Hyperbolic Knot Theory

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These are notes on a mini-course held at Quy Nhon University in June 2019. Most proofs can be found in Thurston's notes [4], as well as in the books by Benedetti and Petronio [2] and Ratcliffe [3]. Basics of hyperbolic geometry can be found in the book by Anderson [1].

Motivation

A knot $K \subset S^3$ is *hyperbolic* if $S^3 \setminus K$ is a hyperbolic manifold, which means:

- either $S^3 \setminus K$ admits a complete Riemannian metric of constant negative sectional curvature,
- or, equivalently, $S^3 \setminus K = \mathbb{H}^3/\Gamma$, where $\mathbb{H}^3$ is the hyperbolic space and Γ is a torsion-free discrete subgroup of the isometries of $\mathbb{H}^3$.

The role of hyperbolic geometry in knot theory is illustrated by the following two theorems:

Theorem 1 (Thurston). *If a knot $K \subset S^3$ is neither a satellite nor a torus knot, then it is hyperbolic.*

After Thurston's theorem there is definitively a strong reason to study hyperbolic knots. This is a particular case of Thurston's geometrization, that he proved much earlier than the Perelman's proof in full generality. I shall not prove this theorem in the notes, but at least I shall try to explain why torus knots or satellite knots cannot be hyperbolic.

Another relevant theorem is:

Theorem 2 (Mostow-Prasad). *Any two hyperbolic metrics on $S^3 \setminus K$ are isometric.*

Isometric means that there is a bijection that preserves the hyperbolic metric (either as distances or as Riemannian metrics). One of the consequences is that metric invariants of $S^3 \setminus K$, like the volume, are also topological invariants of the knot K. I discuss two of these invariants, volume and Reidemeister torsion, in the last part of these notes.

In general, Mostow-Prasad rigidity holds for hyperbolic manifolds of finite volume and dimension ≥ 3.

Contents

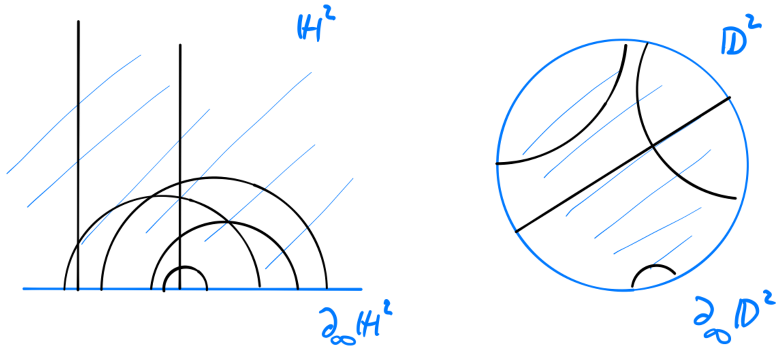
1 Hyperbolic geometry

Let $n \geq 2$. By a theorem of H. Cartan there exists a unique connected, simply-connected, Riemannian manifold of dimension n that is complete and has constant sectional curvature -1. It is called the *hyperbolic n-space*. We shall work with models of the hyperbolic plane (dim 2) and the hyperbolic space (dim 3).

1.1 The hyperbolic plane

We work with two models (see [1, 2, 3, 4] for other models):

	Upper half-plane	Poincaré disc model
	$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im} z > 0\}$	$\mathbb{D}^2 = \{z \in \mathbb{C} \mid \lvert z\rvert < 1\}$
$z = x + y\,i,\;\; x, y \in \mathbb{R}$	$ds^2 = \dfrac{dx^2 + dy^2}{y^2}$	$ds^2 = \dfrac{dx^2 + dy^2}{4(1 - x^2 - y^2)^2}$
Ideal boundary	$\partial_\infty \mathbb{H}^2 = \mathbb{R}^2 \cup \{\infty\}$	$\partial_\infty \mathbb{D}^2 = S^1 = \{z \in \mathbb{C} \mid \lvert z\rvert = 1\}$



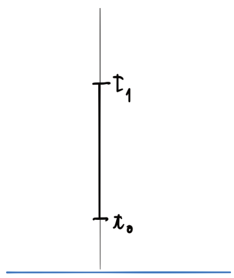
Here are some of the basic properties (see the references above for a proof):

- In both models, geodesics are circles and lines perpendicular to the ideal boundary.
- Both models are conformal: only angles are well represented. Distances are not well represented: the distance to the ideal boundary is infinite.
- Adding the ideal boundary is a compactification: $\mathbb{H}^2$ is not compact, but $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ has a natural topology so that it is homeomorphic to a closed disc (this is more clear for the disc model).
- There is a conformal transformation of $\mathbb{C} \cup \{\infty\}$ that maps $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$ to $\mathbb{D}^2 \cup \partial_\infty \mathbb{D}^2$:
$$z \mapsto \frac{z - i}{-i\, z + 1}$$
It carries the metric from one model to the other, in particular it carries the geodesics, the distances, etc...
- Two points in $\partial_\infty \mathbb{H}^2$ are "joined" by a unique geodesic (in fact they are the "ideal end-points" or limits of a unique geodesic)

The notation may be confusing, because $\mathbb{D}^2$ denotes only the disc model but $\mathbb{H}^2$ denotes both, the hyperbolic plane and the upper-half plane model.

Let us compute a pair of lengths of segments. Firstly, in the upper half plane $\mathbb{H}^2$ consider the segment $\{x = 0,\ t_0 \le y \le t_1\}$. It can be described as

$$\begin{array}{rcl} [t_0, t_1] & \to & \mathbb{H}^2 \\ t & \mapsto & (0, t) = t\, i \end{array}$$



Its length is

$$\text{length} = \int_{t_0}^{t_1} \frac{1}{t} dt = |\log(t_1/t_0)|$$

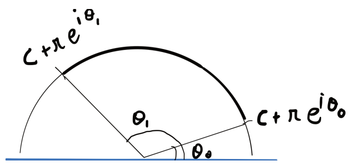
In particular the distance to each ideal end-point of the geodesic is infinite, and

$$s \mapsto e^s i$$

is an arc-parameter (namely $d(e^s i, e^t i) = |s - t|$).

Secondly, consider an arc of half-circle perpendicular to $\partial_\infty \mathbb{H}^3$, with angles between θ_0 and θ_1. Its description is

$$\begin{array}{rcl} [\theta_0, \theta_1] & \to & \mathbb{H}^2 \\ \theta & \mapsto & (c + r\cos\theta, r\sin\theta) = c + r\, e^{i\theta} \end{array}$$



Its length is

$$\text{length} = \int_{\theta_0}^{\theta_1} \frac{1}{\sin t} dt = \left| \log \left(\frac{\tan(\theta_1/2)}{\tan(\theta_0/2)} \right) \right|$$

Again the distance to each ideal end-point of the geodesic is infinite, and

$$s \mapsto c + r\, e^{i2 \arctan e^s} = c + r \frac{\sinh(s) + i}{\cosh(s)}$$

is an arc-parameter.

The group of orientation preserving isometries of the hyperbolic plane is denoted by

$$\mathrm{Isom}^+(\mathbb{H}^2) = \{\text{orientation preserving isometries of } \mathbb{H}^2\}$$

In each model the group of orientation-preserving isometries can be represented by a matrix group:

$$\mathrm{Isom}^+(\mathbb{H}^2) \cong \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm \mathrm{Id}\}$$
$$\mathrm{Isom}^+(\mathbb{D}^2) \cong \mathrm{PSU}(2) = \mathrm{SU}(2)/\{\pm \mathrm{Id}\}$$

A matrix $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on the models $\mathbb{H}^2$ and $\mathbb{D}^2$ by the rational transformation

$$z \mapsto \frac{az + b}{cz + d}.$$

The action is the same, but the matrices differ from one model to the other.

The conformal transformation $z \mapsto \frac{z-i}{-i\,z+1}$ that maps the half-plane $\mathbb{H}^2$ to the half-disc $\mathbb{D}^2$ has matrix:

$$\pm \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

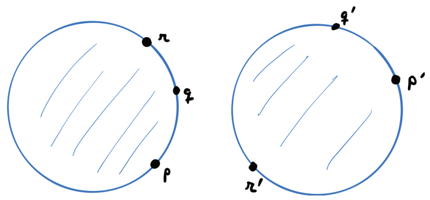
Its conjugation maps matrices in $\mathrm{PSL}_2(\mathbb{R})$ to matrices in $\mathrm{PSU}(2)$.

Proposition 3. *Orientation preserving isometries of $\mathbb{H}^2$ extend continuously to the ideal boundary $\partial_\infty \mathbb{H}^2$*

This proposition follows from the description of isometries as *restriction* of rational transformations of $\mathbb{C} \cup \{\infty\}$. Preserving the orientation is not needed, but in these notes I do not look at orientation reversing isometries.

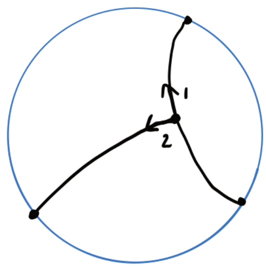
Proposition 4. $\mathrm{Isom}^+(\mathbb{H}^2)$ *acts simply transitively on the space of cyclically ordered triples of pairwise different points in $\partial_\infty \mathbb{H}^2 \cong S^1$.*

In other words: given $p, q, r \in \partial_\infty \mathbb{H}^2$ pairwise different and $p', q', r' \in \partial_\infty \mathbb{H}^2$ also pairwise different, there exists a unique isometry $\gamma \in \mathrm{Isom}^+(\mathbb{H}^2)$ such that $\gamma(p) = p'$, $\gamma(q) = q'$ and $\gamma(r) = r'$, provided that p, q and r induce the same cyclic order as p', q' and r' on $\partial_\infty \mathbb{H}^2 \cong S^1$.



If we do not require the isometries to be orientation preserving, then we do not need the points to have the same cyclic ordering.

The proposition can be proved either just from explicit computation with matrices, or using the fact that three ideal points determine an oriented orthonormal frame, and the fact that $\mathrm{Isom}^+(\mathbb{H}^2)$ acts simply transitively on the space of orthonormal oriented frames.



Proposition 5. *Let* $\gamma \in \mathrm{Isom}^+(\mathbb{H}^2)$*. If* $\gamma \neq \mathrm{Id}$*, then:*

- γ *either has a fixed point in* $\mathbb{H}^2$*, or*
- γ *has exactly 2 fixed points in* $\partial_\infty \mathbb{H}^2$*, or*
- γ *has exactly 1 fixed point in* $\partial_\infty \mathbb{H}^2$*.*

Proof. By Browder fixed point theorem, γ has at least one fixed point in $\mathbb{H}^2 \cup \partial_\infty \mathbb{H}^2$, which is homeomorphic to a compact disc. If γ has 3 fixed points in $\partial_\infty \mathbb{H}^2$, then $\gamma = \mathrm{Id}$. $\square$

According to this proposition, there are three types of non-trivial isometries $\gamma \in \mathrm{Isom}^+(\mathbb{H}^2)$:

- γ *elliptic* when γ has a fixed point in $\mathbb{H}^2$. Then

$$\gamma \sim \pm \begin{pmatrix} \cos\frac{\alpha}{2} & \sin\frac{\alpha}{2} \\ -\sin\frac{\alpha}{2} & \cos\frac{\alpha}{2} \end{pmatrix} \qquad \mathrm{trace}(\gamma) \in (-2, 2)$$

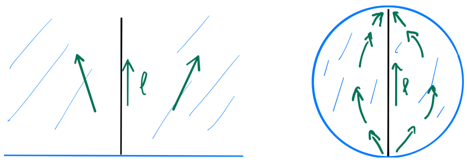
(here $\sim$ denotes matrix conjugation: $A \sim B$ means that $A = C^{-1}BC$ for some $C \in \mathrm{PSL}_2(\mathbb{R})$)

γ is a rotation of angle α.

- γ *hyperbolic* (also called *loxodromic*) when γ has two fixed points in $\partial_\infty \mathbb{H}^2$. Then

$$\gamma \sim \pm \begin{pmatrix} e^{\frac{l}{2}} & 0 \\ 0 & e^{-\frac{l}{2}} \end{pmatrix} \qquad \text{trace}(\gamma) \in \mathbb{R} \setminus [-2, 2]$$

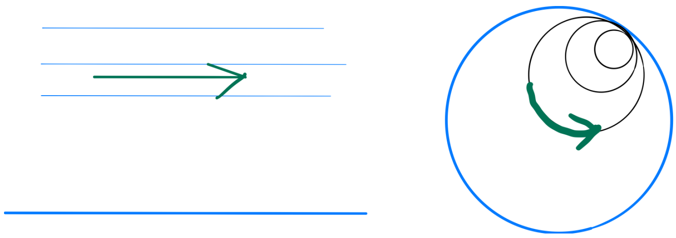
It preserves a geodesic (its ideal end-points are the points fixed by γ). In this geodesic, the displacement function $x \mapsto d(\gamma(x), x)$ reaches its minimum, that is positive (it is l).



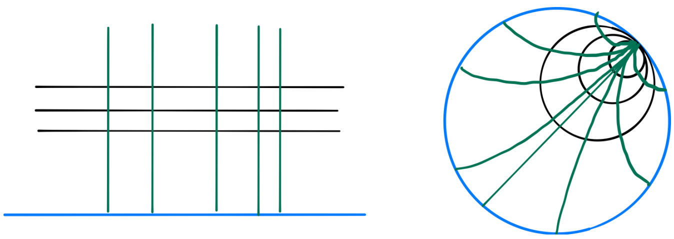
- γ *parabolic* when γ has a unique fixed point in $\partial_\infty \mathbb{H}^2$. Then

$$\gamma \sim \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{trace}(\gamma) = \pm 2$$

It preserves the *horocycles* centered at the ideal point. The displacement function $x \mapsto d(\gamma(x), x)$ has no minimum, and the infimum is zero.



Definition 6. A *horocycle* centered at an ideal point $p \in \partial_\infty \mathbb{H}^2$ is a bi-infinite curve orthogonal to all geodesics asymptotic to p.

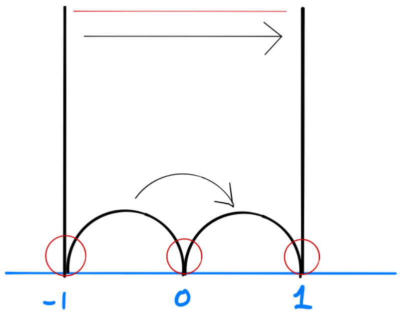


Horocycles centered at $p \in \partial_\infty \mathbb{H}^2$ are limits of metric circles $\{x \in \mathbb{H}^2 \mid d(x, c_n) = r_n$ of center c_n and radius r_n, with $c_n \to p$ and radius $r_n = d(c_n, x_0)$ for some given $x_0 \in \mathbb{H}^2$.

In the upper half-plane model, *horocycles* centered at ∞ are represented by horizontal lines. In the disc model, or for points in $\mathbb{R}$ for the upper half-plane model, *horocycles* are circles or lines tangent to that point.

1.2 Examples of hyperbolic surfaces

Consider the ideal quadrilateral Q with side identifications $z \mapsto z+2$ and $z \mapsto \frac{z}{2z+1}$:



The matrices of these transformations are:

$$\pm\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \qquad \pm\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}$$

$$\begin{array}{cc} \infty \mapsto \infty & 0 \mapsto 0 \\ -1 \mapsto 1 & -1 \mapsto 1 \end{array}$$

The area of a region $R \subset \mathbb{H}^2$ is computed as $\int_R \frac{dx \wedge dy}{y^2}$. The quadrilateral Q in the figure has area 2π.

The group generated by these isometries

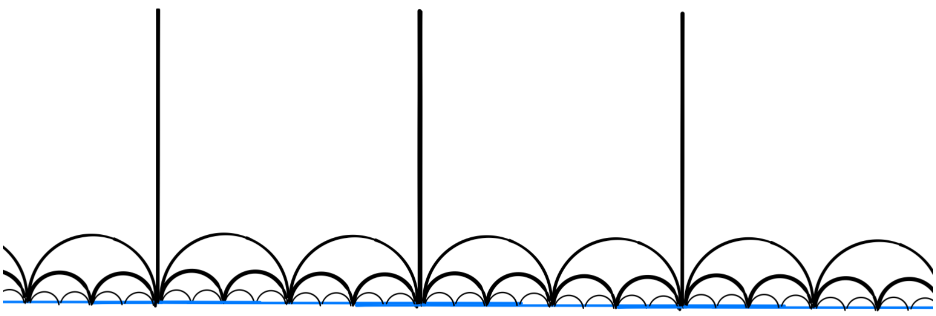
$$\Gamma = \langle \pm\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}, \pm\begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}\rangle$$

is a subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ therefore it is discrete (discrete means that it has no accumulation points in $\mathrm{Isom}^+(\mathbb{H}^2) = \mathrm{PSL}_2(\mathbb{R})$).

The quadrilateral Q is a *fundamental domain*:

- every point in $\mathbb{H}^2$ can be written as γx for some $x \in Q$ and some $\gamma \in \Gamma$, and
- for every $x \neq y \in \mathring{Q}$, $\gamma(x) \neq y$ for all $\gamma \in \Gamma$. (Here $\mathring{Q}$ means the interior of Q).

Being a fundamental domain means also that the images of Q define a *tessellation* of $\mathbb{H}^2$:

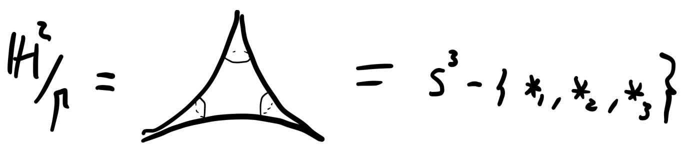


That it is a fundamental domain can be checked by direct computation, or using a theorem of Poincaré. This theorem of Poincaré requires:

(a) that the generators of the group correspond to side pairings, and

(b) that horocycles centered at ideal points of Q are preserved.

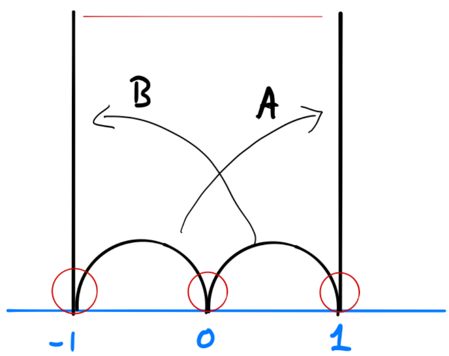
Point (b) holds true for the horocycles centered at ∞ because trace $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = 2$. Item (b) also holds for the horocycles centered at 0 because trace $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = 2$. For the other horocycles, as -1 is identified to 1 via those isometries, it follows from the fact that $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -2 & 1 \end{pmatrix}$ has trace -2. The conclusion of Poincaré theorem is that Q is a fundamental domain and that $\mathbb{H}^2/\Gamma$ is the result of gluing the sides of Q by the side pairing. Thus $\mathbb{H}^2/\Gamma$ is a three times punctured sphere:



Let us change the side parings:

$$A = \pm \begin{pmatrix} a + 1/a & a \\ a & a \end{pmatrix} \qquad B = \pm \begin{pmatrix} a + 1/a & -a \\ -a & a \end{pmatrix}$$

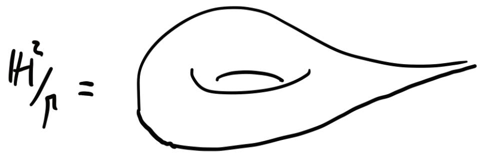
$$\begin{array}{cc} -1 \mapsto \infty & 1 \mapsto \infty \\ 0 \mapsto 1 & 0 \mapsto -1 \end{array}$$



Discreteness and that the quadrilateral is a fundamental domain requires Poincaré theorem. Here the four ideal points are in the same orbit of the group, and the stabilizer of ∞ is generated by

$$ABA^{-1}B^{-1} = -\begin{pmatrix} 1 & 2(2 + \frac{1}{a^2}) \\ 0 & 1 \end{pmatrix}$$

which is parabolic. Hence, by Poincaré theorem Q is a fundamental domain, and $\mathbb{H}^2/\Gamma$ here is a punctured torus:



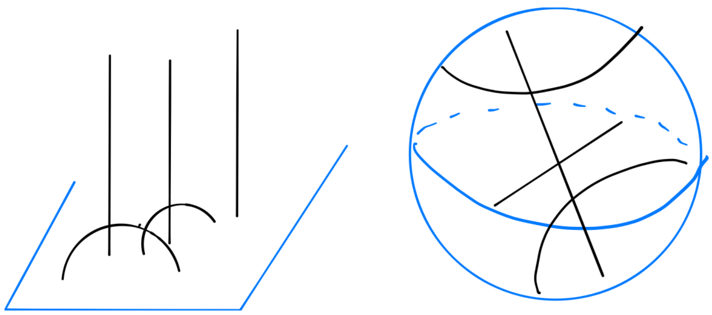
The hyperbolic structure is not unique, as we have a parameter $a \in \mathbb{R} \setminus \{0\}$ (we could also deform one vertex of the quadrilateral to have a further parameter).

One can use Poincaré's Theorem to construct compact hyperbolic surfaces. For this purpose, one must consider isometric side pairings on compact polygons, with the condition that the angles at equivalent vertices add to 2π. The existence of such polygons is not difficult to prove with elementary hyperbolic geometry, however the explicit matrices in the compact case are more complicated to give. In this way, one can construct hyperbolic structures on all surfaces with negative Euler characteristic. The reader can find examples in the references given specially in the book by Anderson [1] that focuses in dimension 2.

1.3 Hyperbolic space

We use again two models:

Upper half-space	Poincaré ball model
$\mathbb{H}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid t > 0\}$	$\mathbb{B}^3 = \{(x, y, t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 < 1\}$
$ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{t^2}$	$ds^2 = \dfrac{dx^2 + dy^2 + dt^2}{4(1 - x^2 - y^2 - t^2)^2}$
$\partial_\infty \mathbb{H}^3 = \mathbb{R}^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\}$	$\partial_\infty \mathbb{B}^3 = S^2 = \{(x, y, t) \in \mathbb{R}^3 \mid x^2 + y^2 + t^2 = 1\}$
$z = x + y\,i,\ x, y \in \mathbb{R}$	



Again, geodesics are lines and circles perpendicular to the ideal boundary, and both models are conformal.

Any two points in the ideal boundary $\partial_\infty \mathbb{H}^3$ are joined by a unique geodesic.

The isometry group of the upper half-space model is:

$$\mathrm{Isom}^+(\mathbb{H}^3) = \mathrm{Conf}^+(\partial_\infty \mathbb{H}^3) = \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm \mathrm{Id}\}.$$

A matrix $\pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on $\partial_\infty \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$ by the rational transformation

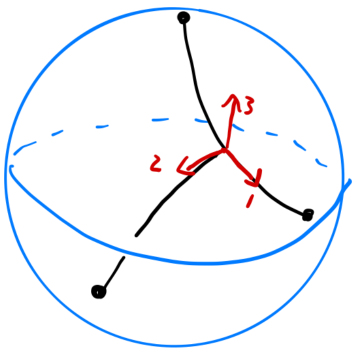
$$z \mapsto \frac{az + b}{cz + d}$$

The action of an isometry on $\mathbb{H}^3$ can be understood from the action on the ideal boundary $\partial_\infty \mathbb{H}^3$ by conformal extension. To describe explicitly the action on $\mathbb{H}^3$ one may use quaternions.

Proposition 7. $\mathrm{Isom}^+(\mathbb{H}^3)$ *acts simply transitively on the space of triples of pairwise different points in* $\partial_\infty \mathbb{H}^3 \cong S^2$.

If we do not require the isometries to be orientation preserving, then there are precisely two isometries tan map a triple of different point to another, one orientation preserving, the other orientation reversing-

As for the plane, the proposition can be proved either just from explicit computation with matrices, or using the fact that three ideal points determine an oriented orthonormal frame, and the fact that $\mathrm{Isom}^+(\mathbb{H}^3)$ acts simply transitively on the space of oriented orthonormal frames.



Thus, by the same argument as for the hyperbolic plane we have:

Proposition 8. *For any orientation preserving isometry* $\gamma \in \mathrm{Isom}^+(\mathbb{H}^3)$ *other than the identity:*

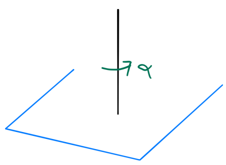
- *either* γ *has a fixed point in* $\mathbb{H}^3$, *or*
- γ *has exactly 2 fixed points in* $\partial_\infty \mathbb{H}^3$ *(and no fixed point in* $\mathbb{H}^3$*), or*
- γ *has exactly 1 fixed point in* $\partial_\infty \mathbb{H}^3$.

Again, there are three kinds of non-trivial isometries $\gamma \in \mathrm{Isom}^+(\mathbb{H}^3)$:

- γ is *elliptic* when γ has a fixed point in $\mathbb{H}^2$. Then

$$\gamma \sim \pm \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 \\ 0 & e^{-i\frac{\alpha}{2}} \end{pmatrix} \qquad \mathrm{trace}(\gamma) \in (-2, 2)$$

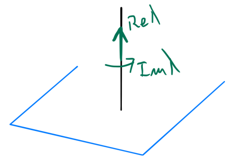
It is a rotation of angle α around an axis of fixed points.



- γ *hyperbolic* (also called *loxodromic*) when γ has two fixed points in $\partial_\infty \mathbb{H}^2$. Then

$$\gamma \sim \pm \begin{pmatrix} e^{\frac{\lambda}{2}} & 0 \\ 0 & e^{-\frac{\lambda}{2}} \end{pmatrix} \qquad \text{trace}(\gamma) \in \mathbb{C} \setminus [-2, 2]$$

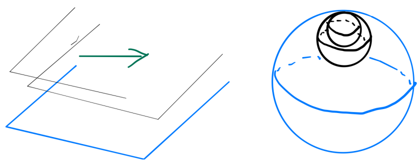
  it preserves a geodesic (its ideal end-points are the points fixed by γ). In this geodesic, the displacement function $x \mapsto d(\gamma(x), x)$ reaches its minimum. γ acts as a translation along this geodesic of displacement $\text{Re}(\lambda)$ composed with a rotation of angle $\text{Im}(\lambda)$ around its axis.



- γ *parabolic* when γ has a unique fixed point in $\partial_\infty \mathbb{H}^2$:

$$\gamma \sim \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{trace}(\gamma) = \pm 2.$$

  It preserves the *horosphere* centered at the ideal point (circles or lines tangent to that point). The displacement function $x \mapsto d(\gamma(x), x)$ has no minimum.



We next discuss subgroups of isometries.

A subgroup $\Gamma < \text{Isom}^+(\mathbb{H}^3)$ is *discrete* if it has the discrete topology, namely if it has no accumulation points in $\text{Isom}^+(\mathbb{H}^3)$.

The action of a group Γ on a topological space X is proper if, for each compact subset $K \subset X$, the set $\{\gamma \in \Gamma \mid \gamma K \cap K \neq \emptyset\}$ is finite. The next two lemmas are easy results of topology, see [3] for a proof:

Lemma 9. *$\Gamma < \text{Isom}^+(\mathbb{H}^3)$ is discrete iff Γ acts properly discontinuously on $\mathbb{H}^3$.*

We say that Γ *acts freely* if every $\gamma \in \Gamma$ with $\gamma \neq \text{Id}$ acts on $\mathbb{H}^3$ without fixed points.

Lemma 10. *When $\Gamma < \text{Isom}^+(\mathbb{H}^3)$ is discrete and acts freely, then the quotient $\mathbb{H}^3/\Gamma$ is a manifold and the fundamental group of $\mathbb{H}^3/\Gamma$ is isomorphic to Γ: $\pi_1(\mathbb{H}^3/\Gamma) \cong \Gamma$.*

An element $\gamma \in \Gamma$ has finite order or is *a torsion element* if $\gamma^n = \text{id}$ for some natural $n \geq 2$. Γ is *torsion-free* if it has no torsion elements.

Proposition 11. *$\Gamma < \text{PSL}_2(\mathbb{C})$ discrete. Then:*

(a) Γ *torsion-free iff* Γ *has no elliptics iff* Γ *acts freely on* $\mathbb{H}^3$.

(b) If Γ *is abelian and torsion-free, then it is either a group of hyperbolic isometries along a geodesic (and* $\Gamma \cong \mathbb{Z}$*) or it is a group of parabolic isometries that fix an ideal point (and* $\Gamma \cong \mathbb{Z}$ *or* $\Gamma \cong \mathbb{Z} \oplus \mathbb{Z}$*).*

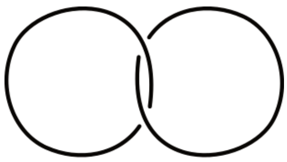
Proof. (a) is a consequence of the classification of isometries, the fact that parabolic and hyperbolic isometries have infinite order, and the fact that elliptic elements of a discrete groups are torsion elements.

For (b) notice that if $\gamma_1\gamma_2 = \gamma_2\gamma_1$, then $\gamma_1(\mathrm{Fix}(\gamma_2)) = \mathrm{Fix}(\gamma_1)$, hence either both γ_1 and γ_2 are hyperbolic with the same axis, or parabolic with the same fixed ideal point. □

Definition 12. M^3 is hyperbolic if $M^3 = \mathbb{H}^3/\Gamma$ with Γ discrete and torsion-free.

Corollary 13. *Torus knots are not hyperbolic.*

To prove the corollary, notice that for a torus knot $K \subset S^3$, the fundamental group $\pi_1(S^3 \setminus K)$ has a nontrivial element that commutes with every other element in $\pi_1(S^3 \setminus K)$ (ie, it hs nontrivial center). By the discussion on abelian subgroups of a torsion-free discrete group of isometries in Proposition 11, if $S^3 \setminus K = \mathbb{H}^3/\Gamma$ with $\Gamma \cong \pi_1(S^3 \setminus K)$ discrete and torsion-free, then either (a) $\Gamma \cong \mathbb{Z}$ is a group of hyperbolic isometries with a given axis, or that (b) $\Gamma \cong \mathbb{Z}$ or $\mathbb{Z}^2$ is a group of parabolic isometries with a common fixed ideal point. When $\Gamma \cong \mathbb{Z}$ then $\mathbb{H}^3/\Gamma \cong S^1 \times \mathbb{R}^2$ and K would be the trivial knot, and when $\Gamma \cong \mathbb{Z}^2$, then $\mathbb{H}^3/\Gamma \cong S^1 \times S^1 \times \mathbb{R}$ and it is not a knot exterior because it has two ends (in fact it is the exterior of the Hopf link, see the picture).

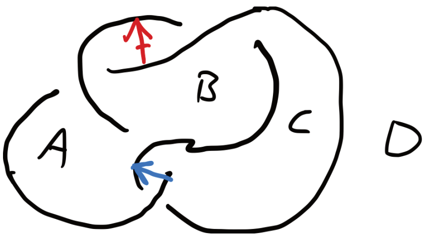


Remark 14. We have shown that the trivial knot and the Hopf link are the only links that are torus links and also hyperbolic. But their hyperbolic structure has infinite volume and we show below that all other hyperbolic knots and links have finite hyperbolic volume. In general, the trivial knot and the Hopf link are not considered as hyperbolic links. They are also special cases among torus knots or links, as they are too simple.

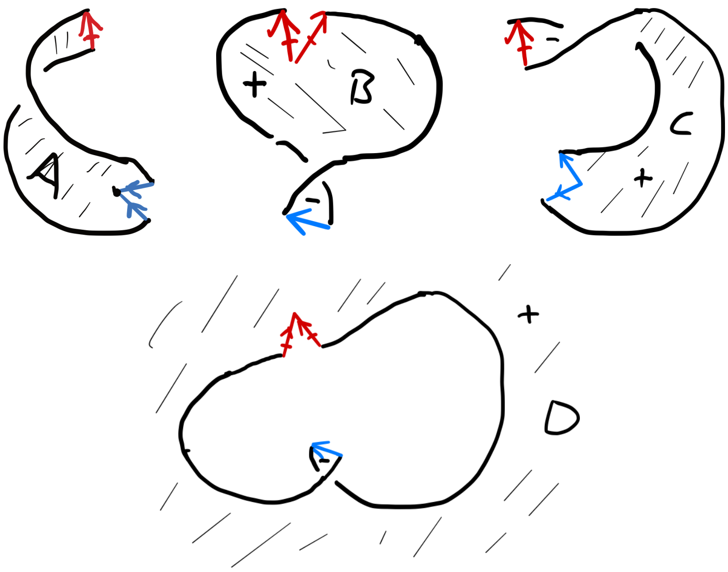
1.4 Examples of hyperbolic knots and links

The figure eight knot. Following Thurston [4], we view $S^3\setminus$(figure eight knot) as the union of two ideal tetrahedra.

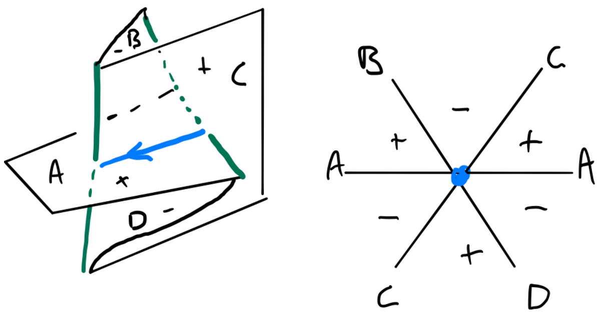
We cut $S^3\setminus$(figure eight knot) open along the surfaces A, B, C and D:



Here we represent the cells A, B, C and D, with a sign $+$ or $-$ according to the face:

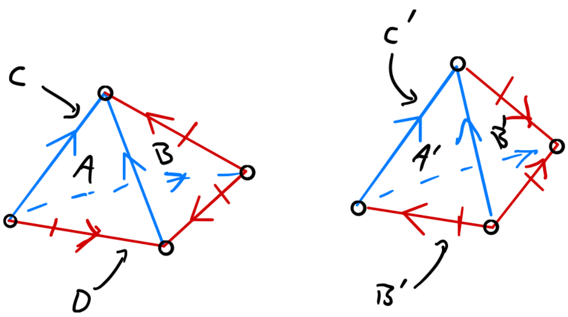


We also represent how the cells meet at the blue arrow, the part of the knot is represented in green (the right hand picture is a cross section of this intersection)

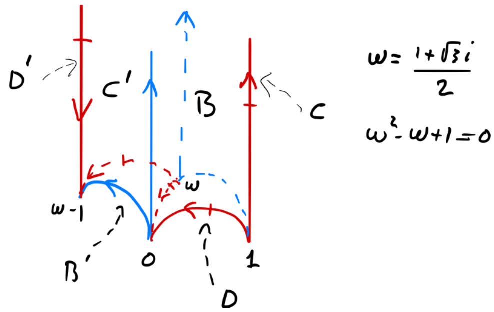


After removing the knot and cutting open along the arrows, the cells A, B, C and D are ideal triangles (a triangle without vertices). The first time the visualization requires some effort, and it is an example of extraordinary vision of Thurston.

We then obtain two ideal tetrahedra (two tetrahedra without vertices):



We glue them along A/A' and put the ideal vertices in the ideal boundary $\partial_\infty \mathbb{H}^3 = \mathbb{C} \cup \{\infty\}$: 0, 1, ∞, $\omega = \frac{1+\sqrt{3}i}{2}$ and $\omega - 1 = \omega^2 = \frac{-1+\sqrt{3}i}{2}$.



The side pairings are

$$\begin{array}{ccc} b & c & d \\ B \to B' & C \to C' & D \to D' \\ & & \\ \infty \mapsto \omega - 1 & \infty \mapsto \omega - 1 & 0 \mapsto \omega - 1 \\ 1 \mapsto \omega & \omega \mapsto 0 & \omega \mapsto \omega \\ 0 \mapsto 0 & 1 \mapsto \infty & 1 \mapsto \infty \\ & & \\ \pm\begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} & \pm\begin{pmatrix} -1 & \omega \\ \omega & -\omega \end{pmatrix} & \pm\begin{pmatrix} 2-\omega & -1 \\ -\omega & \omega \end{pmatrix} \end{array}$$

$$\Gamma = \langle \pm\begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix}, \pm\begin{pmatrix} -1 & \omega \\ \omega & -\omega \end{pmatrix}, \pm\begin{pmatrix} 2-\omega & -1 \\ -\omega & \omega \end{pmatrix} \rangle \subset \mathrm{PSL}_2(\mathbb{Z}[\omega])$$

where $\mathbb{Z}[\omega] = \mathbb{Z} \oplus \mathbb{Z}\omega$. This immediately implies that Γ is discrete.

We can apply a 3-dimensional version of Poincaré theorem. One of the conditions for Poincaré theorem is that, after side identifications, the dihedral angles at edges add to 2π. Notice that in the two tetrahedra together there are 6 red edges and 6 blue edges, and each dihedral angle is $2\pi/6$. The other condition is that horospheres must be preserved. There is only one equivalence class of ideal vertices, and it can be checked that we need to look at the following elements:

$$b = \pm\begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \qquad \text{and} \qquad db^{-1}d^{-1}b^2d^{-1}b^{-1}d = \begin{pmatrix} -1 & 0 \\ -4+2\omega & -1 \end{pmatrix}$$

that are parabolic. So Γ is discrete and torsion free, and $\mathbb{H}^3/\Gamma$ is the result of these side identifications (eg the $S^3 \setminus$ (the figure eight knot)). In addition, one can obtain a presentation for the group. Following the side pairings along the blue edge induces an algebraic relation:

$$dc^{-1}b^{-1}c = 1$$

and the red edge:

$$b^{-1}dbd^{-1}c = 1.$$

These relations are equivalent to

$$c = db^{-1}d^{-1}b, \qquad cd = bc.$$

Thus we get the presentation

$$\langle b, d \mid db^{-1}d^{-1}bd = bdb^{-1}d^{-1}b \rangle.$$

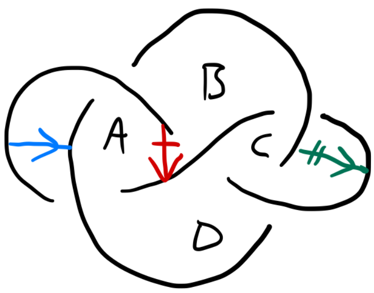
It follows from this that the parabolic elements

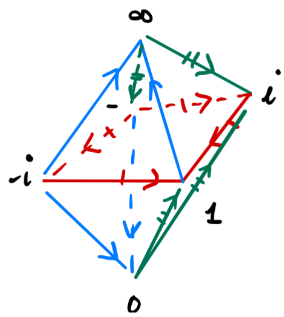
$$b = \pm \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} \quad \text{and} \quad db^{-1}d^{-1}b^2d^{-1}b^{-1}d = \begin{pmatrix} -1 & 0 \\ -4+2\omega & -1 \end{pmatrix}$$

are a meridian and a longitude.

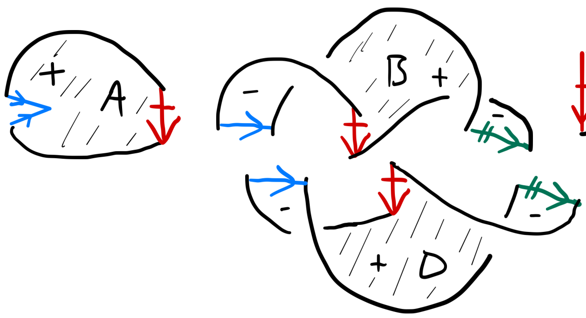
The Whitehead link.

Following again Thurston's notes [4], we view $S^3 \setminus$ (Whitehead link) as the result of side pairing of an ideal octahedron, and we put its vertices at the ideal boundary:

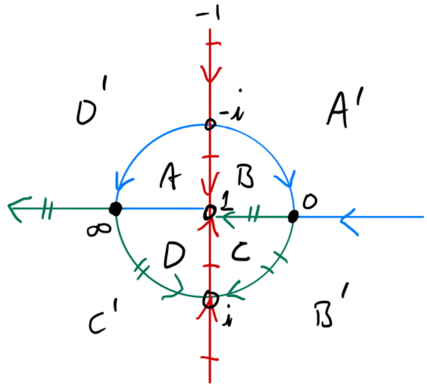




We represent also the cells:



Notice that there is no unbounded cell, so the complement is connected (an octahedron). The side-parings are represented as:



The the side pairings are the isometries:

$A \to A'$	$B \to B'$	$C \to C'$	$D \to D'$
$0 \mapsto \infty$	$i \mapsto 1$	$\infty \mapsto 0$	$\infty \mapsto \infty$
$-i \mapsto 1$	$0 \mapsto 0$	$i \mapsto 1$	$-i \mapsto 1$
$-1 \mapsto -i$	$-1 \mapsto -1$	$-1 \mapsto i$	$-1 \mapsto i$
$\pm\begin{pmatrix} i & 1 \\ i & 0 \end{pmatrix}$	$\pm\begin{pmatrix} 1 & 0 \\ i+1 & 1 \end{pmatrix}$	$\pm\begin{pmatrix} 0 & i \\ i & 1+i \end{pmatrix}$	$\pm\begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix}$

The group generated by these four isometries is a subgroup of $\mathrm{SL}_2(\mathbb{Z}[i])$ and therefore it is discrete. Here Poincaré theorem also applies, and concludes that $S^3 \setminus$ (Whitehead link) is hyperbolic.

2 Properties of hyperbolic manifolds

2.1 Thin-thick decomposition

Let M^3 be a hyperbolic and orientable three-manifold. For $x \in M^3$ we define the (open) metric ball of radius $r > 0$ as

$$B(x, r) = \{y \in M^3 \mid d(x, y) < r\}.$$

We also define:

$$\begin{aligned} \text{inj}(x) &= \sup\{r \mid B(x,r) \text{ is isometric to a ball in } \mathbb{H}^3\} \\ &= \inf\{\frac{1}{2}\text{length}(\sigma) \mid \sigma \text{ geodesic loop at } x\}. \end{aligned}$$

A geodesic loop σ is a continuous path that starts and finishes at x and it is locally geodesic. In addition we define:

- The ε-*thin* part is $M^{[0,\varepsilon)} = \{x \in M \mid \text{inj}(x) < \varepsilon\}$.
- The ε-*thick* part is $M^{[\varepsilon,\infty)} = \{x \in M \mid \text{inj}(x) \geq \varepsilon\}$.

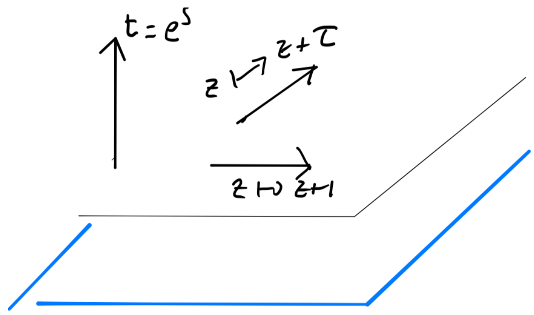
A *cusp* is the quotient of a horoball H by a discrete group of parabolic isometries that preserve the center of the horoball and is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. To describe a cusp, in the upper half space we may assume that the horoball is centered at ∞, then

$$H = \{(x, y, t) \in \mathbb{H}^3 \mid t > c\}$$

for some constant $c > 0$. Then, writing $z = x + iy$, the cusp is

$$H/\langle z \mapsto z + 1, z \mapsto z + \tau\rangle = H/\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \rangle$$

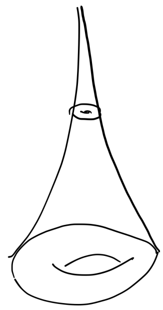
for some $\tau \in \mathbb{C} \setminus \mathbb{R}$.



Every horosphere t =constant is a plane, and projects to the cusp as a 2-torus, as it is divided by the action of $\langle z \mapsto z + 1, z \mapsto z + 2\rangle$. These torus shrink as t grows. More precisely, writing $t = e^s$, the metric becomes

$$\frac{dx^2 + dy^2 + dt^2}{t^2} = e^{-2s}(dx^2 + dy^2) + ds^2$$

Here s can be understood as the length parameter of a geodesic that goes into the cusp, and the torus defined by s = *constant* becomes exponentially small with s.



In particular:

- the injectivity radius in a cusp becomes arbitrarily small, and
- the volume of a cusp is finite.

Theorem 15 (Margulis)**.** *There exists a universal constant $\mu_3 > 0$ such that, for every $0 < \varepsilon < \mu_3$ and for every M^3 hyperbolic, orientable, and of finite volume, the ε-thin part $M^{(0,\varepsilon)}$ is the disjoint union of the following (finitely many) regions:*

- *cusps,*
- *tubular neighborhoods of geodesics of length $< \varepsilon/2$.*

Finiteness of the number of components of the thin part follows from the fact that each such component contributes with a definitive amount of volume. The proof relies on Lie groups and the characterization of nilpotent and abelian subgroups of Γ.

Remark 16. The constant μ_3 is called the Margulis constant.

In every dimension $n \geq 2$ there exists a Margulis constant $\mu_n > 0$.

If we do not assume that the volume is finite, there are other kinds of neighborhoods (rank one cusps).

Corollary 17. *Given M^3 hyperbolic with $\mathrm{vol}(M^3) < \infty$, there exists $\delta = \delta(M^3) > 0$ such that $M^{[0,\delta)}$ is the disjoint union of cusps.*

For the corollary, take $\delta > 0$ less than the Margulis constant and also less than 1/2 the length of the shortest closed geodesic of M (the shortest geodesic exists by Theorem 15).

Proposition 18. *Let $T^2 \subset M^3$ be a torus in a hyperbolic 3-manifold such that the inclusion induces an injection of fundamental groups $\pi_1(T^2) \to \pi_1(M^3)$. Then, after isotopy T^2 bounds a cusp.*

This proposition follows from the classification of discrete abelian subgroups of hyperbolic isometries (Proposition 11) and from some results in topology of three manifolds. A consequence is:

Corollary 19. *A hyperbolic knot is not a satellite.*

2.2 Mostow-Prasad rigidity

Definition 20. Two continuous maps $f\colon M \to N$ are *homotopic* if there is a continuous map $H\colon M \times [0,1] \to N$ such that, writing $H(x,t) = H_t(x)$, $H_0 = f$ and $H_1 = g$.

When f and g are homotopic, write $f \sim g$.

H as above is called a *homotopy* between f and g.

When H_t is a homeomorphism for each $t \in [0,1]$, then H is called an *isotopy*.

A continuity map $f\colon M \to N$ is called a *homotopy equivalence* when there exists $g\colon N \to M$ such that $g \circ f \sim \mathrm{Id}_M$ and $f \circ g \sim \mathrm{Id}_N$.

Theorem 21 (Mostow-Prasad)**.** *Let M^3 and N^3 be two hyperbolic 3-manifolds of finite volume.*

Every homotopy equivalence between M^3 and N^3 is homotopic to a unique isometry.

An isometry is a bijection that preserves distances (or equivalently, that preserves the Riemannian metrics), therefore it is a homeomorphism.

Corollary 22. *If $f\colon \mathbb{H}^3/\Gamma_1 \to \mathbb{H}^3/\Gamma_2$ is a homotopy equivalence of finite volume hyperbolic manifolds, then there exists $\eta \in \mathrm{Isom}(\mathbb{H}^3)$ such that $\Gamma_2 = \eta\Gamma_1\eta^{-1}$.*

The conclusion of the corollary is that Γ_1 and Γ_2 are conjugate by an isometry of $\mathbb{H}^3$ that may be non-orientable. Notice that η is not unique, as for any $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$, if η satisfies the conclusion of the corollary, the so does $\gamma_2\eta\gamma_1$, but this is the unique indeterminacy we encounter.

Proof. By the theorem, f is homotopic to a unique isometry $h\colon \mathbb{H}^3/\Gamma_1 \to \mathbb{H}^3/\Gamma_2$ that lifts to the universal covering

$$\begin{array}{ccc} \mathbb{H}^3 & \xrightarrow{\widetilde{h}} & \mathbb{H}^3 \\ \downarrow & & \downarrow \\ \mathbb{H}^3/\Gamma_1 & \xrightarrow{h} & \mathbb{H}^3/\Gamma_2 \end{array}$$

Then $\eta = \widetilde{h}$ satisfies the conclusion of the corollary. □

From basic results on algebraic topology, we have that a group isomorphism is realized by a homotopy equivalence. Therefore:

Corollary 23. *Let $\Gamma_1, \Gamma_2 < \mathrm{Isom}(\mathbb{H}^3)$ be discrete and with finite covolume. Any group isomorphism $\Gamma_1 \cong \Gamma_2$ is realized by conjugation of some isometry $\eta \in \mathrm{Isom}(\mathbb{H}^3)$.*

Namely, if $g\colon \Gamma_1 \to \Gamma_2$ is a group isomorphism, then there exists $\eta \in \mathrm{Isom}^+(\mathbb{H}^3)$ such that $g(\gamma) = \eta\gamma\eta^{-1}$ for every $\gamma \in \Gamma_1$.

Theorem 21 is due to Mostow in the compact case, and the generalization to finite volume, to Prasad.

It is false for $n = 2$ or when the volume is infinite. It holds true for $n > 2$ and finite volume.

See Thurston's notes [4, Chapter 5] for a proof.

Corollary 24. *Metric invariants are topological invariants (even homotopic invariants).*

2.3 Hyperbolic Dehn surgery

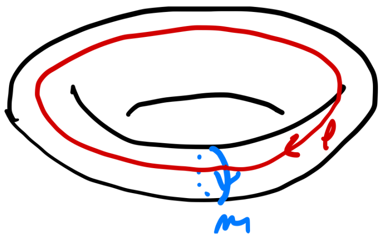
Dehn surgery on a knot $K \subset S^3$ consists in removing from S^3 an open tubular neighborhood of the knot $\mathring{N}(K)$, homeomorphic to an open solid torus $\mathring{N}(K) \cong \mathring{D}^2 \times S^1$, and glue again a (closed) solid torus using a homeomorphism of boundaries $h\colon \partial(D^2 \times S^1) \to \partial(S^3 \setminus \mathring{N}(K))$:

$$S^3 \setminus \mathring{N}(K) \cup_h D^2 \times S^1 = S^3 \setminus \mathring{N}(K) \sqcup D^2 \times S^1 / x \sim h(x),$$

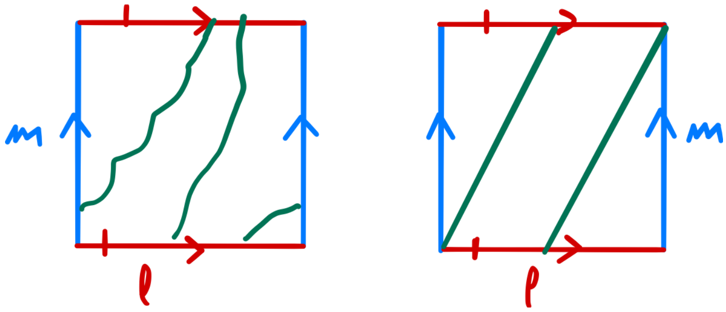
$\forall x \in \partial(D^2 \times S^1)$.

Lemma 25. *The homeomorphism type of the Dehn surgery is parameterized by the slope* $h(\partial D^2 \times \{*\})$.

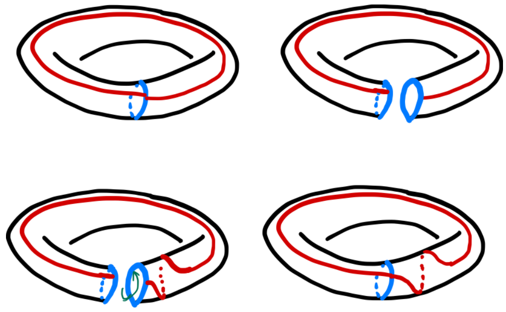
We describe what a *slope* means. In the torus $\partial D^2 \times S^1$ we fix a meridian $m = \partial D^2$ and a longitude l.



If we cut open along m and l, every simple closed curve in the torus $\partial D^2 \times S^1$ can be represented by a line of constant slope $p/q \in \mathbb{Q} \cup \{\infty\}$, rational or infinity. This curve is homotopic to $pm + ql$, with $p, q \in \mathbb{Z}$ coprime.



The *idea of the proof* of the lemma is as follows. For two homeomorphisms $h, h'\colon \partial(D^2 \times S^1) \to \partial(S^3 \setminus \mathring{N}(K))$ such that $h(D^2 \times \{*\})$ is homotopic to $h'(D^2 \times \{*\})$, we consider $\phi = h^{-1}h'$, which is a homeomorphism of $\partial(D^2 \times S^1)$, with the property that it maps the meridian $\partial D^2 \times \{*\}$ to a curve homotopic to itself. By standard arguments of surfaces (due to Nielsen), ϕ can be isotoped to be the identity on $\partial D^2 \times \{*\}$, and by standard arguments of low dimensional topology, it extends to a homeomorphism of the solid torus $D^2 \times S^1$. Such a homeomorphism is a composition of the so called Dehn twists. The following picture is intended to represent a Dehn twist (its inverse is also called a Dehn twist):



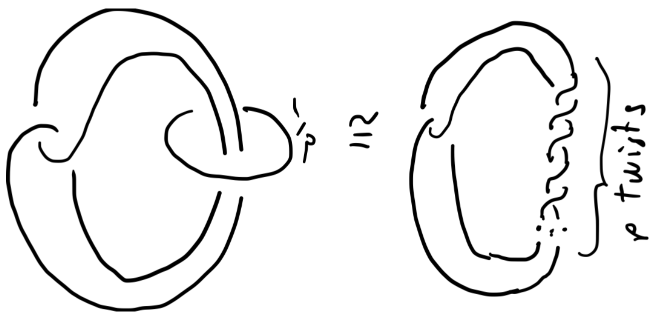
Remark 26. The result of Dehn surgery with slope $1/0 = \infty$ is S^3 itself: $K_\infty = S^3$.

Theorem 27 (Thurston)**.** *Let K be a hyperbolic knot, then for almost every slope $p/q \in \mathbb{Q} \cup \{\infty\}$, $K_{p/q}$ is hyperbolic*

Remark 28. The theorem is true not only for knots but for manifolds with cusps. When there are several cusps, the surgery/filling can also be partial, in some of the cusps.

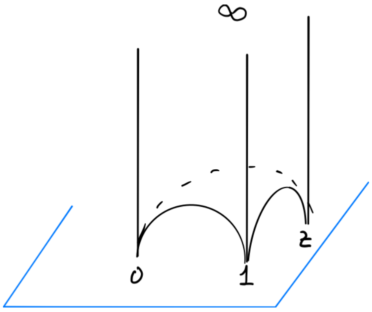
Example 29. For the figure eight knot, $K_{p/q}$ is hyperbolic except for $p/q \in \{\infty, 0, \pm 1, \pm 2, \pm 3, \pm 4\}$. It is the hyperbolic knot with the largest amount of non-hyperbolic surgeries.

Example 30. On the Whitehead link, the result of $1/p$-surgery on one of the components is again a knot exterior, a twist link. It is hyperbolic except for finitely many $p \in \mathbb{Z}$

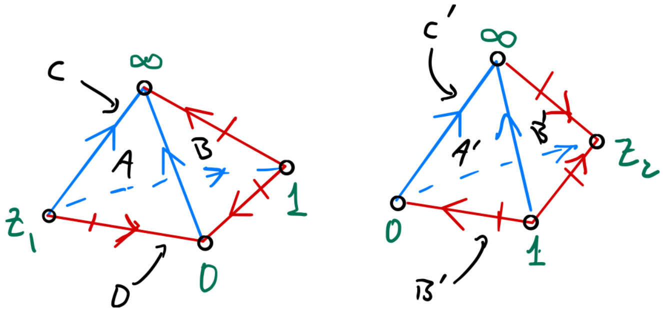


Instead of giving a proof of the Dehn filling theorem, we work explicitly an example, the figure eight knot. This example was first explained in Thurston's notes [4], see also [3] for this example, or [2] for a proof in general.

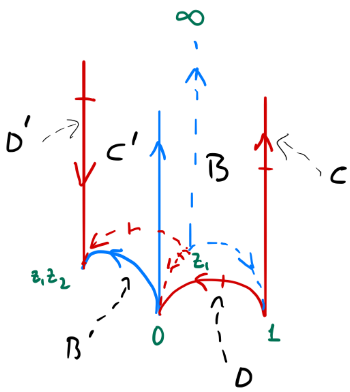
First we need a remark on the hyperbolic structure of ideal tetrahedra in $\mathbb{H}^3$. Given any three different points in the ideal boundary $\partial_\infty \mathbb{H}^3$, there is a unique orientation preserving isometry that maps the three points to 0, 1 and ∞. Thus an ideal tetrahedron in $\mathbb{H}^3$ can be placed with three ideal vertices 0, 1, and ∞. The fourth ideal vertex is a point $z \in \mathbb{C} \setminus \{0, 1\}$, hence this gives a parameter space of deformations of complex dimension one:



For the figure eight knot complement, we have two ideal tetrahedra:



There are two parameters z_1 and z_2 for these tetrathedra. Now we glue the faces A and A', the isometry that maps A to A' acts on the ideal boundary $\partial_\infty \mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$ as multiplication by the parameter z_1 (as it maps $\infty \mapsto \infty$, $0 \mapsto 0$, and $1 \mapsto z_1$):



Then we get the union of the tetrahedra located in $\mathbb{H}^3$. The side pairings are given by the isometries:

b	c	d
$B \to B'$	$C \to C'$	$D \to D'$
$\infty \mapsto z_1 z_2$	$\infty \mapsto z_1 z_2$	$0 \mapsto z_1 z_2$
$1 \mapsto z_1$	$z_1 \mapsto 0$	$z_1 \mapsto z_1$
$0 \mapsto 0$	$1 \mapsto \infty$	$1 \mapsto \infty$

We just compute b and d, as we know $c = db^{-1}d^{-1}b$:

$$b = \sqrt{\frac{z_2 - 1}{z_1 z_2}} \begin{pmatrix} \frac{z_1 z_2}{z_2 - 1} & 0 \\ \frac{1}{z_2 - 1} & 1 \end{pmatrix} \qquad d = \frac{1}{\sqrt{(z_1 - 1)(z_2 - 1)}} \begin{pmatrix} 1 - z_1 - z_2 & z_1 z_2 \\ -1 & 1 \end{pmatrix}$$

When we require that c maps C to C', or the relation $cd = dc$, we get

$$-z_2 z_1^2 + z_2^2 - z_1 z_2 - 2z_2 + 1 = 0.$$

This yields a deformation space of complex dimension one. We know from Mostow-Prasad rigidity that only one point corresponds to the complete structure. The other points yield a metrically non-complete space (ie we may have Cauchy sequences that do not converge). The metric completion (the result of adding the limits of Cauchy sequences) in general is not a hyperbolic manifold, but for countable many points the metric completion consist in adding a simple closed curve, and the result is a manifold, the Dehn surgery $K_{p/q}$.

To understand the deformation, we need to look at the cusp, at the meridian and longitude:

$$m = b^{-1} = \sqrt{\frac{z_1 z_2}{z_2 - 1}} \begin{pmatrix} \dfrac{z_2 - 1}{z_1 z_2} & 0 \\ -\dfrac{1}{z_1 z_2} & 1 \end{pmatrix},$$

$$l = db^{-1}d^{-1}bbd^{-1}b^{-1}d = \begin{pmatrix} \dfrac{(z_2 - 1)^2}{z_2} & 0 \\ -\dfrac{(z_1 z_2 + z_2 - 1)(z_1 z_2 - z_1 - z_2)}{(z_2 - 1)^2 z_1 z_2} & \dfrac{z_2}{(z_2 - 1)^2} \end{pmatrix}$$

If we want l and m to be parabolic, then $z_1 = z_2 = \dfrac{1 \pm \sqrt{3}i}{2}$, which correspond to the complete structure (complex conjugation corresponds to a change of orientation).

To understand the metric completion, following Thurston's notes, write

$$m = \begin{pmatrix} e^{u/2} & 0 \\ * & e^{-u/2} \end{pmatrix} \qquad \text{and} \qquad l = -\begin{pmatrix} e^{v/2} & 0 \\ * & e^{-v/2} \end{pmatrix}$$

Thus we take logarithms

$$u = \log(\frac{z_2 - 1}{z_1 z_2})$$
$$v = 2 \log \frac{-(z_2 - 1)^2}{z_2}$$

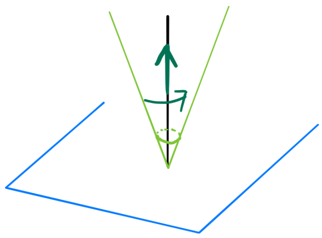
Take resolutions of logarithms so that when $z_1 = z_2 = \dfrac{1 \pm \sqrt{3}i}{2}$, $u = v = 0$. Then

$$v = 2\sqrt{3}i\, u + O(u^3)$$

and write the equation

$$pu + qv = 2\pi i$$

When $(p, q) \in \mathbb{Z}^2$ coprime is a solution of this equation, we view the metric completion as a Dehn surgery with slope p/q.



This figure represents a tubular neighborhood of the geodesic invariant by m and l. On the geodesic, m acts as a translation of length $\mathrm{Re}(u)$ plus a rotation of angle $\mathrm{Im}(u)$, and the same for l and v. Then when we write $pu + qv = 2\pi i$, the curve $pm + ql$ (with slope p/q) acts as a "rotation of angle 2π". By continuity, we view that the images of tetrahedra cover of a neighborhood of the invariant geodesic in the picture, and the image of $pm + ql$ gives precisely one turn. For $r, z \in \mathbb{Z}$ satisfying $ps - qr = 1$ (so that the slopes p/q and r/s generate the peripheral torus) one can check that the real part of $ru + sv$ is $\mathrm{Re}(ru + sv) = \mathrm{Re}(v)/p = -\mathrm{Re}(u)/q \neq 0$, hence it has non-zero displacement along the geodesic.

In addition, the equations

$$\begin{cases} pu + qv = 2\pi i \\ v = 2\sqrt{3}i\, u + O(u^3) \end{cases}$$

define a homeomorphism

$$\begin{array}{rcl} \text{nbhd of } 0 \text{ in } \mathbb{C} & \to & \text{nbhd of } \infty \text{ in } \mathbb{R}^2 \cup \{\infty\} \\ u & \mapsto & (p, q) \end{array}$$

This reaches all but finitely many slopes $p/q \in \mathbb{Q} \cup \{\infty\}$.

This process has been implemented in software, that deals with hyperbolic 3-manifolds. Snap Pea was written by Jeff Weeks:

http://www.geometrygames.org/SnapPea/index.html

Using the kernel of Snap Pea, Marc Culler, Nathan Dunfield, and Matthias Goerner wrote SnapPy, that can be found in:

https://snappy.math.uic.edu/

3 Invariants of hyperbolic manifolds

By Mostow-Prasad rigidity, metric invariants of the hyperbolic manifolds $S^3 \setminus K$ are also topological invariants of $S^3 \setminus K$. Here we describe the volume and Reidemeister torsion.

3.1 The volume

Cusps have finite volume, so for a knot $K \subset S^3$, by Proposition 18,

$$\mathrm{vol}(S^3 \setminus K) < \infty.$$

For the figure eight knot, the volume is twice the volume of the regular ideal tetrahedron, it is $\approx 2.0298832\ldots$

The regular ideal tetrahedron maximizes the volume among all ideal tetrahedra, hence for any Dehn surgery $K_{p/q}$ on the figure eight knot K we get:

$$\mathrm{vol}(K_{p/q}) < \mathrm{vol}(S^3 \setminus K).$$

In addition, using the computations for the figure eight knot in the previous section, by the approximation argument, we have that when $p^2 + q^2 \to \infty$, the parameter $u \to 0$, $z_2, z_2 \to \dfrac{1+\sqrt{3}i}{2}$, and the ideal tetrathedra converge to the regular ones. Hence

$$\lim_{p^2+q^2} \mathrm{vol}(K_{p/q}) = \mathrm{vol}(S^3 \setminus K).$$

Neumann and Zagier have proved that, for any hyperbolic knot K:

$$\mathrm{vol}(K_{p/q}) = \mathrm{vol}(S^3 \setminus K) - \frac{c_1}{p^2 + q^2 c_2} + O(\frac{1}{p^4 + q^4})$$

for some constants $c_1, c_2 > 0$ that depend on the knot. In fact we have the same behavior for every cusped manifold, but only for $p^2 + q^2$ large. For general (p, q) and for any cusped manifold, the bound $\mathrm{vol}(K_{p/q}) < \mathrm{vol}(S^3 \setminus K)$. is due to Gromov, using the so called simplicial volume.

This behavior of the volume under Dehn surgery/filling with the following theorem of Jørgensen gives a picture of the set of volumes.

Theorem 31 (Jørgensen)**.** *Given a constant $C > 0$, the set of (homeomorphism classes of) thick parts of manifolds with volume $< C$ is finite.*

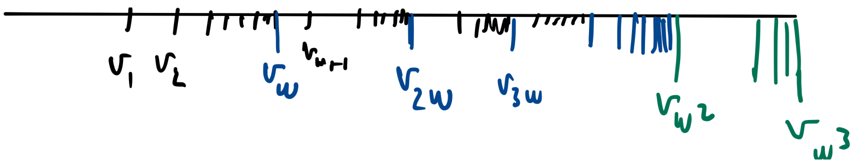
Jørgensen theorem follows from compactness theorems “à la Cheeger” on spaces of Riemannian manifolds with (a) diameter bounded above, (b) injectivity radius bounded below and (c) bounded curvature. However Jørgensen proved it before those theorems where available.

Corollary 32. *Given $C > 0$, all manifolds with volume $< C$ are obtained by Dehn filling on finitely many cusped manifolds.*

Corollary 33. *The set of volumes is well ordered (every nonempty subset has a minimum).*

In addition, it has cardinality ω^ω.

Cardinality ω^ω means the following. There is a minimal volume v_1, a second minimal volume v_2, and in this way a sequence v_n =the n-th smallest volume. The sequence v_n grows and converges to v_w, which is the smallest volume of a cusped manifold. After v_ω there is a next volume $v_{\omega+1}$, followed by $v_{\omega+2}$, and therefore a sequence $v_{\omega+n}$ that accumulates to $v_{2\omega}$, the second smallest volume of manifolds with one cusp. This yields a sequence $v_{n\omega}$ that accumulates to v_{ω^2}, the smallest volume of manifolds with two cusps. This process is iterated, so every volume is codified by a polynomial on ω with coefficients in $\mathbb{N} \cup \{0\}$.



There is one manifold of minimal volume, the Matveev-Fomenko-Weeks manifold, it is the result of $(5/2, 5)$-surgery on the Whitehead link. It has volume $v_1 \approx 0.94270736\ldots$.

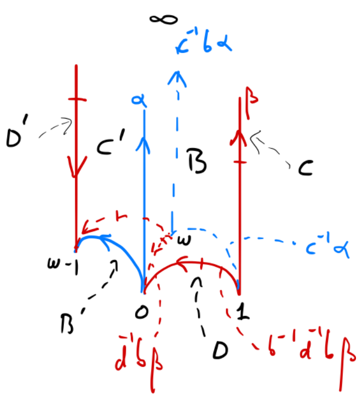
There are two manifolds that minimize the volume among cusped manifolds (and have volume $v_\omega \approx 2.0298832\ldots$): the figure eight knot complement and the $5/2$-surgery in one component of the Whitehead link (it is called also the figure eight knot sibling, as it is obtained from two regular ideal tetrahedra by different side identifications).

The volume $v_{\omega^2} \approx 3.663862\ldots$ is realized by two manifolds with two cusps, that are the exterior of two links, the Whitehead link and the Pretzel link $(2, 3, 8)$.

Remark 34. Hyperbolic volume equals the so called simplicial volume, introduced by Gromov and using only a topological definition. In addition it generalizes to any 3-manifold (in particular to any knot complement). See [4, 2, 3].

3.2 Reidemeister torsion

We do not give the definition of Reidemeister torsion, we just compute it for our favorite example, the figure eight knot complement. We start with the side pairing of two regular ideal tetrahedra, and we consider the corresponding cell structure.



We start with E^3 a 3-cell, union of the two ideal tetrahedra, as in the figure. The 2-cells are B, C, D, $B' = b\,B$, $C' = c\,C$, and $D' = d\,D$. Call the 1-cells α and β as in the picture, so that every other 1-cell is the image by the group of one of them. As the tetrahedra are ideal, there are no 0-cells.

The boundary map of the three cell is:

$$\partial_3 E^3 = (1-b)B + (1-c)C + (1-d)D$$

The boundary of the 2-cells is:

$$\begin{aligned} \partial_2 B &= -\alpha + (1 - b^{-1}d^{-1}b)\beta \\ \partial_2 C &= (c^{-1}b - c^{-1})\alpha + -\beta \\ \partial_2 D &= c^{-1}\alpha + (b^{-1}d^{-1}b - d^{-1}b)\beta \end{aligned}$$

Recall the side identifications

$$\begin{array}{ccc} b & c & d \\ B \to B' & C \to C' & D \to D' \\ & & \\ \infty \mapsto \omega - 1 & \infty \mapsto \omega - 1 & 0 \mapsto \omega - 1 \\ 1 \mapsto \omega & \omega \mapsto 0 & \omega \mapsto \omega \\ 0 \mapsto 0 & 1 \mapsto \infty & 1 \mapsto \infty \\ & & \\ \begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} & -\begin{pmatrix} -1 & \omega \\ \omega & -\omega \end{pmatrix} & \begin{pmatrix} 2-\omega & -1 \\ -\omega & \omega \end{pmatrix} \end{array}$$

We have chosen signs so that the relations $cd = bc$ and $c = db^{-1}d^{-1}b$ hold. In other words, we have lifted the group $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ to $\mathrm{SL}_2(\mathbb{C})$.

Next we "twist the chain complex by the representation". In the previous boundary operators, we replace b, c and d by a representation. Before, we need to replace each element by its inverse (because we are constructing a chain complex by a right action):

$$\partial_3 \to \begin{pmatrix} 1-b^{-1} \\ 1-c^{-1} \\ 1-d^{-1} \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ \omega & 1 \\ \omega & \omega \\ \omega & 1 \\ \omega & 1 \\ \omega & 2-\omega \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -\omega & 0 \\ -\omega+1 & -\omega \\ -\omega & 0 \\ -\omega+1 & -1 \\ -\omega & \omega-1 \end{pmatrix} = d_3$$

and

$$\partial_2 \to \begin{pmatrix} -1 & b^{-1}c - c & c \\ 1 - b^{-1}db & -1 & b^{-1}db - b^{-1}d \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 & 0 & 1 & -\omega \\ 0 & -1 & \omega & 1-\omega & -\omega & \omega \\ -1 & 1 & -1 & 0 & \omega & 0 \\ -1 & 1 & 0 & -1 & 0 & 0 \end{pmatrix} = d_2$$

Of course $d_2 d_3 = 0$. Reidemeister torsion consist in taking determinants of complementary minors. We divide the determinant of the result of removing the third and fourth column to d_2, by the determinant of the matrix formed by third and fourth rows of d_3:

$$\frac{\begin{vmatrix} -1 & 0 & 1 & -\omega \\ 0 & -1 & -\omega & \omega \\ -1 & 1 & \omega & 0 \\ -1 & 1 & 0 & 0 \end{vmatrix}}{\begin{vmatrix} -\omega+1 & -\omega \\ -\omega & 0 \end{vmatrix}} = -2$$

This is a topological invariant of K, that does not depend on the choices made, but it depends on the lift of $\Gamma < \mathrm{PSL}_2(\mathbb{C})$ to $\mathrm{SL}_2(\mathbb{C})$. There is only another choice of sign (that consists in changing the change of the meridian). If we change the sign of b and d (but not of c), we get: 6. There are no further choices of sign, so we get the numbers

$$\{-2, 6\}.$$

This can be codified with a polynomial: if we multiply the matrices of b and d by a variable t (but not c because it is in the kernel of the abelianization map):

$$\begin{array}{ccc} b & c & d \\ t\begin{pmatrix} 1 & 0 \\ -\omega & 1 \end{pmatrix} & -\begin{pmatrix} -1 & \omega \\ \omega & -\omega \end{pmatrix} & t\begin{pmatrix} 2-\omega & -1 \\ -\omega & \omega \end{pmatrix} \end{array}$$

then we obtain:

$$\Delta_K(t) = t^2 - 4t + 1.$$

This is well defined only up to multiplication by a factor t^{2n}, $n \in \mathbb{Z}$. Notice that this polynomial recovers the previous results:

$$\Delta_K(1) = -2 \qquad \text{and} \qquad \Delta_K(-1) = 6.$$

This polynomial is in fact a *twisted Alexander polynomial* (twisted by the representation of Γ in $\mathrm{SL}_2(\mathbb{C})$).

One can consider other representations of Γ. For instance $\mathrm{PSL}_2(\mathbb{C})$ acts by conjugation on the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \cong \mathbb{C}^3$, which yields a representation of Γ in $\mathrm{SL}_3(\mathbb{C})$. For this representation, the twisted polynomial of the figure eight knot is

$$(t-1)(t^2 - 5t + 1).$$

One can consider further representations of $\mathrm{SL}_2(\mathbb{C})$ and obtain further polynomials. There are a lot of results on those polynomials that we do not describe here. Just mention that, for the trivial representation (that maps every element in Γ to 1), Milnor proved that the torsion polynomial of a knot K is

$$\mathrm{A}_K(t)/(t-1),$$

where A_K is the Alexander polynomial of K. Notice that for the trivial representation the knot does not need to be hyperbolic.

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