HOMEWORK 4, DUE THURSDAY FEBRUARY 6

Problem 1, (5 points): Please do Problem 2.2.39 in Heil.
Solution: Fix $\alpha < 1$ and suppose that there is no cube with $|E \cap Q|_e \geq \alpha |Q|$. Hence for every cube we have that $|E \cap Q|_e < \alpha |Q|$. For any $\varepsilon > 0$ there exists a countable collection of cubes Q_k such that $E \subset \cup_k Q_k$ and such that

$$\sum_k |Q_k| \leq |E|_e + \varepsilon \ .$$

This follows from the definition of the exterior measure. We may assume that $E \cap Q_k \neq \emptyset$ because otherwise we could drop it from the sum while preserving the above inequality. By assumption

$$\alpha \sum_k |Q_k| > \sum_k |E \cap Q_k|_e$$

and by subadditivity

$$\sum_k |E \cap Q_k|_e \geq |E|_e \ .$$

Hence

$$\alpha(|E|_e + \varepsilon) \geq |E|_e$$

which, since ε can be chosen as small as we like and $\alpha < 1$ is a contradiction.

Problem 2, (5 points): Please work Problem 2.3.17 in Heil.
Solution: Consider the sets $E \setminus A_n$. Since $|E| < \infty$ and $|A_n| \to |E|$ as $n \to \infty$ we find

$$|E \setminus A_n| \to 0$$

as $n \to \infty$. Hence there exists a subsequence n_k such that $\sum_k |E \setminus A_{n_k}| < |E|$. Now,

$$\cup_k (E \setminus A_{n_k}) = E \setminus (\cap_k A_{n_k}) \ .$$

and hence

$$|E \setminus (\cap_k A_{n_k})| + |\cap_k A_{n_k}| = |E| \ .$$

Therefore,

$$|\cap_k A_{n_k}| = |E| - |\cup_k (E \setminus A_{n_k})| \geq |E| - \sum_k |E \setminus A_{n_k}| > 0 \ ,$$

using subadditivity.

Problem 3, (7 points): Please work Problem 2.3.18 in Heil.
Solution: If E is measurable then

$$|Q| = |Q \cap E|_e + |Q \setminus E|_e$$

using Caratheodory's criterion. Conversely, suppose the above equation holds for every box Q. Let A be an arbitrary set. By subadditivity of the exterior measure

$$|A|_e \leq |A \cap E|_e + |A \setminus E|_e \ .$$

Pick $\varepsilon > 0$. There exists a countable collection of boxes such that $A \subset \cup_k Q_k$ and

$$|A|_e \ge \sum_k |Q_k| - \varepsilon = \sum_k |Q_k \setminus E| + |Q_k \cap E| - \varepsilon \ .$$

and by countable subadditivity and monotonicity

$$|A \setminus E| \le |(\cup_k Q_k) \setminus E| \le \sum_k |Q_k \setminus E| \ ,$$

$$|A \cap E| \le \sum_k |Q_k \setminus E|$$

so that

$$|A|_e \ge |A \setminus E| + |A \cap E| - \varepsilon$$

which proves the claim because ε is arbitrary.

Problem 4, (3 points): Please work Problem 2.3.19 in Heil.

Solution: The function $f(t) = |E \cap B_t|$ is increasing by monotonicity. Pick any increasing sequence t_k that converges to t. Then $E \cap B_{t_k}$ is and increasing sequence of nested sets and

$$\lim_{k \to \infty} |E \cap B_{t_k}| = | \cup_k E \cap B_{t_k}| = |E \cap B_t| \ .$$

So $f(t)$ is continuous from the left. If t_k is any sequence decreasing towards t then

$$\lim_{k \to \infty} f(t_k) = | \cap_k E \cap B_{t_k}| = f(t) \ .$$

Thus f is continuous on $(0, \infty)$. c) follows by taking $t_k \to \infty$ and b) follows from $f(t) \le B_t| \to 0$ as $t \to 0$. The last assumption implies that $f(t)$ is bounded and any continuous monotone increasing function that is bounded is uniformly continuous.

Problem 5, (5 points): Please do problem 2.4.8 in Heil.

Solution: a) For any set $E \subset \mathbb{R}^d$ there exists a G_δ set G with $E \subset G$ and $|E|_e = |G|$. Thus we have $E_k \subset G_k$ and $|E_k|_e = |G_k|$ for $k = 1, 2, \ldots$. We do not know, however, whether the G_k are nested. Instead of G_k we consider for any $m = 1, 2, \ldots$

$$B_m := \cap_{k=m}^{\infty} G_k$$

We obviously have that $B_m \subset B_{m+1}$. Further, since $E_m \subset E_k \subset G_k$ for all $k \ge m$ we have that $E_m \subset B_m$. Thus, by continuity

$$| \cup_m B_m| = \lim_{m \to \infty} |B_m|$$

and hence by monotonicity

$$| \cup_m E_m|_e \le | \cup_m B_m| = \lim_{m \to \infty} |B_m| \le \liminf_{m \to \infty} |G_m| = \lim_{m \to \infty} |E_m|_e$$

Hence

$$| \cup_m E_m|_e \le \lim_{m \to \infty} |E_m|_e$$

By monotonicity

$$| \cup_m E_m|_e \ge |E_k|_e$$

for all $k = 1, 2, \ldots$ and hence

$$| \cup_m E_m|_e \ge \lim_{m \to \infty} |E_m|_e \ .$$

b) Recall the non-measurable sets we constructed in class. There were countably many congruent subsets of the circle of circumference 1. Call them M_k. They had the property that $M_k \cap M_\ell = \emptyset$ for $k \neq \ell$ and $\cup_k M_k = C$ the circle. Consider the sets $B_m = \cup_{k=m}^{\infty} M_k$. We have that $B_1 \supset B_2 \supset B_3 \cdots$ which are nested. Each set has finite exterior measure because it is a subset of C. Further

$$\cap_{m=1}^{\infty} B_m = \emptyset$$

because it consists precisely of the points that are in infinitely of the sets M_k which are disjoint. The sets M_k all have the same exterior measure, since they are congruent. Call this number K. This number cannot be zero since by subadditivity

$$1 = |C| \leq \sum_k |M_k|_e \ .$$

Hence

$$|B_m|_e \geq |M_m|_e = K$$

and

$$\liminf_{m \to \infty} |B_m|_e \geq K$$

whereas

$$| \cap_{m=1}^{\infty} B_m|_e = 0 \ .$$