HOMEWORK , DUE THURSDAY APRIL 9. PLEASE UPLOAD THE HOMEWORK ON CANVAS

Problem 1, (5 points): Please do Problem 5.5.12. in Heil (Use the Lebesgue differentiation theorem.)
Solution: Since $f \in L^1[a,b]$ we know, by the Lebesgue differentiation theorem, that

$$\lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(y)dy = f(x)$$

for almost every $x \in [a,b]$. Since $\int_a^x f(y)dy = 0$ by assumption, we also know that

$$\int_{x-\varepsilon}^{x+\varepsilon} f(y)dy = 0$$

and hence $f = 0$ almost everywhere.

Problem 2, (5 points): Please work Problem 5.5.14 in Heil.
Solution: We have that

$$f \star g_h(x) - f(x) = \int_{\mathbb{R}^d} [f(x-y) - f(x)] h^{-d} g(y/h) dy$$

where we have used that $g \in L^1(\mathbb{R}^d)$ and $\int_{\mathbb{R}^d} g(y)dy = 1$. Note that by a change of variables

$$\int h^{-d} g(y/h) dy = \int g(y) = 1 \ .$$

Recall that the convolution is well defined for a.e.$x \in \mathbb{R}^d$. Changing variables we get

$$f \star g_h(x) - f(x) = \int_{\mathbb{R}^d} [f(x-hy) - f(x)] g(y) dy$$

and hence using Fubini

$$\|f \star g_h(\cdot) - f(\cdot)\|_1 \le \int_{\mathbb{R}^d} \|f(\cdot - hy) - f(\cdot)\|_1 g(y) dy \ .$$

Now,

$$|\|f(\cdot - hy) - f(\cdot)\|_1 g(y)| \le 2\|f\|_1 |g(y)|$$

and the right side is an integrable function. Since $\|f(\cdot - hy) - f(\cdot)\|_1 \to 0$ as $h \to 0$ for every $y \in \mathbb{R}^d$ we have by the Dominated Convergence Theorem that

$$\lim_{h \to 0} \|f \star g_h(\cdot) - f(\cdot)\|_1 = 0 \ .$$

Note that we did not assume that g was supported in a ball. This works for any L^1 function g with $\int g = 1$.

Problem 3, (5 points): Please do Problem 5.5.16 in Heil

Solution: We have clearly that $Mf_n(x)$ is an increasing sequence. Hence

$$\lim_{n\to\infty} Mf_n(x) = \sup_n Mf_n(x)$$

for all x. Thus,

$$\lim_{n\to\infty} Mf_n(x) = \sup_n \sup_{r>0} \frac{1}{|B_r(x)|}\int_{B_r(x)} f_n(y)dy$$

and since two supremums can always be interchanged we find

$$\lim_{n\to\infty} Mf_n(x) = \sup_{r>0} \sup_n \frac{1}{|B_r(x)|}\int_{B_r(x)} f_n(y)dy$$

However, by the monotone convergence theorem

$$\sup_n \frac{1}{|B_r(x)|}\int_{B_r(x)} f_n(y)dy = \frac{1}{|B_r(x)|}\int_{B_r(x)} f(y)dy$$

and hence

$$\lim_{n\to\infty} Mf_n(x) = Mf(x) \ .$$

Problem 4, (5 points): Please work Problem 5.5.18 in Heil
Solution: By Tchebyshev's inequality for every function in $L^1(\mathbb{R}^d)$,

$$|\{|f| > \alpha\}| \le \frac{\|f\|_1}{\alpha}$$

and hence $L^1(\mathbb{R}^d) \subset \mathrm{Weak} - L^1(\mathbb{R}^d)$. The second statement follows from the inequality on the maximal function

$$|\{Mf > \alpha\}| \le \frac{A}{\alpha}\|f|_1 \ .$$

Note: The function $|x|^{-d}$ is in Weak $- L^1(\mathbb{R}^d)$ but not in $|L^1(\mathbb{R}^d)$.

Problem 5, (5 points): Please do problem 5.5.19 a) in Heil.
Solution: The exists an G_δ set H with $A \subset H$ and $|A|_e = |H|$.Consider the characteristic function $\chi_H(x) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and note that

$$\frac{1}{|B_r(x)|}\int_{B_r(x)} \chi_H(y)dy = \frac{|H \cap B_r(x)|}{|B_r(x)|}$$

and by Lebesgue's theorem this converges to 1 for a.e. $x \in H$. It remains to see that

$$|H \cap B_r(x)| = |A \cap B_r(x)|_e$$

Quite generally, if B is measurable

$$|A|_e = |A \cap B|_e + |A \setminus B|_e$$

and

$$|H| = |H \cap B| + |H \setminus B| \ .$$

Since $A \subset H$ we have $|A \cap B|_e \le |H \cap B|$ and $|A \setminus B|_e \le |H \setminus B|$. Since $|A|_e = |H|$ it follows that $|A \cap B|_e = |H \cap B|$ and $|A \setminus B|_e = |H \setminus B|$. Hence

$$\frac{|H \cap B_r(x)|}{|B_r(x)|} = \frac{|A \cap B_r(x)|_e}{|B_r(x)|}$$