On Radial Symmetry of Steady and Stationary Solutions to 2D Euler Equation

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Abstract

We study whether all stationary solutions of 2D Euler equation must be radially symmetric, if the vorticity is compactly supported or has some decay at infinity. Our main results are the following:
(1) On the one hand, we are able to show that for any non-negative smooth stationary vorticity that is compactly supported (or has certain decay as $|x| \to \infty$), it must be radially symmetric up to a translation.
(2) On the other hand, if we allow vorticity to change sign, then by applying bifurcation arguments to sign-changing radial patches, we are able to show that there exists a compactly-supported, sign-changing smooth stationary vorticity that is non-radial.
We have also obtained some symmetry results for uniformly-rotating solutions for 2D Euler equation, as well as stationary/rotating solutions for the SQG equation. The symmetry results are mainly obtained by calculus of variations and elliptic equation techniques.

Problem:

- 2D Euler Equation :

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0 \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2 \\ u = \nabla^\perp \Delta^{-1} \omega \\ \omega(0,\cdot) = \omega_0(\cdot). \end{cases}$$

- Stream Function:

$$\phi := \omega * \mathcal{N}, \text{ where } \mathcal{N} = \frac{1}{2\pi} \log|x|$$

- Biot-Savart Law:

$$u = \nabla^\perp \mathcal{N} * \omega$$

- Boundary Equation: We say that a vortex patch is uniformly rotating about the origin with angular velocity Ω if

$$\phi(x) - \frac{\Omega}{2}|x|^2 = Const. \tag{1}$$

on each connected component of the boundaries.
Also we say a smooth vorticity ω is uniformly rotating about the origin with angular velocity Ω if

$$\phi(x) - \frac{\Omega}{2}|x|^2 = Const. \tag{2}$$

on each connected component of regular level curves of ω.

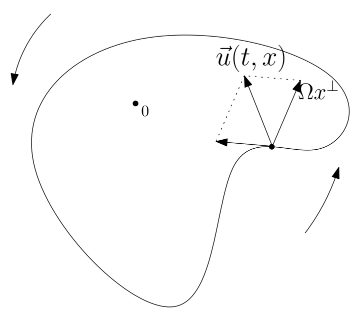


Figure 1: Rotating Vortex Patch

- If $\Omega = 0$, ω is said to be stationary.
- Constants in (1) and (2) may vary on each of disconnected boundaries.

- For generalized Surface Quasi Geostrophic equations, we replace u by

$$u_\alpha := \nabla^\perp \left(C(\alpha)|x|^{-\alpha} * \theta \right) \text{ and } C(\alpha) = -\frac{1}{2\pi} \frac{\Gamma(\frac{\alpha}{2})}{2^{1-\alpha}\Gamma(1-\frac{\alpha}{2})}.$$

- Existence of non-radial patches exist for $\Omega \in (0, \frac{1}{2})$ [1].

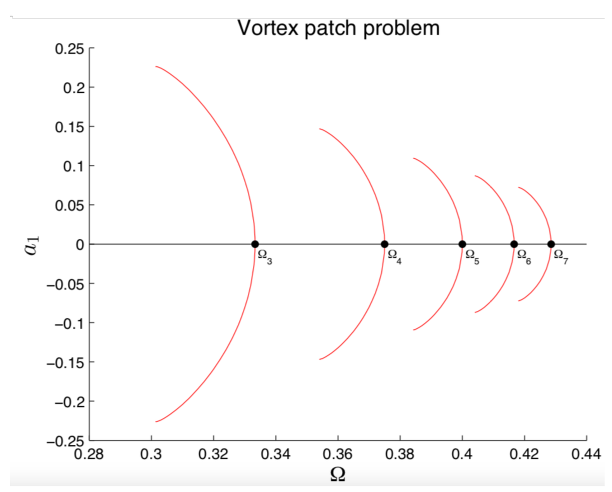
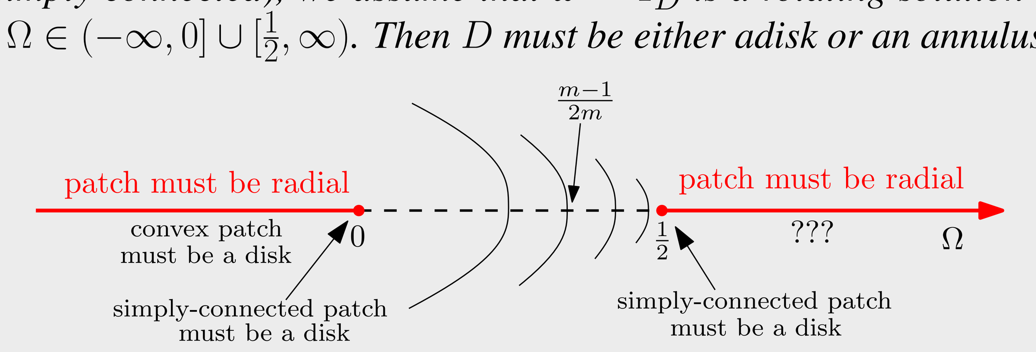


Figure 2: Bifurcation curves of rotating vortex [4]

Main Theorems [Gómez-Serrano, P, Shi, Yao '19]

Theorem 1 (Non-sign changing steady/ stationary solution). *For a connected domain D with smooth boundary (not necessarily simply connected), we assume that $\omega = 1_D$ is a rotating solution to 2D Euler equation with angular velocity $\Omega \in (-\infty, 0] \cup [\frac{1}{2}, \infty)$. Then D must be either adisk or an annulus.*



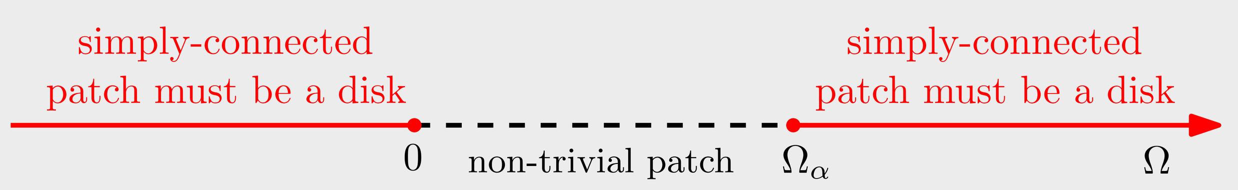
- $\omega \geq 0$ is necessary condition to have radial symmetry. For stationary solutions, we are able to find non-radial solutions in Theorem 2.

Theorem 2 ((In progress) Sign changing stationary soluton). *There exist stationary nontrivial patch solutions to the 2D Euler equations, even with finite energy. These solutions are close to nested annuli with vorticity of different signs.*

- Theorem 1 (for $\Omega \leq 0$) and Theorem 2 hold true for smooth solutions.
- Our proof of Theorem 2 is based on Crandall Rabinowitz theorem.

Theorem 3 (gSQG). *• If $\omega = 1_D$ is rotating patch solution for a simply connected D, with angular velocity $\Omega \leq 0$ for $\alpha \in [0,2)$, then D must be a disk.*

• Let $\omega = 1_D$ be a simply connected patch solution to gSQG equation where $|D| = |B_1|$. Then there exists Ω_α such that if ω is a rotating solution with $\Omega \geq \Omega_\alpha$, then D must be radially symmetric, that is $D = B_1$.



- For $|B_R| = |D|$,

$$\Omega_{R,\alpha} := R^{-\alpha} \frac{2^{\alpha-1}\Gamma(1-\alpha)\Gamma(\frac{\alpha}{2}+1)}{\Gamma(1-\frac{\alpha}{2})^2\Gamma(2-\frac{\alpha}{2})}.$$

- $\Omega_{R,\alpha} \to \frac{1}{2}$ as $\alpha \to 0$ (Euler case).
- Proof is based on continuous Steiner symmetrization as in [2] and maximum principle argument.
- Euler equation is critical in the sense that for any $\alpha > 0$, there exist non-radial, non-simply connected stationary patch solutions [5].
- Euler is also critical in the sense that the upper threshold Ω_α does not depend on the size of the patch, while gSQGs has dependence on the size as in the above formula.

Main ideas of proof of Theorem 1

- Idea of the proof (simply connected case):

Observe that a rotating solution is a "critical point" of a functional under a divergence free vector field.
↓
We choose a vector field of the flow so that the solution can be a "critical point" only if the solution is radially symmetric.

- Energy functional:

$$\mathcal{E}(\rho) := \frac{1}{2}\int_{\mathbb{R}^2} \rho(x)(\rho * \mathcal{N})(x)dx - \frac{\Omega}{2}\int_{\mathbb{R}^2} \rho(x)|x|^2 dx.$$

- Perturbation: For a divergence free vector field V, consider a continuity equation,

$$\begin{cases} \rho_t + \nabla \cdot (V\rho) = 0 \\ \rho(0,\cdot) = 1_D \end{cases}$$

- Formally, one can compute

$$\frac{d}{dt}\mathcal{E}(\rho)\Big|_{t=0} = \int_D V \cdot \nabla \left(\phi - \frac{\Omega}{2}|x|^2 \right) dx =: \mathcal{I} = 0.$$

- We choose $V = -\nabla \left(\frac{|x|^2}{2} + p \right) = -(x + \nabla p)$ where p is a solution to

$$\begin{cases} \Delta p = -2 & \text{in } D \\ p = 0 & \text{on } \partial D. \end{cases} \tag{3}$$

- One can show that

$$\mathcal{I} = (1-2\Omega)\int_D p(x)dx - \frac{|D|^2}{4\pi} + \Omega \int_D |x|^2 dx$$

- Talenti (1976) [6] : For the solution p to (3) it holds that

$$\int_D p(x)dx \leq \frac{|D|^2}{4\pi},$$

and the equality holds if and only if D is a disk.

- Since $\int_D |x|^2 dx \geq \frac{|D|^2}{2\pi}$ and equality holds only if D is a disk, one can show $\mathcal{I} = 0$ only if D is a disk for $\Omega \in (-\infty, 0] \cup [\frac{1}{2}, \infty)$.

Further Discussions

- Must every vortex patch on circular domains, for example a unit disk or an annulus, be radially symmetric?
- Instability of vortex patch for small angular velocity. Numerically, it is observed that

$$\sup_{x \in \partial D} |x| \gtrsim \Omega^{-\frac{1}{2}} \text{ for } \Omega \ll 1. \text{ See Figure 1.}$$

- Is a sign-changing vorticity with compactly supported velocity radially symmetric?

References

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