t)}{\tau(n;t)} x p^*(n+1,x). \tag{5.5.26}$$

We shall denote by $(a_{ij};x^{b_i})$ the $(K+1)\times(K+1)$ matrix with entry (i,j) equal to a_{ij} if $j \neq K+1$ and entry $(i,K+1)$ equal to x^{b_i}. From (5.5.8), (5.5.14), (5.5.15), (5.5.23) and (5.5.25) we compute that

$$p(n,x) = \frac{x^n}{(x-1)^K\tau(n;t)} \det\left(\phi_j(n+i-1;t);x^{i-1}\right), \tag{5.5.27}$$

$$p^*(n,x) = \frac{x^{-n}}{(1-x)^K\tau(n;t)} \det\left(\phi_j(n+i-2;t);x^{K+1-i})\right), \tag{5.5.28}$$

from which (5.5.26) follows immediately.

From (5.5.27) and (5.5.28) $p(n,x)$ and $p^*(n,x)$ are rational functions of x on the Riemann sphere with poles only at $x=0$, $x=1$ and $x=\infty$. Thus, using (5.5.26), to establish (5.5.24), it is enough to prove that

$$\mathrm{res}_{x=0}\, p(n,x)p^*(m+1,x) = \delta_{nm}, \qquad \text{for all } n,m\in\mathbb{Z}, \tag{5.5.29}$$

$$\mathrm{res}_{x=\infty}\, p(n,x)p^*(m+1,x) = -\delta_{nm}, \qquad \text{for all } n,m\in\mathbb{Z}, \tag{5.5.30}$$

which forces $\mathrm{res}_{x=1}\, p(n,x)p^*(m+1,x) = 0$, for all $n,m\in\mathbb{Z}$.

From (5.5.27) and (5.5.28), remembering the definition of $\tau(n;t)$ in (5.5.9), we obtain by a straightforward computation that, around $x=0$, we have the expansion

$$p(n,x)p^*(m+1,x) = \frac{x^{n-m-1}}{\tau(n;t)\tau(m+1;t)}\left(\tau(n+1;t)\tau(m;t)+O(x)\right),$$

which establishes (5.5.29) for $m \le n$. The discrete KP bilinear identities (5.1.11) tell us (remembering that the formal residue there means the residue at infinity) that

$$\mathrm{res}_{z=\infty}\Psi(n+i;t,z)\Psi^*(n;t,z) = 0,\ \forall i\ge 0 \Leftrightarrow \mathrm{res}_{x=\infty}\, p(n+i,x)p^*(n,x) = 0,\ \forall i \ge 0,$$

where the equivalence follows from the definitions (5.5.23) and (5.5.25). Thus (5.5.30) holds for $m \leq n-1$.

It remains to establish (5.5.29) for $m > n$ and (5.5.30) for $m \geq n$. Making the change of variable $y = 1/x$, from (5.5.26) we obtain that

$$p(n,x)p^*(m+1,x)dx = -\frac{\tau(n+1;t)\tau(m;t)}{\tau(n;t)\tau(m+1;t)}p(m,y)p^*(n+1,y)dy,$$

thus implying (5.5.29) for $m \geq n+1$ and (5.5.30) for $n \leq m$. This completes the proof of Theorem 5.5.5. □

5.6 Some explicit formulae for the bispectral ring

In this last section we give an explicit construction for the bispectral ring $A_{W'}$ in Theorem 5.4.2. Proposition 5.6.2 below can be looked up as an alternative proof of the bispectral property in Theorem 5.4.2. Although this approach is less geometric, it allows to obtain easily bispectral operators $B_\theta(z, d/dz)$.

For a discrete function $f(n)$ we define $\left(\int f\right)(n)$ in such a way that

$$\Delta\left(\int f\right)(n) = f(n),$$

that is, up to an additive constant, we have

$$\left(\int f\right)(n) = \sum_{j=0}^{n-1} f(j). \tag{5.6.1}$$

The following lemma is an easy adaptation of Lemma 3.2.1.

Lemma 5.6.1. *Given $g_0, g_1, \ldots, g_{K+1}$, let us define*

$$G(n) = \sum_{j=1}^{K+1}(-1)^{K+1+j}g_j(n)\left(\int g_0 \mathrm{Wr}_\Delta(g_1,\ldots,\hat{g}_j,\ldots,g_{K+1})\right)(n). \tag{5.6.2}$$

Then

$$\mathrm{Wr}_\Delta(g_1,\ldots,g_K,G)(n) = \theta(n)\mathrm{Wr}_\Delta(g_1,\ldots,g_{K+1})(n) \tag{5.6.3}$$

with

$$\theta(n) = \left(\int g_0 \mathrm{Wr}_\Delta(g_1,\ldots,g_K)\right)(n+1). \tag{5.6.4}$$

Using this lemma we can prove using the notations of Theorem 5.4.2 the following proposition.

Proposition 5.6.2. *Let $W \in Gr^{ad}$ and let $\tau_W(n;t)$ be the corresponding discrete tau function. If $\theta(n)$ is a polynomial in n, such that $\theta(n) - \theta(n-1)$ is divisible by the tau function $\tau_W(n;t)$, then $\theta(n)$ belongs to the bispectral ring $A_{W'}$, i.e. there exists a differential operator $B_\theta(z, d/dz)$ with coefficients independent of n, such that*

$$B(z, d/dz)\tilde{\Psi}(n;z) = \theta(n)\tilde{\Psi}(n;z). \tag{5.6.5}$$

In the next example we show how Proposition 5.6.2 can be used in practice to compute the bispectral ring.

Example 5.6.3. Let us consider the situation in Theorem 5.5.5 of an operator L obtained by performing 2 elementary successive Darboux transformations from the discrete second derivative operator $L_0 = \Lambda - 2 + \Lambda^{-1}$. From (5.5.9) and (5.5.10) one easily computes that the corresponding tau function is given by

$$\tau(n;t) = 2n^3 + (6t_1 + 3)n^2 + (6t_1^2 + 6t_1 + 1)n - 6t_3 - 6t_2 + 2t_1^3 + 3t_1^2. \tag{5.6.6}$$

We choose

$$\theta(n) = \frac{n^4}{2} + (2t_1+2)n^3 + \left(3t_1^2 + 6t_1 + \frac{5}{2}\right) n^2 + (2t_1^3 + 6t_1^2 + 4t_1 + 1 - 6t_2 - 6t_3)n, \tag{5.6.7}$$

such that $\theta(n) - \theta(n-1) = \tau(n,t)$. Now, a direct computation shows that

$$\begin{aligned} B_\theta(z, d/dz) = {} & \frac{(z+1)^4}{2}\frac{d^4}{dz^4} + (2t_1+5)(z+1)^3\frac{d^3}{dz^3} \\ & + \frac{3(z+1)^2}{z^2}\left((t_1+2)^2z^2 - 2z - 2\right)\frac{d^2}{dz^2} \\ & - \frac{z+1}{z^3}\left((6t_3 + 6t_2 - 2t_1^3 - 9t_1^2 - 12t_1 - 6)z^3 + (12t_1+12)z^2 + 12t_1 z - 12\right)\frac{d}{dz} \\ & - \frac{6(z+1)}{z^3}(t_1+1)(t_1 z - 2). \end{aligned} \tag{5.6.8}$$

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