Supersymmetric Analysis of a Simplified Two Dimensional Anderson Model at Small Disorder

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Abstract This work proposes a very simple random matrix model, the Flip Matrix Model, liable to approximate the behavior of a two dimensional electron in a weak random potential. Its construction is based on a phase space analysis, a suitable discretization and a simplification of the true model. The density of states of this model is investigated using the supersymmetric method and shown to be given, in the limit of large size of the matrix by the usual Wigner's semi-circle law.

I Introduction

This paper is devoted to the rigorous study of a random matrix model that is liable to give a good approximation for the density of states (DOS) of the two dimensional Anderson model in the weak disorder regime and in the vicinity of the Fermi level. The main result is that the DOS of this random matrix model converges to the usual Wigner's semi-circle law in the small coupling limit. The Anderson model of an electron in a random potential corresponds to the Hamiltonian

$$H \;=\; (-\Delta + \lambda V(x))$$

acting on the Hilbert space $L^2(\mathbb{R}^2)$, where Δ is the usual Laplacian and V is a real Gaussian process on $\mathbb{R}^2$ with short range correlations.

This model was initially proposed to describe the electron motion in doped semi-conductors at low temperature or in normal disordered metals. It is now the central model for the theory of eletronic transport and wave propagation in disordered systems [1]. It was conjectured by Anderson as soon as 1958 [2] that such a model exhibits a localized phase in which the electrons are trapped by the defects. In 1979 it was argued that this model has a phase transition in dimensions three or more between the localized phase and an extended one [3].

The localized phase is now well under control. In one dimension, localization was rigorously established for any disorder at the end of the seventies [4, 5]. Later localization was established in any dimension at strong disorder or for energies out of the conduction bands [6]. A simplified and more efficient method to get this result was given in [7].

In contrast the weak disorder regime is still poorly understood. In two dimensions it has been argued [3, 8] and numerically established [9] that localization persists at arbitrarily small disorder, with a localization length diverging like $O(e^{c/\lambda^2})$. In dimension three, numerical simulations confirm the existence of the Anderson transition [9], leading to an extended phase. In addition, analytical results [10] and other numerical calculations show that the level spacing distribution follows the Wigner-Dyson distribution for random matrix theory (RMT) [11]. This gave the motivation for a description of mesoscopic system in terms of RMT [12]. This method has been very successful when compared to experiments and was the source of developments of supersymmetric methods [13, 14] in solid state physics. But on the rigorous level it is still a mathematical challenge up to now to prove even regularity of the DOS (e.g. real analyticity in energy in a certain interval) at arbitrarily weak coupling in dimensions two and three.

Using a phase-space analysis inspired by the renormalization group method around Fermi surfaces in condensed matter [15, 16, 17], G. Poirot and coauthors [18, 19] have established that the effective Hamiltonian near the Fermi level is given indeed by a random matrix model. In two dimensions this random matrix model is similar to the GUE (Gaussian Unitary Ensemble), but contains an extra discrete symmetry. Unfortunately in three dimensions and beyond, this extra symmetry becomes continuous, and produces more complicated correlations between matrix elements [18]. Recently a band matrix model in three dimensions has been proposed and its DOS studied through the supersymmetric method [20]. In this note, motivated by these works, the two dimensional case for the Anderson model is reconsidered. A simplified model is constructed on the basis of the phase space analysis. It is a matrix model, although not one of the regular ensembles. We analyze this model rigorously through the supersymmetric method, and proves that its DOS converges to the Wigner semi-circle law as the matrix gets large.

II The Flip Matrix Model

Average spectral properties can be studied through the averaged Green's functions of the model. For some suitable choice of units, these averaged Green's functions are defined as

$$G_{\pm}(y-x) \;=\; \lim_{\epsilon\to 0^{\pm}} \int d\mu(V)\langle x|(-\Delta - E + \imath\epsilon + \lambda V)^{-1}|y\rangle$$

where $d\mu(V)$ is the disorder distribution, λ is a coupling constant controlling the disorder strength, and E is real. The DOS is defined by the imaginary part of the averaged retarded Green's function :

$$\nu(E) \;=\; -\frac{1}{\pi}\,\mathrm{Im}G_{+}(0)$$

For a given non-zero E in the conduction band (e.g. for $|E|$ neither too large nor too small), the free Green's function $(p^2 - E)^{-1}$ (we simply write $-\Delta = p^2$) is singular on a surface, which in two dimensions is a curve. Two regimes have been clearly identified:

1. for $|p^2 - E| \geq 0(1)\lambda^2$, the random potential λV is indeed a weak perturbation of the free Green function. It is statistically unlikely to develop an eigenvalue of the same size than $p^2 - E$, making the denominator of the Green's function singular. So this regime can be controlled by the usual techniques of multiscale cluster and Mayer expansion, with some large/small field analysis. Typical V's belong to the small field regions; exceptional V's in large field regions are controlled through some type of Tchebycheff inequalities, and give rise to small probabilistic factors. These factors can then pay for the necessary complex contour deformations and rough bounds that desingularize the denominator of the Green's function in these regions [19].

2. for $|p^2 - E| \leq 0(1)\lambda^2$, the interesting region, a non trivial imaginary part generated by the potential should stop the renormalization group analysis. This is easily seen at second order perturbation theory in λ: an imaginary term proportional to $i\lambda^2$ appears. Hence $G_\pm$ should decay exponentially with a length scale proportional to λ^{-2}, and the DOS should be regular. The problem is to justify non pertubatively this well-known theoretical argument.

For the moment the main result in this direction is [21]:

Theorem 1 *There exists some finite $\eta > 0$ such that for $\epsilon = \lambda^{2+\eta}$ the averaged Green's function $< G_{+\epsilon}(x, y) >_V$ decays at large $|x - y|$ with a rate $\tau(\lambda)$ which is independent of η, and inverse of λ^2.*

This result was obtained through some difficult non-perturbative *Ward identity* which uses the particular momentum conservation laws in two dimensions. It is non-trivial, since the decay rate in λ^{-2}, as expected from perturbation theory, does not depend on the η cutoff. This result seems difficult to extend to fully perform the limit $\epsilon \to 0$ (this would in fact be implied by a similar theorem but with $\eta = 1$).
To complete the proof of exponential decay of the averaged Green's function and of the regularity of the averaged DOS (without $\lambda^{2+\eta}$ cutoff), the supersymmetric method [13, 14] may be more convenient. Indeed the $|p^2 - E| \leq 0(1)\lambda^2$ regime of the 2D Anderson model is no longer perturbative in the naive sense, since the potential combines non-trivially with the deterministic part. Supersymmetry seems the right tool to control this phenomenon since in this formalism it corresponds simply to contour deformation to a non-trivial saddle point that generates Wigner's law and stops the RG flow. The functional integral away from the saddle point can be controlled non-perturbatively by a small/large field analysis, which is compatible with standard weak-coupling cluster expansions [20].
The *Flip Matrix Model* studied in this paper incorporates four simplifications with respect to the true Anderson 2 dimensional model:

- The regime $|p^2 - E| \geq 0(1)\lambda^2$ has been completely removed from the model. This is justified since this regime being fully perturbative can be added later in the style of [19]. So we restrict our Hilbert space to functions supported in momentum space by a tiny shell around the circle $p^2 = E$ (or $2 - \cos p_1 - \cos p_2 = E$ for a square lattice version of the model).

- The space has been reduced to a single cube of side size λ^{-2}. This is in agreement with the idea of a contour translation generating a $i\lambda^2$ part in the denominator of the Green's function. Then the full model should correspond to weakly coupled such cubes. Once the single cube-model has been understood, the full model should follow from a cluster expansion.

- In this cube the momentum shell has been divided into $0(\lambda^{-2})$ cells, called sectors, and the corresponding infinite dimensional Hilbert space has been replaced by a finite dimensional one, so that the *random operator* corresponding to V is replaced by a random matrix. This discretization step is justified since the phase space analysis is now fine enough, so that each random operator being approximately constant, it is legitimate to replace it by a single random coefficient.

- The probability law for the random coefficients of the matrix has been computed from the 2D momentum conservation rule, with a last simplification. As well known the *rhombus rule* of 2D momentum conservation for four vectors of equal length slightly weakens for almost degenerate *flat* rhombuses and this degeneracy requires an anisotropic slicing of sectors and of the cube into parallelepipeds [17, 19, 21]. We forget this complication to stick with simpler isotropic sectors, so we use a simplified momentum conservation rule. Normally introducing anisotropic analysis is only a (painful) complication of the model that gets rid of the difficulties associated to almost degenerate rhombuses.

Then the subsequent model would be almost the GUE if not for the rhombus *flip*. This flip creates a new symmetry in the random matrix, so that the supersymmetric analysis of this model requires essentially twice as many variables as for the GUE. The full proof of regularity of the DOS for the 2D Anderson model is therefore now reduced to the hard task of fitting together all pieces of the puzzle by removing our four approximations to go from the Flip Model to the true 2D Anderson Model.

III The Flip Matrix Model: notations

Let N be a large number parametrizing the number of sectors. We take E positive of order 1 to sit well inside the conduction band. To model the $2D$

Anderson model in a large spatial cell of side size $O(\lambda^{-2})$, we have divided as indicated above the momentum shell of width $O(\lambda^2)$ around $p^2 = E$ into $2N = O(\lambda^{-2})$ isotropic cells. Since the initial $V(x)$ was real, its Fourier transform $V(p)$ (the hat on Fourier transforms is removed for simplicity) is complex and obeys

$$V(-p) = \bar{V}(p) \ . \tag{III.1}$$

Since $V(x)$ is random independent identically distributed in direct space, $V(p)$ is Gaussian white noise, so that

$$\langle V(p)\bar{V}(q)\rangle \ = \ \delta(p-q) \ .$$

In order to implement the momentum conservation constraint at the vertex the "rhombus rule" will be simplified by forgetting the degeneracy of the rhombus near collapse. Discretization of $V(p)$ follows from it being considered as a random matrix between the cell indices. Collision with the initial potential $V(p)$ transforms a state with momentum r into a state with momentum $r+p$. Therefore the discretized matrix corresponds to V with matrix elements $V(\gamma)$ where γ is some discretized version of p. Its coefficients are

$$V_{\alpha,\beta}$$

where α and β run over the set of sectors and $V_{\alpha,\beta}$ transforms a state with momentum sector α into a state with momentum sector β if and only if $\gamma \simeq \beta - \alpha$. Solving this equation for a known γ gives in a generic case $0 < |\gamma| < 2$ two possible pairs forming a rhombus, namely (α, β) and $(-\beta, -\alpha)$, so that for $\beta \neq \pm\alpha$ there is a single random variable for the two pairs:

$$V_{\alpha,\beta} = V_{-\beta,-\alpha} \tag{III.2}$$

In the degenerate case $\gamma = 0$, it gives $2N$ equal pairs $V_0 = V_{\alpha,\alpha}$, and in the degenerate cases $|\gamma| = 2$ it gives a single pair $V_{\alpha,-\alpha}$. If, as explained in the introduction, the almost degenerate rhombuses are neglected, which should be harmless since they are not generic, as well as the possible effect of nearest neighbor slight overlaps between the momentum definition of cells which may lead to slight complications, V is well approximated by a random Gaussian $2N \times 2N$ complex matrix. This matrix is Hermitian because of (III.1):

$$V_{\alpha,\beta} = \bar{V}_{\beta,\alpha} \tag{III.3}$$

It is therefore convenient to label sectors as $\{1, \cdots, N\} \cup \{-N, \cdots, -1\}$, so that $\{1, \cdots, N\}$ labels projective sectors, as shown in Figure 1.

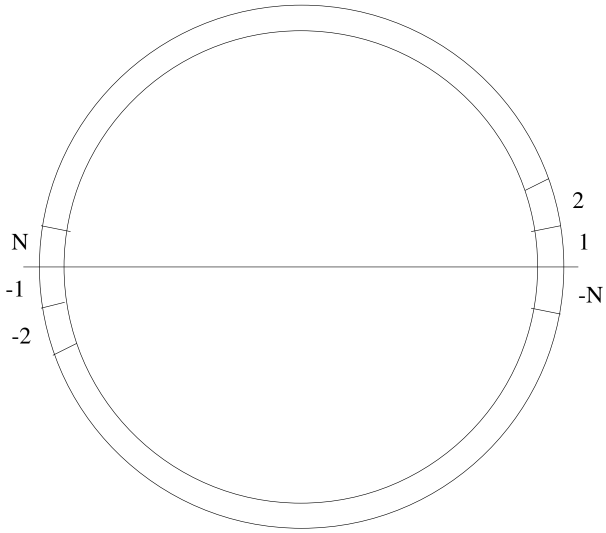

Figure 1. Numbering sectors

This leads to a set of $2N(N-1)/2 = N(N-1)$ complex variables $V_{\alpha,\beta}$ for $\alpha \in \{1, \cdots, N\}$, and $\beta \in \{1, \cdots, N\} \cup \{-N, \cdots, -1\}$, $\alpha < |\beta|$. They fill a quarter of the matrix, made of the upper half of the upper-left quarter and of the lower half of the upper-right quarter. All other coefficients are deduced from these. Thanks to (III.2)-(III.3), and to the particular numbering of Figure 1, there is symmetry around the antidiagonal, and conjugate symmetry around the ordinary diagonal (so that V is hermitian).
The upper anti-diagonal is made of N additional independent complex variables $V_{\alpha,-\alpha}$ for $\alpha \in \{1, \cdots, N\}$, corresponding to momentum transfers of length 2 of V, while the lower antidiagonal is hermitian conjugate to the upper one. Finally the variables $V_{\alpha\alpha}$ are all equal to a single real variable V_0 and that fills the ordinary diagonal of the matrix V.
The total number of real independent variables among ther matrix elements, is therefore $2N(N-1) + 2N + 1 = 1 + 2N^2$. They all have covariance equal to 1. The corresponding normalized Gaussian measure will be denoted $d\mu(V)$. This model is similar to the standard GUE (that would have $4N^2$ independent real variables for a $2N \times 2N$ matrix instead), except for the change on the diagonals and the additional symmetry with respect to the anti-diagonal, corresponding to rhombus flips.

The structure of V can be summarized as follows (remember that the rows and columns indices are labelled as $\{1, \cdots, N, -N, \cdots, -1\}$ hence *not* as $\{1, \cdots, N, -1, \cdots, -N\}$):

$$
V = \left(\begin{array}{cc}
\begin{bmatrix}
V_0 & & \vdots & & \vdots & & \\
& \ddots & \vdots & & \vdots & & \\
& \cdots & V_0 & \cdots & V_{\alpha,\beta} & \cdots & \\
& & \vdots & \ddots & \vdots & & \\
& \cdots & \overline{V_{\alpha,\beta}} & \cdots & V_0 & \cdots & \\
& & \vdots & & \vdots & \ddots & \\
& & \vdots & & \vdots & & V_0
\end{bmatrix}
&
\begin{bmatrix}
& & \vdots & & \vdots & & \\
& \ddots & \vdots & & \vdots & & \\
& \cdots & V_{\alpha,-\beta} & \cdots & V_{\alpha,-\alpha} & \cdots & \\
& & \vdots & \ddots & \vdots & & \\
& \cdots & V_{\beta,-\beta} & \cdots & V_{\alpha,-\beta} & \cdots & \\
& & \vdots & & \vdots & \ddots & \\
& & \vdots & & \vdots & &
\end{bmatrix}
\\ \hdashline
\begin{bmatrix}
& & \vdots & & \vdots & & \\
& \ddots & \vdots & & \vdots & & \\
& \cdots & \overline{V_{\alpha,-\beta}} & \cdots & \overline{V_{\beta,-\beta}} & \cdots & \\
& & \vdots & \ddots & \vdots & & \\
& \cdots & \overline{V_{\alpha,-\alpha}} & \cdots & \overline{V_{\alpha,-\beta}} & \cdots & \\
& & \vdots & & \vdots & \ddots & \\
& & \vdots & & \vdots & &
\end{bmatrix}
&
\begin{bmatrix}
V_0 & & \vdots & & \vdots & & \\
& \ddots & \vdots & & \vdots & & \\
& \cdots & V_0 & \cdots & V_{\alpha,\beta} & \cdots & \\
& & \vdots & \ddots & \vdots & & \\
& \cdots & \overline{V_{\alpha,\beta}} & \cdots & V_0 & \cdots & \\
& & \vdots & & \vdots & \ddots & \\
& & \vdots & & \vdots & & V_0
\end{bmatrix}
\end{array} \right)
$$

Our main result in this note is:

Theorem 2 *In the large N-limit, the DOS of this Flip Matrix Model converges to Wigner's semi-circle distribution. The corrections to the limit are uniformly bounded as $O(1/N)$ as $N \to \infty$.*

The rest of the paper is devoted to the proof of this result. Heuristically, the diagonals should not matter much for the average spectral properties. The main difficulty is to check that the additional symmetry of the rhombus flip does not alter Wigner's semi-circle law. This is made precise using the supersymmetric formalism [13, 14].

IV Mean field analysis

Let the superfields be

$$\Psi_{\sigma\alpha} \ = \ (S_{\sigma\alpha}, \chi_{\sigma\alpha})$$

for $\sigma = \pm 1$ and greek variables such as α, β now running only within $[1, ..., N]$. The conventions are taken from [14]. Hence bosonic fields are complex with

$\bar{S} = S^+$ and pairs of anticommuting variables χ, χ^+ satisfy:

$$\chi_i \chi_j \;=\; -\chi_j \chi_i \,,\; \chi_i^+ \chi_j \;=\; -\chi_j \chi_i^+ \,,\; \chi_i^+ \chi_j^+ \;=\; -\chi_j^+ \chi_i^+$$

$$(\chi_i^+)^+ \;=\; -\chi_i \,, \quad (\chi_i \chi_j)^+ \;=\; \chi_i^+ \chi_j^+ \,,$$

$$\int d\chi_i \;=\; \int d\chi_i^+ = 0, \qquad \int \chi_i d\chi_i \;=\; \int \chi_i^+ d\chi_i^+ \;=\; \frac{1}{\sqrt{2\pi}}$$

so that

$$\int \prod_i d\chi_i^+ d\chi_i e^{-\chi^+ M \chi} \;=\; \det(M/2\pi)$$

whereas ordinary complex bosonic variables give

$$\int \prod_i dS_i^+ dS_i e^{-S^+ M S} \;=\; \det\left(M/2\pi\right)^{-1}$$

(and M has to have a positive real part). By the usual rules of anticommuting integrals each Gaussian term is written as an integral over superfields. Hence the density of states can be written as

$$\nu(E) \;=\; \lim_{ImE \to 0^+} \frac{1}{\pi} \Im \int S_\alpha^+ S_\alpha e^{\,\imath L_0} \prod_{\alpha,\sigma} d\Psi_{\sigma\alpha}^+ \, d\Psi_{\sigma\alpha} \, d\mu(V) \qquad \text{(IV.4)}$$

where the supersymmetric action decomposes as

$$L_0 \;=\; (A_0 + B_0 + C_0)$$

with

$$A_0 \;=\; \sum_{\alpha<\beta,\sigma} \left\{ \lambda V_{\alpha,\sigma\beta} \; \left(\Psi_\alpha^+ \Psi_{\sigma\beta} + \Psi_{-\sigma\beta}^+ \Psi_{-\alpha} \right) + h.c. \right\}$$

$$B_0 \;=\; \sum_\alpha \lambda V_{\alpha,-\alpha} \left(\Psi_\alpha^+ \Psi_{-\alpha} + h.c. \right)$$

$$C_0 \;=\; (E + \lambda V_0) \sum_{\alpha,\sigma} \Psi_{\sigma\alpha}^+ \Psi_{\sigma\alpha}$$

correspond respectively to the main part, the anti-diagonal and the diagonal of the matrix. The Gaussian integration over the V variables will be performed except for V_0 leading to the quartic terms

$$\nu(E) = \lim_{ImE\to 0^+} \frac{1}{\pi} Im \int S^+_\alpha S_\alpha e^{-L} \prod_{\alpha,\sigma} d\Psi^+_{\sigma\alpha} d\Psi_{\sigma\alpha} d\mu_0(V_0)$$

$$L = \lambda^2 (A+B+C)$$

$$A = \sum_{\alpha<\beta,\sigma} (\Psi^+_\alpha \Psi_{\sigma\beta} + \Psi^+_{-\sigma\beta}\Psi_{-\alpha})(\Psi^+_{\sigma\beta}\Psi_\alpha + \Psi^+_{-\alpha}\Psi_{-\sigma\beta}) \tag{IV.5}$$

$$B = \sum_\alpha \Psi^+_\alpha \Psi_{-\alpha}\Psi^+_{-\alpha}\Psi_\alpha$$

$$C = (\lambda^{-1}E + V_0) \sum_{\alpha,\sigma} \Psi^+_{\sigma\alpha}\Psi_{\sigma\alpha} \ .$$

The sum in (IV.5) can be written as

$$A \ = \ \sum_\sigma \Phi^+_{\alpha\beta;\,\sigma}\Phi_{\alpha\beta;\,\sigma} \quad with \quad \Phi_{\alpha\beta;\,\sigma} \ = \ (\Psi^+_{\sigma\beta}\Psi_\alpha + \Psi^+_{-\alpha}\Psi_{-\sigma\beta}) \ .$$

Exchanging α and β in $A = \sum_\sigma \Phi^+_{\alpha\beta;\sigma}\Phi_{\alpha\beta;\sigma}$, both for $\sigma = \pm 1$ does not change the sum. This is because the bosonic part in Φ commute and the rule for the fermionic part leads to $\Phi^+_{\alpha\beta;+} = \Phi_{\beta\alpha;+}$, whereas for $\sigma = -1$, $\Phi_{\beta\alpha;-} = \Phi_{\alpha\beta,-}$. Hence

$$A \ = \ \frac{1}{2}\sum_{\alpha,\beta,\sigma} (\Psi^+_\alpha \Psi_{\sigma\beta} + \Psi^+_{-\sigma\beta}\Psi_{-\alpha})(\Psi^+_{\sigma\beta}\Psi_\alpha + \Psi^+_{-\alpha}\Psi_{-\sigma\beta}) + D_{++} + D_{+-} + D_- \ ,$$

where

$$D_- = -(1/2)\sum_\alpha (\Psi^+_\alpha \Psi_{-\alpha} + \Psi^+_\alpha \Psi_{-\alpha})(\Psi^+_{-\alpha}\Psi_\alpha + \Psi^+_{-\alpha}\Psi_\alpha) = -2B$$

$$D_{++} = -(1/2)\sum_{\alpha,\sigma} (\Psi^+_{\sigma\alpha}\Psi_{\sigma\alpha})^2 \qquad D_{+-} = -\sum_\alpha (\Psi^+_\alpha \Psi_\alpha \Psi^+_{-\alpha}\Psi_{-\alpha})$$

$$D_{+-} = -\sum_\alpha (S^+_\alpha S_\alpha + \chi^+_\alpha \chi_\alpha)(S^+_{-\alpha}S_{-\alpha} + \chi^+_{-\alpha}\chi_{-\alpha}) \ .$$

The decomposition of D_{+-} into its boson and fermion parts and adding B above, eventually leads to $L = \lambda^2(\mathcal{A}+\mathcal{B}+\mathcal{C})$ with

$$\mathcal{A} = (1/2)\sum_{\alpha,\beta,\sigma} (\Psi^+_\alpha \Psi_{\sigma\beta} + \Psi^+_{-\sigma\beta}\Psi_{-\alpha})(\Psi^+_{\sigma\beta}\Psi_\alpha + \Psi^+_{-\alpha}\Psi_{-\sigma\beta})$$

$$\mathcal{B} = \sum_\alpha \mathcal{B}_\alpha$$

$$\mathcal{B}_\alpha = -2S^+_\alpha S_{-\alpha}S^+_{-\alpha}S_\alpha - \frac{1}{2}\sum_\sigma (\Psi^+_{\sigma\alpha}\Psi_{\sigma\alpha})^2 - (\sum_\sigma S^+_{\sigma\alpha}\chi^+_{-\sigma\alpha})(\sum_\sigma S_{\sigma\alpha}\chi_{-\sigma\alpha})$$

$$\mathcal{C} = C = (\lambda^{-1}E + V_0)\sum_{\alpha,\sigma}\Psi^+_{\sigma\alpha}\Psi_{\sigma\alpha}. \qquad \text{(IV.6)}$$

The term $\mathcal{B}$ being a sum of *diagonal quartic* terms, will be neglected for a while. For indeed it cannot be written as the *square* of a sum over α. However, the bosonic part of this term has the *wrong sign*, which may create some difficulties when performing the integration in (IV.4). This problem will be addressed in section VI. The other terms, however, can be reorganized so as to get the square of a sum over the α's by pushing the terms with index α on the left and the ones with index β on the right. This is possible thanks to the commutation rules for bosons and for fermions. The calculation is tedious but straightforward and gives:

$$\begin{aligned} \mathcal{A} &= -(1/2)(\sum_{\alpha,\sigma} S^+_{\sigma\alpha}S_{\sigma\alpha})^2 + (1/2)(\sum_{\alpha,\sigma}\chi^+_{\sigma\alpha}\chi_{\sigma\alpha})(h.c.) \qquad \text{(IV.7)} \\ &\quad -2(\sum_\alpha S^+_\alpha S^+_{-\alpha})(h.c.) - (\sum_{\sigma\alpha} S_{\sigma\alpha}\chi^+_{\sigma\alpha})(h.c.) - (\sum_{\sigma\alpha} S^+_{\sigma\alpha}\chi^+_{-\sigma\alpha})(h.c.)\ . \end{aligned}$$

The squares can be *unfolded* by mean of an integration over auxiliary gaussian fields. This amount to introduce two real fields a_0 and b_0, one complex field $a, \bar{a}$ and two pairs of fermionic fields $\xi, \bar{\xi}$ and $\eta, \bar{\eta}$ with Gaussian measure

$$\begin{aligned} d\mu(a_0, b_0, a, \bar{a}, \bar{\xi}, \xi, \bar{\eta}, \eta) &= e^{-V_0^2/2 - a_0^2/2 - b_0^2/2 - |a|^2/2 - \xi^*\xi - \eta^*\eta} \times \cdots \\ &\cdots \ \frac{dV_0\, da_0\, db_0\, d^2a}{(2\pi)^{5/2}} d\xi^* d\xi d\eta^* d\eta\ . \qquad \text{(IV.8)} \end{aligned}$$

Therefore

$$e^{\lambda^2(\mathcal{A}+\mathcal{C})} = \int d\mu \;\; e^{i\lambda \sum_\alpha \Phi_\alpha^+ R\Phi_\alpha}$$

with

$$\begin{aligned}
\sum_\alpha \Phi_\alpha^+ R\Phi_\alpha &= (a_0 + V_0 + \lambda^{-1}E)\sum_{\alpha,\sigma} S_{\sigma\alpha}^+ S_{\sigma\alpha} \\
&+ (ib_0 + V_0 + \lambda^{-1}E)\sum_{\alpha,\sigma} \chi_{\sigma\alpha}^+ \chi_{\sigma\alpha} \\
&+ \left(\bar{a}\sum_\alpha S_\alpha S_{-\alpha} + h.c.\right) \\
&+ \left(\xi^*\sum_{\sigma\alpha} S_{\sigma\alpha}^+ \chi_{\sigma\alpha} + h.c.\right) + \left(\eta^*\sum_{\sigma\alpha} S_{\sigma\alpha}\chi_{-\sigma\alpha} + h.c.\right)
\end{aligned} \qquad \text{(IV.9)}$$

where Φ is the superfield $\Phi_\alpha^+ = (S_\alpha^+, S_{-\alpha}, \chi_\alpha^+, \imath\chi_{-\alpha})$ and R is a 4×4 supermatrix. Setting $A_0 = (a_0 + V_0 + \lambda^{-1}E)$ and $\imath B_0 = (\imath b_0 + V_0 + \lambda^{-1}E)$ this matrix is given by

$$R \;=\; \begin{pmatrix} A_0 & a & \xi^* & -\imath\eta \\ \bar{a} & A_0 & \eta^* & -\imath\xi \\ \xi & \eta & \imath B_0 & 0 \\ -\imath\eta^* & -\imath\xi^* & 0 & -\imath B_0 \end{pmatrix} \;=\; \begin{pmatrix} A & \rho^* \\ \rho & B \end{pmatrix} . \qquad \text{(IV.10)}$$

In order that the integration over the primitive fields be given in term of Φ it will be convenient to introduce the new fermionic fields

$$\tilde{\chi}_{-\alpha} \;=\; \imath\chi_{-\alpha}^+ , \;\; \tilde{\chi}_{-\alpha}^+ \;=\; \imath\chi_{-\alpha} , \quad d\tilde{\chi}_{-\alpha} \;=\; -\imath d\chi_{-\alpha}^+ , \;\; d\tilde{\chi}_{-\alpha}^+ \;=\; -\imath d\chi_{-\alpha} .$$

This leads to the formula

$$\prod_\alpha \int d^2S_\alpha d^2S_{-\alpha} d\chi_\alpha^+ d\chi_\alpha d\tilde{\chi}_{-\alpha}^+ d\tilde{\chi}_{-\alpha} e^{\imath\lambda\Phi_\alpha^+ R\Phi_\alpha} \;=\; [Sdet(\imath\lambda R)]^{-N} \;\; .$$

The *super determinant* can be computed from the *Schur complement* formula

$$Sdet \begin{pmatrix} A & \rho^* \\ \rho & B \end{pmatrix} \;=\; \frac{1}{\det B}\; \det(A - \rho^* B^{-1}\rho)\;.$$

Defining $\hat{a}_0 = a_0 + V_0 + \lambda^{-1}E$, $\hat{b}_0 = ib_0 + V_0 + \lambda^{-1}E$ and using the Pauli matrices $\sigma_2, \sigma_2, \sigma_3$ the superdeterminant of R is given by gives

$$\begin{aligned} SdetR \;&=\; \frac{1}{(\hat{b}_0)^2}\;\det\Bigg(\hat{a}_0\mathbf{1} + a_1\sigma_1 + a_2\sigma_2 \\ &- \begin{pmatrix} \xi^* & -\imath\eta \\ \eta^* & -\imath\xi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \xi & \eta \\ -\imath\eta^* & -\imath\xi^* \end{pmatrix}\hat{b}_0^{-1}\Bigg) \\ &=\; \frac{1}{(\hat{b}_0)^2}\Bigg(\Big\{(a_0 + V_0 + \lambda^{-1}E + (\eta^*\eta + \xi^*\xi)\hat{b}_0^{-1}\Big\}^2 \\ &-\; \Big\{a + 2\xi^*\eta\hat{b}_0^{-1}\Big\}\Big\{\bar{a} + 2\eta^*\xi\hat{b}_0^{-1}\Big\}\Bigg)\;. \end{aligned} \tag{IV.11}$$

Including back the scaling factors, taking into account the relation $2\lambda^2 = N^{-1}$ and ignoring the $\mathcal{B}$ term, the density of states is given by

$$\begin{aligned} \int da_0 dV_0 \;\; \cdots \;\; &\left(e^{-(a_0^2+V_0^2+|a|^2+b_0^2)}\frac{(E+V_0+ib_0)^2}{(E+V_0+a_0)^2 - |a|^2}\right)^N \\ \cdots \;\; &\times(1+O(1/N))\;\; \frac{E+V_0+a_0}{(E+V_0+a_0)^2 - |a|^2}\;. \end{aligned} \tag{IV.12}$$

For indeed, taking into account the part of the 2-point function involved in the integral (IV.4), the fermionic part gives a $O(1/N)$ correction in the form

$$\langle(1 + r\eta^*\eta\xi^*\xi)^{-N-1}\rangle \;=\; 1 - r\frac{N+1}{4N^2}\,,$$

thanks to the integral

$$\frac{\pi}{2N}\int d\xi^* d\xi e^{-2N\xi^*\xi}\xi^*\xi \;=\; -\frac{1}{4N}\;.$$

V The saddle points at leading order

In this section the $O(1/N)$ and $\mathcal{B}$ terms are still neglected. As $N \to \infty$, the integral (IV.12) can be treated by the saddle point method, since no fermion variable are involved anymore. The saddle point equations can be written $\partial S = 0$ where S is the *effective action* given by

$$S = -(a_0^2 + V_0^2 + |a|^2 + b_0^2) + 2\ln(E + V_0 + ib_0) - \ln((E + V_0 + a_0)^2 - |a|^2)) \ .$$

More precisely, differentiating with respect to the independant variables appearing in the integral (IV.12) gives

$$-(1/2)\frac{\partial S}{\partial a_0} \ = \ a_0 + \frac{a_0 + V_0 + E}{(E + V_0 + a_0)^2 - |a|^2} \ = \ 0 \ , \tag{V.13}$$

$$-(1/2)\frac{\partial S}{\partial |a|} \ = \ |a|[1 - \frac{1}{(E + V_0 + a_0)^2 - |a|^2}] \ = \ 0 \ , \tag{V.14}$$

$$-\frac{\imath}{2}\frac{\partial S}{\partial b_0} \ = \ ib_0 + \frac{1}{E + V_0 + ib_0} \ = \ 0 \ , \tag{V.15}$$

$$(1/2)\frac{\partial S}{\partial V_0} = -V_0 + \frac{1}{E + V_0 + ib_0} - \frac{a_0 + V_0 + E}{(E + V_0 + a_0)^2 - |a|^2} = 0 \ . \tag{V.16}$$

Thanks to (V.13) & (V.15), (V.16) can be written as

$$V_0 + ib_0 - a_0 = 0 \, , \tag{V.17}$$

so that, thanks to (V.15),

$$ib_0 = \frac{-1}{E + a_0} \ . \tag{V.18}$$

Then two cases occur:

Case 1 : $|a| = 0$. Substituting in (V.13) leads to

$$(E + 2a_0)(a_0^2 + a_0 E + 1) = 0 \, ,$$

giving two solutions

$$a_0 = \frac{-E \pm i\sqrt{4 - E^2}}{2} \, , \ \ \imath b_0 = \frac{-E \pm i\sqrt{4 - E^2}}{2} \, , \ \ V_0 = 0 \ , \tag{V.19}$$

$$a_0 = \frac{-E}{2}\,, \quad \imath b_0 = \frac{-2}{E}\,, \quad V_0 = \frac{4-E^2}{2E}\,. \tag{V.20}$$

Case 2 : $|a| \neq 0$. Then, (V.13), (V.14) & (V.17) give

$$a_0 = -\frac{V_0+E}{2}\,,\ \ |a|^2 = (\frac{V_0+E}{2})^2 - 1\,,\ \imath b_0 = -\frac{3V_0+E}{2}\,,$$
$$3V_0^2 - 2EV_0 + 4 - E^2 = 0\ .$$

It leads to the following solution

$$\begin{aligned} V_0 &= \frac{E \pm 2i\sqrt{3-E^2}}{3}\,, \\ a_0 &= -\frac{2E \pm i\sqrt{3-E^2}}{3}\,, \\ \imath b_0 &= -E \mp i\sqrt{3-E^2}\,, \\ |a|^2 &= \frac{5E^2 - 12 \pm i4E\sqrt{3-E^2}}{9}\,. \end{aligned} \tag{V.21}$$

As $N \to \infty$ the leading contributions are given by the value of the action at the saddle, and we find, for the first saddle point

$$S = 0\ .$$

For the second saddle point, if $E^2 < 4$,

$$S = \frac{4-E^2}{2} + 4\ln(|E|/2) < 0\,,$$

and, for the third saddle point, if $E^2 \leq 3$,

$$S = E^2/3 - \ln 3 \pm 2i(-\phi + \sin 2\phi)\,, \quad \Rightarrow \quad \Re S < 0\,,$$

where $E/\sqrt{3} = \cos\phi$, $\sqrt{3-E^2}/\sqrt{3} = \sin\phi$. Consequently, in the large N limit, only the first saddle point contributes to the density of states, and gives the usual semi-circle law. Therefore our model as expected is in the class of universality of the semi-circle law as $N \to \infty$. The saddle points are degenerate and one should bound the difference between saddle point contribution and the correction corresponding to out-of-the-saddle part of the integral. This should be done e.g. like in [20].

VI The diagonal quartic terms

In this section, the corrections due to the $\mathcal{B}$ term are shown to be small as $N \to \infty$. As noticed in Section IV, the bosonic part of the quartic term $\mathcal{B}$ has the wrong sign. For this reason before integrating over superfields it will be convenient to write a Taylor expansion with integral remainder successively for each of the N sectors appearing in the sum for $\mathcal{B}$:

$$\mathcal{B} = \sum_\alpha \mathcal{B}_\alpha \ .$$

To first order the expansion gives

$$e^{-\mathcal{B}_\alpha} = 1 - \int_0^1 \mathcal{B}_\alpha e^{-\mathcal{B}_\alpha} e^{+t\mathcal{B}_\alpha} dt \ .$$

This Taylor expansion either suppresses $\mathcal{B}_\alpha$ from the exponential of the action or generates a remainder term $R_\alpha = -\int_0^1 \mathcal{B}_\alpha e^{-\mathcal{B}_\alpha} e^{+t\mathcal{B}_\alpha} dt$. Let $p(N)$ be the integer part of $N/\sqrt{\log N}$. The expansion is stopped to the order $p(N) = p$. This gives:

$$e^{-\mathcal{B}} = 1 + \sum_{\substack{P \subset [1,\ldots,N] \\ 0 < |P| \le p(N)}} R_P \ ,$$

where

$$R_P = \prod_{\alpha \in P} R_\alpha \quad \text{if } |P| < p(N) \ ,$$

$$R_P = \prod_{\alpha \in P} R_\alpha \prod_{\alpha > \max P} e^{-\mathcal{B}_\alpha} \quad \text{if } |P| = p(N) \ . \tag{VI.22}$$

Let Q be the complement of P in $[1, ..., N]$. The term 1 was treated in the previous sections. The remainders terms R_P must be shown to be $O(1/N)$ as $N \to \infty$. Let $R_{p(N)} = \sum_{\substack{P \subset [1,\ldots,N] \\ |P| = p(N)}} R_P$ be considered first. For this term it is not necessary to perform any saddle point analysis and it is sufficient to return to the original expression given in (IV.4), hence to a functional integral over V. However some $e^{-\mathcal{B}_\alpha}$-factors are missing or appear with reduced weights. To correct for this, all the $e^{-\mathcal{B}_\alpha}$-terms are recombined with the $\mathcal{A}$-term to reproduce the initial functional integral (IV.4). However, there are quartic correction terms $e^{t\mathcal{B}_\alpha}$ or $e^{\mathcal{B}_\alpha}$. The important remark is that the bosonic part of these terms has now the right sign ! Therefore they can be represented as a well defined integral over a new auxiliary field W_α. For instance

$$e^{-2tS^+_\alpha S_{-\alpha} S^+_{-\alpha} S_\alpha} = e^{-2t|S_\alpha S_{-\alpha}|^2} = \int dW_\alpha d\bar{W}_\alpha e^{-|W_\alpha|^2} e^{\imath\sqrt{2t}W_\alpha S_\alpha \bar{S}_{-\alpha}+cc} \ .$$

With slightly condensed notations, this leads to

$$R_{p(N)} = \sum_{\substack{P\subset[1,\dots,N]\\ |P|=p(N)}} \int S^+ S \Big(\prod_{\alpha\in P} \int_0^1 \mathcal{B}_\alpha dt \Big) e^{i\Psi^+(E-V+\sqrt{2t}W)\Psi} d\Psi^+ d\Psi d\mu(V,W) \ .$$

Then a complex translation $V_0 \mapsto V_0 \pm \imath\lambda^{-1}$ is performed, with the same sign as the imaginary part of E in order to avoid crossing of singularities. In other words

$$\int_{-\infty}^{+\infty} e^{-V_0^2} F(V_0) dV_0 = \int_{-\infty}^{+\infty} e^{-V_0^2} e^{-2i\lambda^{-1}V_0^2} e^{\lambda^{-2}} F(V_0 + i\lambda^{-1}) dV_0 \ .$$

The functional integral can now be bounded by its absolute values everywhere, namely the following contributions are bounded

- by 2^N, for the sum over P, that is the total number of subsets of $[1,N]$;
- by 1, for the integrals such as $\int_0^1 dt$;
- by 1, for the oscillating imaginary integrals;
- by Gram's inequality for fermions or the Schwarz inequality for bosons, for the remainders terms $\mathcal{B}_\alpha$;
- by 1, for every propagator since the imaginary translation in V_0 has created an imaginary part proportional to the identity in the denominator of the Green's function.

This means that each $\mathcal{B}_\alpha$ term give rise to a small factor $\lambda^2 = 1/N$ for each sector in P, hence a factor $1/N^{p(N)}$. The two source terms are bounded by 1. The normalization determinants are then easily bounded by c^N, even without using the supersymmetry cancellations, since the operators considered are bounded in a finite $2N$ dimensional space thanks to the imaginary part of E which is no longer infinitesimal. Combining all factors leads to

$$R_{p(N)} \le c^N e^{-cN\sqrt{\log N}} \ ,$$

showing that this correction term is small indeed.

It remains to prove that the sum of the terms R_P with $1 \leq |P| \leq p(N)$ is small. Let $p = |P|$. To bound these terms a mean-field analysis will be performed (see Section IV & V) in terms of the a and b fields, but only for the sectors of the theory in the complement Q of P. The functional integral to be bounded for a single term is (VI.22). Now, the $e^{-\mathcal{B}_\alpha}$-terms are recombined only for $\alpha \in P$ with the $\mathcal{A}$ term to reproduce the initial functional integrals over the V fields and the terms $e^{t\mathcal{B}_\alpha}$, also for $\alpha \in P$, are again given by defined integrals over new auxiliary fields W_α. Finally, the quartic terms, with sector sums reduced to Q, are treated exactly as in the previous section, hence a mean-fields a, a_0, b_0 are correspondingly introduced. This leads to a representation

$$R_P = \int dV_{P,P}\, dV_{P,Q}\, d^2a\, dV_0\, da_0\, db_0 \prod_{\alpha \in P} R_\alpha\, e^{\mathcal{L}_Q} \ ,$$

where Q is the complement of P in $[1, ..., N]$. In addition, $V_{P,P}$ is the part of the matrix V corresponding to rows and columns in P, including the new fields of the W type. $V_{P,Q}$ correspond to one entry in P and the other in Q, and the mean field computation is now retricted to Q. The integral over superfields gives rise again to a superdeterminant and an additional corrections, of the type

$$\exp\left[Tr \log(1 + CV)\right] \ ,$$

where $C = R^{-1}$ is the matrix for the $N - p$ analogous problem, as in the previous Section IV (see eq. (IV.10)), and $V = V_{PP} + V_{P,Q}$ is the perturbation. This correction term is bounded by

$$|\exp\left[Tr \log(1 + K)\right]| \leq \exp\left(TrK + K^* + KK^*\right) \ .$$

Evaluating $C = R^{-1}$ at the saddle point costs a factor $\exp(Np/2N) = e^{2p}$ at most. Each term $\mathcal{B}_\alpha$ naively gives a factor $\lambda^2 = 1/(2N)$ when evaluated, but this simply compensates the sum over α when p is nonzero but small, so we have to gain an additional $1/N$ factor. Adding a few expansion steps gives such a small additional factor $1/N$ (already for the first non zero value $p = 1$). This can be seen by integrating by parts the superfields in the *vertex* $\mathcal{B}_\alpha$ which has been taken down the exponential. By supersymmetry, the *vaccum graph* corresponding to a contraction of the four fields at the vertex vanishes. This is absolutely necessary since this graph by simple scaling is proportional $1/N$ and cannot have any additional $1/N$ factor. Its vanishing can be checked by hand:

- the self-contractions of the bosonic piece $2S^+_\alpha S_{-\alpha}S^+_{-\alpha}S_\alpha$ give a factor $+2$;

- the self-contractions of the boson-fermion piece

$$(\sum_\sigma S^+_{\sigma\alpha}\chi^+_{-\sigma\alpha})(\sum_{\sigma'} S_{\sigma\alpha}\chi_{-\sigma\alpha})$$

give a factor -2 (since there is one fermionic loop giving the minus sign and one sum over σ giving a factor 2 only after the contractions);

- the selfcontractions of the term $\frac{1}{2}\sum_\sigma(\Psi^+_{\sigma\alpha}\Psi_{\sigma\alpha})^2$ are clearly supersymmetric and also add up to 0.

Consequently, at least one field of the *vertex* $\mathcal{B}_\alpha$ has to contract to the exponential. Performing two contractions in turn, gives always at least a factor p/N^2 at the end instead of the naive $1/N$ factor. Indeed the worst case corresponds to the non trivial contraction term being of the type V_{PQ}. This generates a new factor $1/N$ but a new sum over $\beta \in Q$, which costs $N-p \simeq N$ so nothing is gained yet. But contracting this β field again either generates a diagonal term, hence a new $1/N$ factor, with no new sum, or returns to a $V_{Q,P}$ term. This last situation generates a new $1/N$ factor and a new sum over sectors γ but this time this new sum is restricted to P, so it costs only a factor p instead of N! Hence at worst, after these two contraction steps, a total factor p/N^2 instead of the naive factor $1/N$ is associated to each vertex, as announced. Now the sum over P costs a total factor $N!/p!(N-p)!$, hence is bounded by $c^p[N/p]^p$. Combining all factors, the sum of contributions of such terms with $p \neq 0$ is bounded by $\sum_{p=1}^{+\infty}[c/N]^p \leq c'/N$. It is therefore at least as small as $1/N$.

This technique could also apply to the term $\mathcal{C}$ in the formula (IV.6). Indeed its presence does not modify the large N density of states. This remark might be used to simplify section 3. However it is quite natural to consider the diagonal part of V on the same footing that the other mean field terms in the supermatrix R.

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