1 Abstract

In Cilanne Boulet and Igor Pak’s article, they give a combinatorial proof of the first Rogers-Ramanujan identity by using two symmetries. These symmetries are established by direct bijections. Then I introduce an application of Rogers-Ramanujan identity in analysis, which I found in George E. Andrews’ book.
2 Algebraic Part

The first Rogers-Ramanujan identity:

\[
1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2) \cdots (1-t^k)} = \prod_{i=0}^{\infty} \frac{1}{(1-t^{5i+1})(1-t^{5i+4})}
\]

Recall the Jacobi triple product identity:

\[
\sum_{k=-\infty}^{\infty} z^k q^{k(k+1)/2} = \prod_{i=1}^{\infty} (1 + zq^i) \prod_{j=0}^{\infty} (1 + z^{-1}q^j) \prod_{i=1}^{\infty} (1 - q^i)
\]

Set \( q \leftarrow t^5, z \leftarrow (-t^{-2}) \) and divide both sides by

\[
\prod_{i=0}^{\infty} (1 - t^{5i+1}) \prod_{i=0}^{\infty} (1 - t^{5i+4}),
\]

we have

\[
\prod_{i=0}^{\infty} (1 - t^{5i+1}) \prod_{i=0}^{\infty} (1 - t^{5i+4}) = \sum_{m=-\infty}^{\infty} (-1)^m t^{m(5m-1)/2} \prod_{i=1}^{\infty} \frac{1}{(1-t^i)}
\]

It remains to prove Schur’s identity, which is equivalent to Rogers-Ramanujan identity:

\[
(1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2) \cdots (1-t^k)}) = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)} \sum_{m=-\infty}^{\infty} (-1)^m t^{m(5m-1)/2}
\]

Some combinatorial definitions: denote by \( \mathcal{P}_n \) the set of all partitions \( \lambda \) of \( n \), and let \( \mathcal{P} = \cup_n \mathcal{P}_n \), \( p(n) = |\mathcal{P}_n| \). Let \( l(\lambda), e(\lambda) \) be the number of parts and smallest part of the partition, respectively. Say \( \lambda \) is a Rogers-Ramanujan partition if \( e(\lambda) \geq l(\lambda) \). Denote by \( \mathcal{R}_n \) the set of Rogers-Ramanujan partitions, and let \( \mathcal{R} = \cup_n \mathcal{R}_n \), \( q(n) = |\mathcal{R}_n| \).

We have

\[
P(t) := 1 + \sum_{n=1}^{\infty} p(n) t^n = \prod_{i=1}^{\infty} \frac{1}{(1-t^i)}; \]

\[
Q(t) := 1 + \sum_{n=1}^{\infty} q(n) t^n = 1 + \sum_{k=1}^{\infty} \frac{t^{k^2}}{(1-t)(1-t^2) \cdots (1-t^k)}.
\]
Let $\lambda$ be conjugate of $\lambda$. For $m \geq 0$, define an \textit{m-rectangle} to be a rectangle whose height minus its width is $m$. Define the \textit{first m-Durfee rectangle} to be the largest $m-$rectangle which fits in diagram $[\lambda]$. Denote by $s_m(\lambda)$ the height of the first $m-$Durfee rectangle. Define the \textit{second m-Durfee rectangle} to be the largest $m-$rectangle which fits in diagram $[\lambda]$ below the first $m-$Durfee rectangle, and let $t_m(\lambda)$ to be its height. We allow an $m-$Durfee rectangle to have width 0 but never height 0. Finally, denote by $\alpha, \beta, \gamma$ the three partitions to the right of, in the middle of, and below the $m-$Durfee rectangles.

Define $(2, m)-\text{rank}$, $r_{2,m}(\lambda)$ of a partition $\lambda$ by the formula:

$$r_{2,m}(\lambda) := \beta_1 + \alpha s_m(\lambda) - t_m(\lambda) - \beta_1 + 1 - \gamma'_1$$

Let $H_{n,m,r}$ be the set of partitions of $n$ with $(2, m)-\text{rank} r$. Define $h(n, m, r) = |H_{n,m,r}|$. Note that $(2,0)$-rank is only defined for non-Rogers-Ramanujan partitions because otherwise $\beta_1$ does not exist.

Denote by $h(n, m, r)$ the number of partitions $\lambda$ of $n$ with $r_{2,m}(\lambda) = r$. Similarly, let $h(n, m, \leq r)$ and $h(n, m, \geq r)$ be the number of partitions with the $(2,m)$-rank $\leq r$ and $\geq r$, respectively. The following is from the definition:

$$h(n, m, \leq r) + h(n, m, \geq r + 1) = p(n), m > 0,$$

$$h(n, 0, \leq r) + h(n, 0, \geq r + 1) = p(n) - q(n)$$

for all $r \in \mathbb{Z}$ and $n \geq 1$.

The main part of the proof is:

\textbf{(first symmetry)} $h(n, 0, r) = h(n, 0, -r)$ and

\textbf{(second symmetry)} $h(n, m, \leq -r) = h(n - r - 2m - 2, m + 2, \geq -r)$

The first symmetry holds for any $r$ and the second symmetry holds for $m, r > 0$ and for $m = 0$ and $r \geq 0$.

For $j \geq 0$ let

$$a_j = h(n - jr - 2jm - j(5j - 1)/2, m + 2j, \leq -r - j)$$

and

$$b_j = h(n - jr - 2jm - j(5j - 1)/2, m + 2j, \geq -r - j + 1).$$

We have $a_j + b_j = p(n - jr - 2jm - j(5j - 1)/2)$, for all $r, j > 0$. 

3
By second symmetry,
\[ a_j = h(n - jr - 2jm - j(5j - 1)/2, m + 2j, \leq -r - j) = h(n - jr - 2jm - j(5j - 1)/2 - r - j, -2m - 4j - 2, \geq -r - j) = b_{j+1}. \]

We have
\[ h(n, m, \leq -r) = a_0 = b_1 = (a_1 - b_2) - (a_2 - b_3) + (a_3 - b_4) - \cdots = (b_1 + a_1) - (b_2 + a_2) + (b_3 + a_3) - (b_4 + a_4) + \cdots = p(n - r - 2m - 2) - p(n - 2r - 4m - 9) + p(n - 3r - 6m - 21) - \cdots = \sum_{j=1}^{\infty} (-1)^{j-1} p(n - jr - 2jm - j(5j - 1)/2). \]

By first symmetry,
\[ H_{0, \leq -r}(t) = \prod_{n=1}^{\infty} \frac{1}{1 - t^n} \sum_{j=1}^{\infty} (-1)^{j-1} t^{5j-1}/2, \]
\[ H_{0, \leq -1}(t) = \prod_{n=1}^{\infty} \frac{1}{1 - t^n} \sum_{j=1}^{\infty} (-1)^{j-1} t^{5j+1}/2. \]

By first symmetry, \( H_{0, \leq 0}(t) + H_{0, \leq -1}(t) = H_{0, \leq 0}(t) + H_{0, \geq 1}(t) = P(t) - Q(t) \). Therefore
\[ \prod_{n=1}^{\infty} \frac{1}{1 - t^n} \sum_{j=1}^{\infty} (-1)^{j-1} t^{5j-1}/2 + \sum_{j=1}^{\infty} (-1)^{j-1} t^{5j-1}/2 = \prod_{n=1}^{\infty} \frac{1}{1 - t^n} - (1 + \sum_{k=1}^{\infty} \frac{t^k}{(1 - t)(1 - t^2) \cdots (1 - t^k)}), \]

which proves the Schur identity.
3 Proof of First and Second Symmetry

3.1 Proof of the first symmetry

Define an involution $\phi : H_{n,0,r} \to H_{n,0,-r}$, which preserves the size of partitions as well as their Durfee squares.

Let $\lambda$ be a partition with two Durfee square and partitions $\alpha, \beta, \gamma$.

Denote $s = s_0(\lambda), t = t_0(\lambda)$.

$\phi : \lambda \mapsto \bar{\lambda}$ by first mapping $(\alpha, \beta, \gamma)$ to $(\mu, \nu, \rho, \sigma)$, then to $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$.

(1) Firstly, let $\mu = \beta$.

Secondly, remove the following parts from $\alpha$:

$\alpha_{s-t-\beta_j + j}$ for $1 \leq j \leq t$.

Let $\nu$ be the partition comprising of parts removed from $\alpha$ and $\pi$ be the partitions comprising of the parts which were not removed.

Thirdly, for $1 \leq j \leq t$, let $k_j = \max\{k \leq s - t | \gamma_j' - k \geq \pi_{s-t-k+1}\}$. Let $\rho$ be the partition with parts $\rho_j = k_j$ and $\sigma$ be the partition with parts $\sigma_j = \gamma_j' - k_j$.

(2) Let $\hat{\gamma}' = \nu + \mu$ be the sum of partitions $(\sigma_j = \gamma_j' - k_j)$.

Secondly, $\hat{\alpha} = \sigma \cup \pi$ be the union of partitions.

Thirdly, $\hat{\beta} = \rho$.

Note that $k$ is the unique integer $k$ which satisfies $\pi_{s-t-k+1} \leq \gamma_j' - k \leq \pi_{s-t-k}$.

To show $\phi$ is an involution, check:

1. $\rho$ is a partition.
2. $\sigma$ is a partition.
3. $\bar{\lambda} = \phi(\lambda)$ is a partition.
4. $\phi^2$ is the identity.
5. $r_{2,0}(\bar{\lambda}) = -r_{2,0}(\lambda)$.

Proof.

1. If $k_j \leq k_{j+1}$,
   then $\pi_{s-t-k_j+1} + k_j \leq \pi_{s-t-k_{j+1}+1} + k_{j+1} \leq \gamma_j' \leq \pi_{s-t-k_j} + k_j$.
   Thus $\pi_{s-t-k_j+1} \leq \gamma_j' - k_j \leq \pi_{s-t-k_j}$.
   The uniqueness implies that $k_j = k_{j+1}$.
   Therefore $k_j \geq k_{j+1}$, then $\rho$ is a partition.
2. If \( k_j > k_{j+1} \), then we have \( s - t - k_j + 1 \leq s - t - k_{j+1} \) and therefore \( \pi_{s-t-k_j+1} \leq \pi_{s-t-k_{j+1}} \).

We conclude that \( \gamma'_k - k_j \geq \gamma'_{j+1} - k_{j+1} \).

If \( k_j = k_{j+1} \), since \( \gamma' \) is a partition, \( \gamma'_j - k_j \geq \gamma'_{j+1} - k_{j+1} \).

Therefore \( \sigma \) is a partition.

3. By definition \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \) are also partitions and fit into \( s, t \) Durfee rectangles.

4. Let \( \hat{\lambda} = \phi(\lambda) \). Similarly, let \( \hat{\mu}, \hat{\nu}, \hat{\pi}, \hat{\rho}, \hat{\sigma} \) be the partitions occuring in the intermediate stage of the second application of \( \phi \) and let \( \alpha^*, \beta^*, \gamma^* \) be the partitions to the right of, in the middle of and below the Durfee squares of \( \phi^2(\lambda) = \phi(\lambda) \).

We have \( \hat{\mu} = \hat{\beta} = \rho \). It can be shown that \( \alpha^* = \mu \cup \pi = \alpha, \beta^* = \mu = \beta, \) and \( (\gamma^*)' = \rho + \sigma = \gamma' \).

5. We have \( r_{2,0}(\lambda) = \beta_1 + \alpha_{s-t-\beta_1+1} - \gamma'_1 = \mu_1 + \nu_1 - \rho_1 - \sigma_1 \).

\( r_{2,0}(\hat{\lambda}) = \hat{\beta}_1 + \hat{\alpha}_{s-t-\hat{\beta}_1+1} - \hat{\gamma}'_1 = \rho_1 + \sigma_1 - \mu_1 - \nu_1 \).

Therefore \( r_{2,0}(\hat{\lambda}) = -r_{2,0}(\lambda) \).

### 3.2 Proof of the second symmetry

We present a bijection \( \psi_{m,r} : \mathcal{H}_{n,m,\leq-r} \to \mathcal{H}_{n-r-2m-2,m+2,\geq-r} \) for \( m, r > 0 \) and for \( m = 0, r \geq 0 \).

Describe the action of \( \psi := \psi_{m,r} \) by giving the sizes of the Durfee rectangles of \( \hat{\lambda} := \psi_{m,r}(\lambda) = \psi(\lambda) \) and \( \hat{\alpha}, \hat{\beta}, \hat{\gamma} \).

1. If \( \lambda \) has two \( m \)-Durfee rectangles of height \( s := s_m(\lambda), t := t_m(\lambda) \). Then \( \hat{\lambda} \) has two \((m+2)\)-Durfee rectangles of height \( s' := s_{m+2}(\hat{\lambda}) = s + 1 \) and \( t' := t_{m+2}(\hat{\lambda}) = t + 1 \).

2. Let \( k_1 = \max\{k \leq s - t | \gamma'_1 - r - k \geq \alpha_{s-t-k+1} \} \).

Obtain \( \hat{\alpha} \) from \( \alpha \) by adding a new part of size \( \gamma'_1 - r - k_1 \), \( \hat{\beta} \) from \( \beta \) by adding a new part of size \( k_1 \), and \( \hat{\gamma} \) from \( \gamma \) by removing its first column.

Since \( k_j \) is the unique integer \( k \) which satisfies \( \pi_{s-t-k+1} \leq \gamma'_k - k \leq \pi_{s-t-k} \).

By considering \( k = \beta_1 \), we see \( k_1 \) is defined and indeed have \( k_1 \leq \beta_1 \). And \( k_1 \) is the unique \( k \) such that \( \alpha_{s-t-k+1} \leq \gamma'_1 - r - k \leq \alpha_{s-t-k} \).

Check the map \( \psi = \psi_{m,r} \) defined above is a bijection by 4 parts.
1. We prove that $\hat{\lambda} = \psi(\lambda)$ is a partition.

Note that $\lambda$ has $m - \text{Durfee rectangles of nonzero width, } \hat{\lambda}$ may have $(m + 2) - \text{Durfee rectangles of width } s - 1$ and $t - 1$. Also, the partitions $\hat{\alpha}$ and $\hat{\beta}$ have at most $s + 1$ and $t + 1$ parts. While the partitions $\hat{\beta}$ and $\hat{\gamma}$ have parts of size at most $s - t$ and $t - 1$. This means they can sit to the right of, in the middle of, and below the two $m + 2 - \text{Durfee rectangles of } \hat{\lambda}$.

2. The size of $\hat{\lambda}$ is $n - r - 2m - 2$.

Note the sum of the sizes of the rows added to $\alpha$ and $\beta$ is $r$ less than the size of the column removed from $\gamma$, and that both the first and second $(m + 2) - \text{Durfee rectangles of } \hat{\lambda}$ have size $m + 1$ less than the size of the corresponding $m - \text{Durfee rectangle of } \lambda$.

3. $r_{2,m+2}(\hat{\lambda}) \geq -r$

The part we insert to $\beta$ will be the largest part of the resulting partition, i.e. $\beta = k_1$.

We have $\alpha_{s-t-k_1+1} \leq \gamma'_1 - r - k_1 \leq \alpha_{s-t-k_1}$.

Therefore, we have $\hat{\alpha}_{s'-t'-\hat{\beta}_1+1} = \hat{\alpha}_{s-t-k_1+1} = \gamma'_1 - r - k_1$.

We have chosen $k_1$ in the unique way so that the rows we insert into $\alpha$ and $\beta$ are $\hat{\alpha}_{s'-t'-\hat{\beta}_1+1}$ and $\hat{\beta}_1$, respectively.

Therefore $r_{2,m+2}(\hat{\lambda}) = \hat{\alpha}_{s'-t'-\hat{\beta}_1+1} + \hat{\beta}_1 - \gamma'_1 = \gamma'_1 - r - k_1 + k_1 - \gamma'_1 \geq -r$.

4. Present the inverse map $\psi^{-1}$.

The above characterization of $k_1$ also shows us that to recover $\alpha, \beta, \gamma$ from $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$, we remove $\hat{\alpha}_{s'-t'-\hat{\beta}_1+1}$ from $\hat{\alpha}$, remove $\hat{\beta}_1$ from $\hat{\beta}$, and add a column of height $\hat{\alpha}_{s'-t'-\hat{\beta}_1+1} + \hat{\beta}_1 + r$ to $\hat{\gamma}$. Since we can also easily recover the sizes of the previous $m - \text{Durfee rectangles, we conclude that } \psi \text{ is a bijection between the desired sets.}$
4 An Application in Analysis

We wish to consider in as simple a manner as possible a two-variable generalization \( f(q,t) \) that has the following properties:

1. \( f(q,t) = \sum_{n=0}^{\infty} D_n(q)t^n \), where \( D_n(q) \) are polynomials.

2. \( \lim_{n \to \infty} D_n(q) = \prod_{n=0}^{\infty} 1/(1 - q^{5n+1})(1 - q^{5n+4}) \)

3. \( f(q,t) \) satisfies a first-order nonhomogeneous \( q \)-difference equation.

Let

\[
f(q,t) = \sum_{n=0}^{\infty} \frac{t^{2n}q^{n^2}}{(1 - t)(1 - tq) \cdots (1 - tq^n)}
\]

Firstly, \( (1 - t)f(q,t) = 1 + t^2qf(q, tq) \), thus 3 is satisfied.

Next we note

\[
f(q,t) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} t^{2n}q^{n^2} t^m \binom{n+m}{m}_q = \sum_{N=0}^{\infty} t^N \sum_{0 \leq 2n \leq N} q^{n^2} \binom{N-n}{n}_q.
\]

Thus \( D_n(q) = \sum_{0 \leq 2n \leq N} q^{n^2} \binom{N-n}{n}_q \), we have 1.

\[
\lim_{n \to \infty} D_n(q) = \lim_{t \to 1} (1-t)f(q,t) = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(1-q)(1-q^2) \cdots (1-q^n)} = \prod_{n=0}^{\infty} \frac{1}{(1-q^{5n+1})(1-q^{5n+4})}.
\]

We have 2.
5 Reference

