Throughout, $R$ denotes a commutative ring with $1 \neq 0$.

1. Let $R$ be a Noetherian Jacobson ring, and $M$ an $R$-module. Show that Nakayama’s Lemma holds without finiteness hypotheses on $M$: if $I \subseteq \text{Rad}(R)$, then $M = IM$ iff $M = 0$.

2. (a) Let $R, S$ be domains and $\varphi : R \to S$ a surjection. If $\dim R = \dim S < \infty$, show that $\varphi$ is an isomorphism.
   (b) Let $k$ be a field, and $R = k[\alpha_1, \ldots, \alpha_n]$ a finitely generated $k$-algebra which is a domain. Show that $n \geq \dim R$, and that equality holds iff $\alpha_1, \ldots, \alpha_n$ are algebraically independent over $k$.

3. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d$.
   (a) Show that $|\text{Spec } R| = \infty$ iff $d \geq 2$.
   (b) For any $n < d$, show that $\bigcap_{p \in \text{Spec } R} \text{ht } p = n \mathfrak{p} = \text{nil } R$.

4. Let $(R, \mathfrak{m})$ be a Noetherian local ring, and $f \in \mathfrak{m}$ a nonzerodivisor.
   (a) Show that $\bigcap_{i=1}^{\infty} (f^i) = 0$.
   (b) If $R/(f)$ has no embedded primes, show that $R$ has no embedded primes. (Hint: if $p$ is an embedded prime of $R$, show that there is a witness $x$ for $p$ with $x \notin (f)$, and deduce that the extension of $p$ to $R/(f)$ consists of zerodivisors.)

Note: part (a) is a special case of Krull’s Intersection Theorem.

5. (a) In general, the height of a prime ideal can be much smaller than its minimal number of generators. In the ring $R = \mathbb{Q}[x_0..x_3]$, use the command $\text{monomialCurveIdeal}$ to find examples of prime ideals of codimension 2 with a large number of generators (using the commands $\text{isPrime}$, $\text{codim}$, and $\text{mingens}$). Can you find examples of such primes that require at least 5 generators? Exactly 11 generators?
   (b) Let
   
   $S = \mathbb{Q}[x_0..x_5]$
   $J = \text{minors}(2, \text{genericSymmetricMatrix}(S, 3))$

Find all subsets $X \subseteq \{x_0, \ldots, x_5\}$ of size 3 such that the image of $X$ in $S/J$ is \textit{not} algebraically independent over $\mathbb{Q}$. (The variety defined by $J$ is called the Veronese surface in $\mathbb{P}^5$.)