Hilbert Transform on $C^{1+\varepsilon}$ Families of Lines

Michael T. Lacey

Georgia Institute of Technology
IPAM Workshop

November 18, 2004
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

Outline

1. The Background of the Main Theorem
   - Besicovitch Set
   - Zygmund Conjecture

2. Main Results
   - Main Theorem and Key Proposition
   - Key Proposition Implies Main Theorem
   - Corollary: Carleson’s Theorem on Fourier Series
Besicovitch Set

- Besicovitch set is a compact set of zero measure, that contains a line in every direction of the plane.
Besicovitch set is a compact set of zero measure, that contains a line in every direction of the plane.

This set necessarily has Hausdorff dimension two, a fundamental fact in this subject.
Besicovitch Set

- Besicovitch set is a compact set of zero measure, that contains a line in every direction of the plane.
- This set necessarily has Hausdorff dimension two, a fundamental fact in this subject.
- One constructs highly eccentric rectangles which have small union, but the translates along their long direction, their “reach,” are essentially disjoint.
Besicovitch Set

- Besicovitch set is a compact set of zero measure, that contains a line in every direction of the plane.
- This set necessarily has Hausdorff dimension two, a fundamental fact in this subject.
- One constructs highly eccentric rectangles which have small union, but the translates along their long direction, their "reach," are essentially disjoint.
- We briefly outline the construction of this set.
The triangle contains unit length line segments in a full angle of directions.
The thirds of the triangle are moved so that they share a common base.
Reflect the triangles about their vertexes.
The red triangles are essentially disjoint, and are called the “reach” of the set.
The red triangles are essentially disjoint, and are called the “reach” of the set. A vector field, which points into the set, defined in the “reach” can be Hölder continuous, of any index strictly less than one.
The red triangles are essentially disjoint, and are called the “reach” of the set. A vector field, which points into the set, defined in the “reach” can be Hölder continuous, of any index strictly less than one. Conversely, if \( v \) is Lipschitz, then the “Besicovitch set” is has can’t have zero Lebesgue measure.
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

The Background of the Main Theorem

Zygmund Conjecture

If $v$ is Lipschitz, then for all $f \in L^2(\mathbb{R}^2)$,

$$f(x) = \lim_{\epsilon \downarrow 0} (2\epsilon)^{-1} \int_{-\epsilon}^{\epsilon} f(x - yv(x)) \, dy \quad \text{a.e.}(x)$$
**Zygmund Conjecture**

If $\nu$ is Lipschitz, then for all $f \in L^2(\mathbb{R}^2)$,

$$f(x) = \lim_{\epsilon \downarrow 0} (2\epsilon)^{-1} \int_{-\epsilon}^{\epsilon} f(x - y\nu(x)) \, dy \quad \text{a.e.}(x)$$

**E.M. Stein’s Conjecture**

E.M. Stein’s Conjecture: For all Lipschitz $\nu$ the operator below maps $L^2$ into itself.

$$H_{\nu} f(x) = \int_{-1}^{1} f(x - y\nu(x)) \frac{dy}{y}$$
**Zygmund Conjecture**

If $v$ is Lipschitz, then for all $f \in L^2(\mathbb{R}^2)$,

$$f(x) = \lim_{\epsilon \downarrow 0} (2\epsilon)^{-1} \int_{-\epsilon}^{\epsilon} f(x - yv(x)) \, dy \quad \text{a.e.}(x)$$

---

**E.M. Stein’s Conjecture**

E.M. Stein’s Conjecture: For all Lipschitz $v$ the operator below maps $L^2$ into itself.

$$H_v f(x) = \int_{-1}^{1} f(x - yv(x)) \frac{dy}{y}$$

Both conjectures are open. They represent very subtle statements about the nature of the Besicovitch set.
Main Results of Xiaochun Li and L.

**Theorem (L. & Li)**

For all $\epsilon > 0$, if $v$ has $1 + \epsilon$ derivatives, then

$$\| H_v \|_2 \lesssim (1 + \log \| v \|_{C^{1+\epsilon}})^2.$$
Main Results of Xiaochun Li and L.

Theorem (L. & Li)

For all $\epsilon > 0$, if $\nu$ has $1 + \epsilon$ derivatives, then

$$\|H\nu\|_2 \lesssim (1 + \log \|\nu\|_{C^{1+\epsilon}})^2.$$ 

Let $\lambda$ be a smooth Schwartz function on the plane supported in frequency in $1 \leq |\xi| \leq 2$, and $\lambda_k(y) := 2^{-2k}\lambda(y2^{-k})$, so that we state things in dilation invariant setting.
Main Results of Xiaochun Li and L.

Theorem (L. & Li)
For all $\epsilon > 0$, if $v$ has $1 + \epsilon$ derivatives, then

$$\|H_v\|_2 \lesssim (1 + \log v\|_C^{1+\epsilon})^2.$$

Let $\lambda$ be a smooth Schwartz function on the plane supported in frequency in $1 \leq |\xi| \leq 2$, and $\lambda_k(y) := 2^{-2k}\lambda(y2^{-k})$, so that we state things in dilation invariant setting.

Key Proposition (Scale Invariant Formulation)
If $v$ is Lipschitz, then

$$\|H_v \lambda_k\|_2 \lesssim 1 + \log 2^k v\|_{Lip}.$$
Prior Results

- Previously, these results were known if \( \nu \) were analytic, a result of Nagel, Stein and Wainger, or real-analytic a result of Bourgain.
- There is a rich and beautiful theory of Radon Transforms, as developed by Christ, Nagel, Stein, and Wainger.
Prior Results

- Previously, these results were known if $v$ were analytic, a result of Nagel, Stein and Wainger, or real–analytic a result of Bourgain.
- There is a rich and beautiful theory of Radon Transforms, as developed by Christ, Nagel, Stein, and Wainger.
- Nets Katz has a partial result on Zygmund conjecture.
Prior Results

- Previously, these results were known if $\nu$ were analytic, a result of Nagel, Stein and Wainger, or real-analytic a result of Bourgain.
- There is a rich and beautiful theory of Radon Transforms, as developed by Christ, Nagel, Stein, and Wainger.
- Nets Katz has a partial result on Zygmund conjecture.
- The main point is that these results are true in absence of (a) geometric conditions on $\nu$ (b) minimal smoothness conditions.
Previously, these results were known if \( \nu \) were analytic, a result of Nagel, Stein and Wainger, or real–analytic a result of Bourgain.

There is a rich and beautiful theory of Radon Transforms, as developed by Christ, Nagel, Stein, and Wainger.

Nets Katz has a partial result on Zygmund conjecture.

The main point is that these results are true in absence of (a) geometric conditions on \( \nu \) (b) minimal smoothness conditions.

Genuinely two dimensional wave packet analysis.
With $1 + \epsilon$ smoothness, one can show this:

$$\|H_v \lambda_k \ast f - \lambda_k \ast (H_v \lambda_k \ast f)\|_2 \lesssim 2^{-\epsilon k} \|f\|_2$$

And this proves the Main Theorem from the Key Proposition.
With $1 + \epsilon$ smoothness, one can show this:

$$\|H_v \lambda_k * f - \lambda_k * (H_v \lambda_k * f)\|_2 \lesssim 2^{-\epsilon k} \|f\|_2$$

And this proves the Main Theorem from the Key Proposition. This orthogonality takes some care to formalize correctly.
Key Proposition Implies Main Theorem

With $1 + \varepsilon$ smoothness, one can show this:

$$\|H_v \lambda_k \ast f - \lambda_k \ast (H_v \lambda_k \ast f)\|_2 \lesssim 2^{-\varepsilon k} \|f\|_2$$

And this proves the Main Theorem from the Key Proposition.

Important Obstacle in Extensions

This decouples $\mathbb{R}^2$ scales in the crudest possible way. We need a far more sophisticated decoupling of 2-dim’l scales to e.g. address Zygmund conjecture.
Hilbert Transform on $C^{1+\varepsilon}$ Families of Lines

Main Results

Corollary: Carleson’s Theorem on Fourier Series

**Role of Carleson’s Theorem**

Our Main Theorem has an implication: Pointwise Convergence of Fourier Series in $L^2(\mathbb{R})$. 
Our Main Theorem has an implication: Pointwise Convergence of Fourier Series in $L^2(\mathbb{R})$.

**Carleson’s Theorem**

For all measurable functions $N(x)$, the operator below is bounded from $L^2$ into itself.

$$C_N f(x) := \text{p.v.} \int_{-1}^{1} f(x - y) e^{iN(x)y} \frac{dy}{y}$$
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

Main Results

Corollary: Carleson’s Theorem on Fourier Series

**ROLE OF CARLESON’S THEOREM**

Our Main Theorem has an implication: Pointwise Convergence of Fourier Series in $L^2(\mathbb{R})$.

**CARLESON’S THEOREM**

For all measurable functions $N(x)$, the operator below is bounded from $L^2$ into itself.

$$C_N f(x) := \text{p.v.} \int_{-1}^{1} f(x - y) e^{iN(x)y} \frac{dy}{y}$$

Assuming our Main Theorem, we can show that for smooth $N(x)$, the operator above is bounded, with norm independent of $\|N\|_{C^2}$. 
Construction of $\nu$

Calculate the symbols of both operators where
\[ \sigma(\xi) = \int_{-1}^{1} e^{ix\xi} dy/y. \]

\[ H_{\nu} f(x) = \int_{\mathbb{R}^2} \sigma(\xi \cdot \nu(x)) \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi \]

\[ C_N g(x) = \int_{\mathbb{R}} \sigma(\theta - N(x)) \hat{g}(\theta) e^{i\theta x} \, dx. \]
Construction of $v$

Calculate the symbols of both operators where
\[ \sigma(\xi) = \int_{-1}^{1} e^{ix\xi} \frac{dy}{y}. \]

\[
H_v f(x) = \int_{\mathbb{R}^2} \sigma(\xi \cdot v(x)) \hat{f}(\xi) e^{ix \cdot \xi} \, d\xi
\]

\[
C_N g(x) = \int_{\mathbb{R}} \sigma(\theta - N(x)) \hat{g}(\theta) e^{i\theta x} \, dx.
\]

View $g \in L^2(\mathbb{R})$ as being on the frequency line $\xi_2 = J$ on the plane, where $J$ is a large constant.
CONSTRUCTION OF $\nu$

Calculate the symbols of both operators where

$$\sigma(\xi) = \int_{-1}^{1} e^{ix\xi} \frac{dy}{y}.$$  

$$H_{\nu} f(x) = \int_{\mathbb{R}^2} \sigma(\xi \cdot \nu(x)) \widehat{f}(\xi) e^{ix \cdot \xi} \, d\xi$$

$$C_N g(x) = \int_{\mathbb{R}} \sigma(\theta - N(x)) \widehat{g}(\theta) e^{i\theta x} \, dx.$$  

View $g \in L^2(\mathbb{R})$ as being on the frequency line $\xi_2 = J$ on the plane, where $J$ is large constant.

Then you choose $\nu(x_1, x_2) \asymp (1, N(x_1)/J)$, so that

$$\xi \cdot \nu(x) = x_1\xi_1 - N(x_1) \quad \text{on the line } \xi_2 = J.$$
Corollary: Carleson’s Theorem on Fourier Series

The Picture for $\nu$

- The Blue line is the function $\sigma(\xi_1 - N(x_1))$. 

\[ \xi_2 = J \]

\[ N(x_1) \]
**Main Results**

**Corollary: Carleson's Theorem on Fourier Series**

**The Picture for \( \nu \)**

- The Blue line is the function \( \sigma(\xi_1 - N(x_1)) \).
- Then \( \nu(x_1) \simeq (1, -N(x_1)/J) \)

\[ \xi_2 = J \]

\[ N(x_1) \]

\[ \nu(x_1) \]
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

Main Results

Corollary: Carleson's Theorem on Fourier Series

The Picture for $\nu$

- The Blue line is the function $\sigma(\xi_1 - N(x_1))$.
- Then $\nu(x_1) \simeq (1, -N(x_1)/J)$, so take $J \gg \|\nu\|_{C^2}$.
- The symbol of $H_{\nu}$ and $C_N$ agree on the line $\xi_2 = J$. 

\[ \xi_2 = J \]
\[ N(x_1) \]
\[ \nu(x_1) \]
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

Main Results

Corollary: Carleson’s Theorem on Fourier Series

**The Picture for $v$**

- The Blue line is the function $\sigma(\xi_1 - N(x_1))$.
- Then $v(x_1) \asymp (1, -N(x_1)/J)$, so take $J \gg \|v\|_{C^2}$.
- The symbol of $H_v$ and $C_N$ agree on the line $\xi_2 = J$.
- Note that small oscillations give oscillations in frequency, that increase as frequency increases.
The Proof of Main Theorem

3 Lemma Related To Carleson’s Theorem
- The Weak $L^2$ estimate is Sharp

4 Annular Tiles
- The Functions associated to a Tile
Using the methods of Carleson’s Theorem, as proved in , one can show that

**Lemma for Measurable Vector Fields**

If $v$ is measurable, one has

\[ \| H_v \lambda_0 \|_{2 \rightarrow 2, \infty} < \infty, \]
\[ \| H_v \lambda_0 \|_p < \infty, \quad 2 < p < \infty. \]
Lemma Related to Carleson’s Theorem

Using the methods of Carleson’s Theorem, as proved in , one can show that

Lemma for Measurable Vector Fields

If $v$ is measurable, one has

$$
\|H_v \lambda_0\|_{2 \to 2, \infty} < \infty,
$$

$$
\|H_v \lambda_0\|_p < \infty, \quad 2 < p < \infty.
$$

- The $L^2$ to weak $L^2$ estimate is optimal.
Lemma Related to Carleson’s Theorem

Using the methods of Carleson’s Theorem, as proved in , one can show that

Lemma for Measurable Vector Fields

If \( v \) is measurable, one has

\[
\|H_v \lambda_0\|_{2\to 2, \infty} < \infty,
\]

\[
\|H_v \lambda_0\|_p < \infty, \quad 2 < p < \infty.
\]

- The \( L^2 \) to weak \( L^2 \) estimate is optimal.
- And these estimate are critical to an interpolation argument that we use to prove the Key Proposition.
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines
Lemma Related To Carleson’s Theorem
The Weak $L^2$ estimate is Sharp

Weak $L^2$ Estimate Is Sharp

Consider the radial vector field $\nu$, and a smooth bump function $\phi$. 

$Hv(\varphi(x)) \approx |x|$
Consider the radial vector field $\nu$, and a smooth bump function $\varphi$. **Modulate** $\varphi$ so that it is supported in frequency in the annulus $1 \leq |\xi| \leq 2$. 
Consider the radial vector field $\nu$, and a smooth bump function $\varphi$. Modulate $\varphi$ so that it is supported in frequency in the annulus $1 \leq |\xi| \leq 2$. Calculate $H_\nu$ at any point inside the lines, and there will be no cancellation.
The proof of the positive result for measurable vector fields follows the Lacey–Thiele approach.
Measurable Vector Fields Below $L^2$

- The proof of the positive result for measurable vector fields follows the Lacey–Thiele approach.
- The proof breaks down completely below $L^2$, as it requires essentially the boundedness of the Kakeya maximal function.
The proof of the positive result for measurable vector fields follows the Lacey–Thiele approach.

The proof breaks down completely below $L^2$, as it requires essentially the boundedness of the Kakeya maximal function.

A key innovation is to replace the Kakeya maximal function by a variant associated to the vector field $v$. 
A Tile is a product of *dual* rectangles.
A Tile is a product of dual rectangles. The Frequency Rectangle $\omega_s$ is tangent to a circle of radius $r$, and spans the annulus $r < |\xi| < 2r$. 
A Tile is a product of *dual* rectangles. The Frequency Rectangle $\omega_s$ is tangent to a circle of radius $r$, and spans the annulus $r < |\xi| < 2r$. But otherwise vary arbitrarily.
A Tile is a product of *dual* rectangles. The Frequency Rectangle $\omega_s$ is tangent to a circle of radius $r$, and spans the annulus $r < |\xi| < 2r$. But otherwise vary arbitrarily. A *tile* is $\omega_s \times R_s$. 
More Pictures of Tiles

$\omega_s$

$R_s$
More Pictures of Tiles
Interval of uncertainty associated with rectangle $R$ is a sub arc of the unit circle. Its center is the long direction of the rectangle. Its length is the width of $R$ divided by length of $R$. 
The Uncertainty Intervals

Interval of uncertainty associated with rectangle $R$ is a sub arc of the unit circle. Its center is the long direction of the rectangle. Its length is the width of $R$ divided by length of $R$. **Basis changes are permitted, up to a tolerance level dictated by the angle of uncertainty.**
The Functions Associated to a Tile

$$\varphi_s = \text{Modulate}_{c(\omega_s)} \text{Dilate}_{R_s}^2 \varphi$$
The Functions Associated to a Tile $s$

\[ \varphi_s = \text{Modulate}_{c(\omega_s)} \text{Dilate}^2_{R_s} \varphi \]

$c(\omega_s) = \text{center of } \omega_s$,

$\text{Dilate}^2_{R_s} = L^2$ norm one dilation adapted to scale and location of $R_s$.
The Functions Associated to a Tile $s$

$$\varphi_s = \text{Modulate}_{c(\omega_s)} \text{Dilate}^2_{R_s} \varphi$$

$c(\omega_s) =$ center of $\omega_s$,

$\text{Dilate}^2_{R_s} =$ $L^2$ norm one dilation adapted to scale and location of $R_s$.

$$\phi_s = \mathbf{1}_{\mu(R_s)}(v(x)) \int_{\mathbb{R}} \varphi_s(x - yv(x)) \text{scl}(s) \psi(\text{scl}(s)y) \, dy$$
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

Annular Tiles

The Functions associated to a Tile

**The Functions Associated to a Tile $s$**

\[ \varphi_s = \text{Modulate}_{c(\omega_s)} \text{Dilate}^2_{R_s} \varphi \]

\[ c(\omega_s) = \text{center of } \omega_s, \]

\[ \text{Dilate}^2_{R_s} = L^2 \text{ norm one dilation adapted to scale and location of } R_s. \]

\[ \phi_s = \mathbf{1}_{\mu(R_s)}(v(x)) \int_{\mathbb{R}} \varphi_s(x - yv(x)) \text{scl}(s) \psi(\text{scl}(s)y) \, dy \]

\[ \text{scl}(s) = \text{the short side of } \omega_s. \psi \text{ is a Schwartz function on } \mathbb{R}, \text{ with Fourier support in a small neighborhood of one.} \]
The Functions Associated to a Tile $s$

\[ \varphi_s = \text{Modulate}_{c(\omega_s)} \text{Dilate}_{R_s}^2 \varphi \]

$c(\omega_s) = \text{center of } \omega_s$,

$\text{Dilate}_{R_s}^2 = L^2 \text{ norm one dilation adapted to scale and location of } R_s.$

\[ \phi_s = 1_{\mu(R_s)}(v(x)) \int_{\mathbb{R}} \varphi_s(x - yv(x)) \text{scl}(s)\psi(\text{scl}(s)y) \, dy \]

$scl(s) = \text{the short side of } \omega_s$. $\psi$ is a Schwartz function on $\mathbb{R}$, with Fourier support in a small neighborhood of one. Let $\mathcal{AT}$ be all annular tiles with $\omega_s$ contained in $1 \leq |\xi| \leq 2$. 
The Main Lemma for Sums Over Tiles

For a measurable vector field, we have the estimate

\[
\left\| \sum_{s \in AT} \langle f, \varphi_s \rangle \phi_s \right\|_{2,\infty} \lesssim \|f\|_2
\]

We also have the \(L^p\) inequality for \(p > 2\).
The Main Lemma for Sums Over Tiles

For a measurable vector field, we have the estimate
\[ \left\| \sum_{s \in AT} \langle f, \varphi_s \rangle \phi_s \right\|_{2,\infty} \lesssim \| f \|_2 \]

We also have the $L^p$ inequality for $p > 2$. If in addition $\| v \|_{Lip} < \infty$, then we have the $L^2$ inequality
\[ \left\| \sum_{s \in AT} \langle f, \varphi_s \rangle \phi_s \right\|_2 \lesssim \| f \|_2 \]
\[ \text{scl}(s) > 100 \| v \|_{Lip} \]
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

Outline

5 Uncertainty and Density of Rectangles

6 Maximal Function Lemma
   - Two Lemmas on Lipschitz vector fields

7 The Covering Lemma Statement
   - Selection of $\mathcal{R}'$
**Density for Tiles**

$$\text{dense}(R) = \frac{|R \cap v^{-1}(\mu(R))|}{|R|}$$

This measures the percentage of time $v(x)$ points in the long direction of $R$. 

Hilbert Transform on $C^{1+\epsilon}$ Families of Lines
Uncertainty and Density of Rectangles
Density for Tiles

\[ \text{dense}(R) = \frac{|R \cap \nu^{-1}(\mu(R))|}{|R|} \]

This measures the percentage of time \( \nu(x) \) point in the long direction of \( R \).

A rectangle and three vectors in the angle of uncertainty
Definition of the Maximal Function

Let $\mathcal{R}$ be a collection of rectangles with dense($R$) > $\delta$ for all $R \in \mathcal{R}$. and they all have the same width, and lengths at most $\|v\|_{Lip}/100$. Let

$$M_{\mathcal{R}} f = \sup_{R \in \mathcal{R}} \frac{1_R}{|R|} \int_R f(y) \, dy$$
**Definition of the Maximal Function**

Let $\mathcal{R}$ be a collection of rectangles with $\text{dense}(R) > \delta$ for all $R \in \mathcal{R}$, and they all have the same width, and lengths at most $\|v\|_{\text{Lip}}/100$. Let

$$M_{\mathcal{R}} f = \sup_{R \in \mathcal{R}} \frac{1_R}{|R|} \int_R f(y) \, dy$$

Previously, one considered maximal function over all rectangles of a given eccentricity. Then the maximal function has norm on $L^2$ that blows up like the log of the eccentricity.
**Definition of the Maximal Function**

Let $\mathcal{R}$ be a collection of rectangles with $\text{dense}(R) > \delta$ for all $R \in \mathcal{R}$, and they **all have the same width**, and lengths at most $\|v\|_{\text{Lip}}/100$. Let

$$M_{\mathcal{R}} f = \sup_{R \in \mathcal{R}} \frac{1_R}{|R|} \int_R f(y) \, dy$$

- Previously, one considered maximal function over all rectangles of a given *eccentricity*. Then the maximal function has norm on $L^2$ that blows up like the log of the eccentricity.
- Here, it is essential that the estimate be *independent of eccentricity*. 
Maximal Function Lemma

For all $0 < \delta < 1$, the maximal function satisfies

$$\|M_R\|_{p \to p, \infty} \lesssim \delta^{-3}, \quad 1 < p < \infty$$
Maximal Function Lemma

For all $0 < \delta < 1$, the maximal function satisfies

$$\|M_R\|_{p \to p, \infty} \lesssim \delta^{-3}, \quad 1 < p < \infty$$

- We need the lemma for some $1 < p < 2$. 
Maximal Function Lemma

For all $0 < \delta < 1$, the maximal function satisfies

$$\|M_R\|_{p \to p, \infty} \lesssim \delta^{-3}, \quad 1 < p < \infty$$

- We need the lemma for some $1 < p < 2$.
- And a norm estimate of $\delta^{-N}$ for any $N < \infty$. 
Maximal Function Lemma

For all $0 < \delta < 1$, the maximal function satisfies

$$\|M_R\|_{p \to p, \infty} \lesssim \delta^{-3}, \quad 1 < p < \infty$$

- We need the lemma for some $1 < p < 2$.
- And a norm estimate of $\delta^{-N}$ for any $N < \infty$.
- The method of proof is a careful analysis, in the style of Fefferman and Cordoba.
First Lemma on Lipschitz Vector Fields

Lemma 1: Suppose there are $R_1, \ldots, R_n \in \mathcal{R}$, all containing a common point. Then, $n < \delta^{-1}$. 
First Lemma on Lipschitz vector fields

**Lemma 1:** Suppose there are $R_1, \ldots, R_n \in \mathcal{R}$, all containing a common point. Then, $n < \delta^{-1}$. Suppose not. Then we can find two rectangles, and points in the rectangles, where the vector field points in the long direction of the rectangle.
**First Lemma on Lipschitz Vector Fields**

**Lemma 1:** Suppose there are $R_1, \ldots, R_n \in \mathcal{R}$, all containing a common point. Then, $n < \delta^{-1}$.

Suppose not. Then we can find two rectangles, and points in the rectangles, where the vector field points in the long direction of the rectangle.

At these two points, vector field is nearly radial. But these two points are very close together. Less than angle $\times$ length. That is a contradiction.
Lemma 2

Consider three rectangle, $R_0$, $R$, and $R'$ as pictured. Key assumption is that there is a point $x \in R$ with $v(x) \in \mu(R)$, and $x' \in \mu(R')$, which share the same projection onto $R_0$. Then the uncertainty intervals $\mu(R)$ and $\mu(R')$ are very close.
The Covering Lemma Statement

Given the collection $\mathcal{R}$, of rectangles of density $\delta$, it suffices to show that there is an $\mathcal{R}' \subset \mathcal{R}$ for which
The Covering Lemma

Given the collection $\mathcal{R}$, of rectangles of density $\delta$, it suffices to show that there is an $\mathcal{R}' \subset \mathcal{R}$ for which

$$\left| \bigcup_{R \in \mathcal{R}} R \right| \lesssim \delta^{-1} \left| \bigcup_{R' \in \mathcal{R}'} R' \right|,$$

Standard arguments then prove the Maximal Function Estimate.
The Covering Lemma

Given the collection $\mathcal{R}$, of rectangles of density $\delta$, it suffices to show that there is an $\mathcal{R}' \subset \mathcal{R}$ for which

$$\left| \bigcup_{R \in \mathcal{R}} R \right| \lesssim \delta^{-1} \left| \bigcup_{R' \in \mathcal{R}'} R' \right|,$$

$$\left\| \sum_{\substack{R \in \mathcal{R}' \atop \text{Length}(R) < \text{Length}(R_0)} } \mathbf{1}_{R \cap R_0} \right\|_n \lesssim \delta^{-2} |R_0|^{1/n} \quad R_0 \in \mathcal{R}'.$$

Standard Arguments then prove the Maximal Function Estimate.
**The Covering Lemma**

Given the collection $\mathcal{R}$, of rectangles of density $\delta$, it suffices to show that there is an $\mathcal{R}' \subset \mathcal{R}$ for which

$$\left| \bigcup_{R \in \mathcal{R}} R \right| \lesssim \delta^{-1} \left| \bigcup_{R' \in \mathcal{R}'} R' \right|,$$

$$\left\| \sum_{R \in \mathcal{R}'} 1_{R \cap R_0} \right\|_n \lesssim \delta^{-2} |R_0|^{1/n} \quad R_0 \in \mathcal{R}'.$$

Standard Arguments then prove the Maximal Function Estimate.
Let $M_{100}$ be a maximal function computed in 100 uniformly distributed directions of the plane. This operator maps $L^1(\mathbb{R}^2)$ to weak $L^1$. 

How To Select $\mathcal{R}'$
HOW TO SELECT $\mathcal{R}'$

Let $M_{100}$ be a maximal function computed in 100 uniformly distributed directions of the plane. This operator maps $L^1(\mathbb{R}^2)$ to weak $L^1$.

Initialize

$$\mathcal{R}^{\text{stock}} := \mathcal{R}, \quad \mathcal{R}' = \emptyset.$$
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

The Covering Lemma Statement

Selection of $\mathcal{R}'$

**HOW TO SELECT $\mathcal{R}'$**

Let $M_{100}$ be a maximal function computed in 100 uniformly distributed directions of the plane. This operator maps $L^1(\mathbb{R}^2)$ to weak $L^1$.

Initialize

$$\mathcal{R}^{\text{stock}} := \mathcal{R}, \quad \mathcal{R}' = \emptyset.$$  

Inductive stage: Select $R' \in \mathcal{R}^{\text{stock}}$ with maximal length.
Let $M_{100}$ be a maximal function computed in 100 uniformly distributed directions of the plane. This operator maps $L^1(\mathbb{R}^2)$ to weak $L^1$.

Initialize

$$\mathcal{R}^{\text{stock}} := \mathcal{R}, \quad \mathcal{R}' = \emptyset.$$  

Inductive stage: Select $R' \in \mathcal{R}^{\text{stock}}$ with maximal length. Update, $\mathcal{R}' := \mathcal{R}' \cup \{R'\}$. Remove from $\mathcal{R}^{\text{stock}}$ any rectangle $R$ such that

$$R \subset \left\{ M_{100} \sum_{R' \in \mathcal{R}'} 1_{R'} \geq \delta^{-1} \right\}.$$
Let $M_{100}$ be a maximal function computed in 100 uniformly distributed directions of the plane. This operator maps $L^1(\mathbb{R}^2)$ to weak $L^1$.

Initialize

$$\mathcal{R}^{\text{stock}} := \mathcal{R}, \quad \mathcal{R}' = \emptyset.$$  

**Inductive stage:** Select $R' \in \mathcal{R}^{\text{stock}}$ with maximal length. Update, $\mathcal{R}' := \mathcal{R}' \cup \{R'\}$. Remove from $\mathcal{R}^{\text{stock}}$ any rectangle $R$ such that

$$R \subset \left\{ M_{100} \sum_{R' \in \mathcal{R}'} 1_{R'} \geq \delta^{-1} \right\}.$$  

Repeat until $\mathcal{R}^{\text{stock}}$ is exhausted.
The main point to prove is

\[ \left\| \sum_{\substack{R \in \mathcal{R}' \ \text{length}(R) < \text{length}(R_0) \ \text{and intersects} \ R_0}} 1_{R \cap R_0} \right\|_n \lesssim \delta^{-2} |R_0|^{1/n} \quad R_0 \in \mathcal{R}'. \]
The main point to prove is

\[ \left\| \sum_{\substack{R \in \mathcal{R}' \atop \text{length}(R) < \text{length}(R_0)}} 1_{R \cap R_0} \right\|_n \lesssim \delta^{-2} |R_0|^{1/n} \quad R_0 \in \mathcal{R}'. \]

Fix $R_0$ as in this inequality. $R_0 = I \times J$, in standard coordinates. Length of $R$ is in the first coordinate.
The main point to prove is

$$\left\| \sum_{\substack{R \in \mathcal{R}' \\text{Length}(R) < \text{Length}(R_0) \\text{and} \\text{Length}(R) < \text{Length}(R_0)}} 1_{R \cap R_0} \right\|_n \lesssim \delta^{-2} |R_0|^{1/n} \quad R_0 \in \mathcal{R}'.$$ 

Fix $R_0$ as in this inequality. $R_0 = I \times J$, in standard coordinates. Length of $R$ is in the first coordinate.

We only consider those $R \in \mathcal{R}'$ that intersect $R_0$ and have a smaller length. Call this collection $\mathcal{R}_0$. 
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines
The Covering Lemma Statement
Selection of $\mathcal{R}'$

**Focusing on the Principal Inequality**

The main point to prove is

$$\left\| \sum_{\substack{R \in \mathcal{R}' \cap R_0 \cap \text{Length}(R) < \text{Length}(R_0) \atop \text{Length}(R) < \text{Length}(R_0)}} 1_{R \cap R_0} \right\|_n \lesssim \delta^{-2} |R_0|^{1/n} \quad R_0 \in \mathcal{R}' .$$

Fix $R_0$ as in this inequality. $R_0 = I \times J$, in standard coordinates. Length of $R$ is in the first coordinate.

We only consider those $R \in \mathcal{R}'$ that intersect $R_0$ and have a smaller length. Call this collection $\mathcal{R}_0$.

We will need to select a distinguished subset of $\mathcal{R}_0$. 
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines
The Covering Lemma Statement
Selection of $\mathcal{R}'$

Selecting a Distinguished Subset of $\mathcal{R}_0$, Part 1
Selecting a Distinguished Subset of $R_0$, Part 1

$I(R)$ is the projection of all of $R$ onto the top side of $R_0$. $I_0$ is the top side of $R_0$. 
Selecting a Distinguished Subset of $R_0$, Part 1

The rectangle $R$ has density at least $\delta$. 
The rectangle $R$ has density at least $\delta$. Project those $x \in R$ with $\nu(x)$ in the angle of uncertainty of $R$, onto the top side of $R_0$. Call that set $F(R)$. 

**Selecting a Distinguished Subset of $R_0$, Part 1**
The rectangle $R$ has density at least $\delta$. Project those $x \in R$ with $\nu(x)$ in the angle of uncertainty of $R$, onto the top side of $R_0$. Call that set $F(R)$. This set is at least as big as $\delta \text{length}(R)$. 
Select $\mathcal{R}_1 \subset \mathcal{R}_0$ by initializing $\mathcal{R}^{\text{stock}} := \mathcal{R}_0$, $\mathcal{R}_1 := \emptyset$. 
Selecting a Distinguished Subset of $\mathcal{R}_0$, Part 2

1. Select $\mathcal{R}_1 \subset \mathcal{R}_0$ by initializing $\mathcal{R}^{stock} := \mathcal{R}_0$, $\mathcal{R}_1 := \emptyset$.

2. In the inductive stage, take the longest $R \in \mathcal{R}^{stock}$ for which $F(R)$ is disjoint from $\{F(R) : R \in \mathcal{R}_1\}$.
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

The Covering Lemma Statement

Selection of $\mathcal{R}'$

Selecting a Distinguished Subset of $\mathcal{R}_0$,

Part 2

1. Select $\mathcal{R}_1 \subset \mathcal{R}_0$ by initializing $\mathcal{R}^{\text{stock}} := \mathcal{R}_0$, $\mathcal{R}_1 := \emptyset$.

2. In the inductive stage, take the longest $R \in \mathcal{R}^{\text{stock}}$ for which $F(R)$ is disjoint from $\{F(R) : R \in \mathcal{R}_1\}$.

3. Then, set $\mathcal{R}(R)$ to be those $R' \in \mathcal{R}^{\text{stock}}$ with $I(R') \subset 2I(R)$ and $F(R') \cap F(R)$ not empty.
Selecting a Distinguished Subset of $\mathcal{R}_0$, Part 2

1. Select $\mathcal{R}_1 \subset \mathcal{R}_0$ by initializing $\mathcal{R}^{stock} := \mathcal{R}_0, \quad \mathcal{R}_1 := \emptyset$.

2. In the inductive stage, take the longest $R \in \mathcal{R}^{stock}$ for which $F(R)$ is disjoint from $\{F(R) : R \in \mathcal{R}_1\}$.

3. Then, set $\mathcal{R}(R)$ to be those $R' \in \mathcal{R}^{stock}$ with $I(R') \subset 2I(R)$ and $F(R') \cap F(R)$ not empty.

4. Remove this collection from $\mathcal{R}^{stock}$, and repeat until $\mathcal{R}^{stock}$ is exhausted.
The Principal Lemma of the Maximal Function Estimate

For any subinterval $I \subset I_0$, we have the two estimates
For any subinterval $I \subset I_0$, we have the two estimates

$$\sum_{R_1 \in \mathcal{R}_1} |I_{R_1} \times J| \lesssim \delta^{-1}|I \times J|,$$
The Principal Lemma of the Maximal Function Estimate

For any subinterval $I \subset I_0$, we have the two estimates

$$\sum_{R_1 \in \mathcal{R}_1} |I_{R_1} \times J| \lesssim \delta^{-1}|I \times J|,$$

This is essentially immediate from the disjointness of the sets $F(R)$ for $R \in \mathcal{R}_1$. 

This is essentially a BMO estimate, so it implies the higher moments condition we need.
The Principal Lemma of the Maximal Function Estimate

For any subinterval $I \subset I_0$, we have the two estimates

$$
\sum_{\substack{R_1 \in \mathcal{R}_1 \\ \text{if } I_{R_1} \subset I}} |I_{R_1} \times J| \lesssim \delta^{-1} |I \times J|,
$$

$$
\sum_{\substack{R \in \mathcal{R}_1(R_1) \\ R \cap R_0 \subset I \times J}} |R \cap R_0| \lesssim \delta^{-1} |I \times J|,
$$

$R_1 \in \mathcal{R}_1$. This is essentially a BMO estimate, so it implies the higher moments condition we need.
Hilbert Transform on $\mathcal{C}^{1+\epsilon}$ Families of Lines

The Covering Lemma Statement

Selection of $\mathcal{R}'$

**The Principal Lemma of the Maximal Function Estimate**

For any subinterval $I \subset I_0$, we have the two estimates

$$\sum_{\substack{R_1 \in \mathcal{R}_1 \\mid R_1 \subset I}} |I_{R_1} \times J| \lesssim \delta^{-1} |I \times J|,$$

$$\sum_{\substack{R \in \mathcal{R}_1(R_1) \\mid R \cap R_0 \subset I \times J}} |R \cap R_0| \lesssim \delta^{-1} |I \times J|, \quad R_1 \in \mathcal{R}_1.$$

This is essentially a $BMO$ estimate, so it implies the higher moments condition we need.
Suppose that there is an interval \( I \subset I_0 \) and a choice of \( R_1 \in \mathcal{R}_1 \) such that

\[
\sum_{\substack{R \in \mathcal{R}(R_1) \\ \text{length}(R) \geq 4|I|}} |R \cap I \times J| \geq 10^3 \delta^{-1} |I \times J|.
\]
Suppose that there is an interval $I \subset I_0$ and a choice of $R_1 \in \mathcal{R}_1$ such that

$$
\sum_{\substack{R \in \mathcal{R}(R_1) \\
\text{length}(R) \geq 4|I|}} |R \cap I \times J| \geq 10^3 \delta^{-1}|I \times J|.
$$

Then, for either $\epsilon = +1$ or $\epsilon = -1$, there can be no $R' \in \mathcal{R}(R_1)$ with $2\text{length}(R') < |I|$ and $R'$ intersects $\frac{1}{2}(I + \epsilon|I|) \times J$. 

A final inductive/recursive scheme will then prove the Lemma.
The Essential Geometric Observation

Suppose that there is an interval $I \subset I_0$ and a choice of $R_1 \in \mathcal{R}_1$ such that

$$\sum_{R \in \mathcal{R}(R_1)} |R \cap I \times J| \geq 10^3 \delta^{-1} |I \times J|.$$  

Then, for either $\varepsilon = +1$ or $\varepsilon = -1$, there can be no $R' \in \mathcal{R}(R_1)$ with $2\text{length}(R') < |I|$ and $R'$ intersects $\frac{1}{2}(I + \varepsilon |I|) \times J$. A final inductive/recursive scheme will then prove the Lemma.
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

The Covering Lemma Statement

Selection of $\mathcal{R}'$

**The Proof of the Essential Geometric Observation**

$I \times J$

The proof is by contradiction to the construction of $\mathcal{R}'$, and in particular, the use of the maximal function in 100 different directions.
The Proof of the Essential Geometric Observation

Select $I$ so that $I \times J$ is covered more and $10^3 \delta^{-1}$ times by longer intervals.
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

The Covering Lemma Statement

Selection of $\mathcal{R}'$

The Proof of the Essential Geometric Observation

Remember that all the rectangles that cover $I \times J$ have to have angles of uncertainty that are very close.
Hilbert Transform on $C^{1+\epsilon}$ Families of Lines

The Covering Lemma Statement

Selection of $\mathcal{R}'$

The Proof of the Essential Geometric Observation

Consider a rectangle that is rotated by $90^\circ$, has width the same as all other rectangles, and is translated by about $|l|$. 
Consider a rectangle that is rotated by 90°, has width the same as all other rectangles, and is translated by about $|l|$. And height approximately angle $\times |l|$.
Consider a rectangle that is rotated by 90°, has width the same as all other rectangles, and is translated by about $|I|$. And height approximately angle $\times |I|$.

This rectangle is contained in the removed set. A contradiction. This proves the Essential Geometric Observation.