THE WEIERSTRASS APPROXIMATION THEOREM

There is a lovely proof of the Weierstrass approximation theorem by S. Bernstein. We shall show that any function, continuous on the closed interval \([0, 1]\) can be uniformly approximated by polynomials. We start with the building blocks, the Bernstein polynomials which are given by the expressions

\[
B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, \quad k = 0, 1, \ldots, n.
\]

As always

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!}.
\]

Note that \(B_{n,k}(x) \geq 0\) on \([0, 1]\). E.g.,

\[
B_{1,0} = (1-x), \quad B_{1,1} = x,
\]

\[
B_{2,0} = (1-x)^2, \quad B_{2,1} = 2x(1-x), \quad B_{2,2} = x^2.
\]

Here are some simple identities.

**Lemma 0.1.** We have the following formulas

\[
\sum_{k=0}^{n} B_{n,k}(x) = 1,
\]

\[
\sum_{k=0}^{n} kB_{n,k}(x) = nx,
\]

and

\[
\sum_{k=0}^{n} k(k-1)B_{n,k}(x) = n(n-1)x^2.
\]

**Proof.** The first formula is just the binomial formula which yields \((x + 1 - x)^n = 1\). The second follows again by the binomial formula by writing

\[
\sum_{k=0}^{n} kB_{n,k}(x) = nx \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!((n-1) - (k-1))!} x^{k-1}(1-x)^{(n-1)-(k-1)}
\]

and shifting the index \(k - 1 \to k\). Likewise

\[
\sum_{k=0}^{n} k(k-1)B_{n,k}(x) = \sum_{k=0}^{n} n(n-1)x^2 \frac{(n-2)!}{(k-2)!((n-2) - (k-2))!} x^{k-2}(1-x)^{(n-2)-(k-2)}
\]

and now shift \(k - 2 \to k\). The above arguments only work provided that \(n \geq 2\). For \(n = 1, 2\) they are trivial to verify. \(\square\)

There is a probabilistic flair to the above formulas that can be expressed in the next lemma.
Lemma 0.2. We have that
\[
\sum_{k=0}^{n} k B_{n,k}(x) = x,
\]
and
\[
\sum_{k=0}^{n} \left( \frac{k}{n} - x \right)^2 B_{n,k}(x) = \frac{x(1-x)}{n}.
\]

If we think of the \( B_{n,k}(x) \) as probabilities assigned to the ‘events’ \( k/n \), the first statement says that the expectation value of \( k/n \) is \( x \) and the second formula says that the variance is \( x(1-x)/n \) which is small for \( n \) large. The proof is easy and follows from the previous lemma by simple manipulations. Now we are ready to state

Theorem 0.3. Let \( f : [0,1] \to \mathbb{R} \) be a continuous function. Define the polynomial
\[
B_n(f)(x) := \sum_{k=0}^{n} f \left( \frac{k}{n} \right) B_{n,k}(x).
\]

Then for any \( \varepsilon > 0 \) there exists \( N \) such that for all \( n > N \) and all \( x \in [0,1] \),
\[
|B_n(f)(x) - f(x)| < \varepsilon.
\]

Note that the polynomial is a Riemann sum but with weights \( B_{n,k}(x) \).

Proof. Since \( f \) is continuous on the closed interval, it is uniformly continuous, i.e., for every \( \varepsilon > 0 \) the exists \( \delta > 0 \) which depends only on \( \varepsilon \) such that
\[
|f(x) - f(y)| < \frac{\varepsilon}{2}
\]
whenever \( |x - y| < \delta \). We may write, because of Lemma 0.1
\[
B_n(f)(x) - f(x) = \sum_{k=0}^{n} [f \left( \frac{k}{n} \right) - f(x)] B_{n,k}(x).
\]

Thus,
\[
|B_n(f)(x) - f(x)| \leq \sum_{|\frac{k}{n} - x| < \delta} |f \left( \frac{k}{n} \right) - f(x)| B_{n,k}(x) + \sum_{|\frac{k}{n} - x| \geq \delta} |f \left( \frac{k}{n} \right) - f(x)| B_{n,k}(x).
\]

Note that we have sued the fact that the Bernstein polynomials are non-negative. Hence, we may estimate the right side by
\[
|B_n(f)(x) - f(x)| \leq \frac{\varepsilon}{2} \sum_{|\frac{k}{n} - x| < \delta} B_{n,k}(x) + 2 \max |f(x)| \sum_{|\frac{k}{n} - x| \geq \delta} B_{n,k}(x)
\]

In the second term we write
\[
\sum_{|\frac{k}{n} - x| \geq \delta} B_{n,k}(x) = \sum_{|\frac{k}{n} - x| \geq \delta} \left( \frac{k}{n} - x \right)^2 B_{n,k}(x) \leq \frac{1}{\delta^2} \sum_{|\frac{k}{n} - x| \geq \delta} \left( \frac{k}{n} - x \right)^2 B_{n,k}(x) \leq \frac{x(1-x)}{n\delta^2}
\]

Thus, we have
\[
|B_n(f)(x) - f(x)| \leq \frac{\varepsilon}{2} + 2 \max |f(x)| \frac{x(1-x)}{n\delta^2} \leq \frac{\varepsilon}{2} + \frac{\max |f(x)|}{2n\delta^2}.
\]
and choosing 

\[ n > \frac{\max |f(x)|}{\varepsilon \delta^2} \]

yields the theorem. \qed