ON THE LAPLACE OPERATOR PENALIZED BY MEAN CURVATURE

Evans M. Harrell II*
School of Mathematics
Georgia Institute of Technology
Atlanta GA 30332-0160
USA
harrell@math.gatech.edu

Michael Loss**
School of Mathematics
Georgia Institute of Technology
Atlanta GA 30332-0160
USA
loss@math.gatech.edu

Abstract

Let $h = \sum_{j=1}^{d} \kappa_j$ where the κ_j are the principal curvatures of a d-dimensional hypersurface immersed in R^{d+1}, and let $-\Delta$ be the corresponding Laplace–Beltrami operator. We prove that the second eigenvalue of $-\Delta - \frac{1}{d}h^2$ is strictly negative unless the surface is a sphere, in which case the second eigenvalue is zero. In particular this proves conjectures of Alikakos and Fusco.

* Work supported by N.S.F. grant DMS-9622730
** Work supported by N.S.F. grant DMS-9500840 and the MSRI

I. INTRODUCTION

This article is concerned with linear differential operators of the form

$$H = -\Delta - q$$

defined on curves, surfaces, and hypersurfaces, where q is a quadratic expression in the principal curvatures κ_j. The particular case where

$$q = \sum_j \kappa_j^2$$

has arisen in a number of previous articles, e.g., [AlBaFu] [AlFu] [Ha]. There, H is the operator applied to a function f that one obtains by linearizing a distortion of the surface Ω according to

$$\frac{d\mathbf{x}}{d\varepsilon} = f(\mathbf{x})\ \mathbf{N}\ ,$$

where $\mathbf{N}$ is the unit normal vector.

As such H plays a role in the evolution of phase interfaces in materials, and stability analyses of these interfaces led Alikakos and Fusco [AlFu] to formulate a spectral-geometric conjecture:

Conjecture (Alikakos and Fusco)

a) Suppose that Ω is a simply connected, smooth, compact surface in R^3. The second eigenvalue of H with $q = \sum_j \kappa_j^2$ is maximized at 0 precisely when Ω is a sphere.

b) Suppose that Ω is a simple, closed, smooth curve in the plane. The second eigenvalue of H with $q = \kappa^2$ is maximized at 0 precisely when Ω is a circle.

Note that the potential has the dimension ($length^{-2}$), the same as the differential operator. As a consequence the result is independent of the area of the surface or the length of the curve.

One of us [Ha] recently proved the conjecture in the two–dimensional case. In this case the claim follows immediately from the corresponding one where $q = 2\kappa_1\kappa_2$, i.e., when q is twice the Gauss curvature. The proof in that case is achieved in a very natural way, namely using a variational characterization of the eigenvalues of H with the help of conformal transplantations [Ha]. This approach certainly works in the case where the surface is of the same topological type as the sphere.

Curiously, the one–dimensional conjecture b), for the ordinary differential operator

$$H(\kappa) := -\frac{d^2}{ds^2} - \kappa^2,$$

has stubbornly resisted attacks along these lines. The second eigenvalue must be estimated with the min-max principle, requiring an orthogonalization, and the estimates to show that it is strictly negative except for the circle have been just too delicately balanced to resolve the conjecture unrestrictedly.

There has been progress: Alikakos [Al] was able to resolve the conjecture under certain symmetry assumptions, and Papanicolaou [Pa] resolved it locally, in the sense that if $\mathcal{C}$ is a sufficiently small perturbation of a circle without being an exact circle, then there are at least two negative eigenvalues. Papanicolaou interpreted $H(\kappa)$ as a Hill operator, that is, he took $\kappa(s)$ as a periodic function with integral 2π, disregarding whether it is the curvature function of a closed curve. He also exhibited an example with a nonconstant $\kappa(s)$, for which the second eigenvalue of $H(\kappa)$ is positive. Thus the assumption that Ω is a closed curve is crucial for the theorem which will be proved in the following section.

In dimensions greater than two, there have been no results available until now. One barrier has been the proper choice of q; e.g., it is clear for dimensional reasons that the Gauss curvature is not a natural choice for q, except in two–dimensions. The 'good' choice for q, and this is a crucial insight of this article, is

$$q = \frac{1}{d} h^2 \ .$$

Here d is the dimension of the surface and h is d times the mean curvature, i.e.,

$$h = \sum_{j=1}^{d} k_j \ .$$

Our main result is the following:

Theorem 1

Let Ω be a smooth compact oriented hypersurface of dimension d immersed in R^{d+1}; in particular self–intersections are allowed. The metric on that surface is the standard Euclidean metric inherited from R^{d+1}. Then the second eigenvalue λ_2 of the operator

$$H = -\Delta - \frac{1}{d} h^2$$

is strictly negative unless Ω is a sphere, in which case λ_2 equals to zero.

Remarks:

(i) Using the Cauchy–Schwarz inequality the Alikakos–Fusco conjecture is an immediate consequence of Theorem 1.

(ii) Because under a change of length scale the operator simply picks up a constant factor, we are free to normalize so that the d-dimensional volume of Ω equals 1. We do this henceforth.

(iii) No assumptions have to be made about the topology of the surface. Moreover, the theorem holds also if Ω is immersed in R^n for any $n > d$.

(vi) Although obvious, it is worth pointing out that Theorem 1 is not an intrinsic statement about the surface; it contains also information about the embedding of Ω in

Euclidean space R^{d+1}. In particular, Theorem 1 and its proof do not make any claims about the two dimensional case when q is twice the Gauss curvature.

The point of view which led us to a solution uses rather different ideas than normally used in differential geometry where one usually deals with intrinsic quantities, independent of the embedding. Hence it may be of some value to give a general description of the strategy of proof.

The key technical idea is *to count the negative eigenvalues rather than estimating them.* There is a method (which is now standard in mathematical quantum mechanics) for counting eigenvalues, due to Birman[Bi] and Schwinger [Sc] which provides the first step. We recall it here, following the review article of Simon [Si]:

Lemma (The Birman–Schwinger principle)

Consider the self-adjoint operator

$$-\Delta - W^2(x),$$

where W^2 *is relatively bounded with respect to* $-\Delta$ *with bound less than 1 (for definition see [ReSi p. 162]). A number* $-\mu < 0$ *is an eigenvalue of* $-\Delta - W^2$ *if and only if 1 is an eigenvalue of the bounded, positive operator*

$$K_\mu \ := \ W\,(-\Delta + \mu)^{-1}\,W.$$

Remarks.

(i) The multiplicities are also equal.

(ii) The eigenvalues of K_μ are monotonically decreasing continuous functions of μ and tend to 0 as $\mu \to \infty$. Therefore, if we can locate an eigenvalue > 1, we can be sure that there is an eigenvalue $= 1$ for some larger value of μ.

(iii) In contrast to most uses of the Birman–Schwinger principle, where W is the positive root of the potential W^2, we do not assume $W \geq 0$.

Thus, the original problem, which is about an *inequality* has been reduced to a problem about the *asymptotics* of K_μ as μ tends to zero. The analysis of that problem is relatively straightforward.

In section II we consider the case of a space curve. Not only does this provide most of the ideas in section III, i.e., for hypersurfaces of dimension d embedded in R^{d+1}, but it also shows that the techniques work for embeddings of higher codimension.

II. An extremal property of the circle

In this section we consider the problem of determining the closed curve $\mathcal{C}$ in R^3, of fixed length normalized to 1, which maximizes the second eigenvalue of the self-adjoint differential operator

$$H(\kappa) := -\frac{d^2}{ds^2} - \kappa^2.$$

Here, s is the arc–length and κ is the curvature, regarded as a given function of s. The domain of self-adjointness for this operator on the Hilbert space $L^2(\mathcal{C}, ds)$ consists of periodic functions with absolutely continuous derivatives.

If $\kappa(s)$ is a constant, then the curve is a circle (with $\kappa = 2\pi$), and it is an elementary observation that the first two eigenvalues are $-4\pi^2$ and 0 (degenerate). We prove the one–dimensional Alikakos-Fusco conjecture, with the less restrictive assumption that $\mathcal{C}$ is a space curve.

Theorem 2

Let $\mathcal{C}$ be a smooth curve in R^3 with curvature κ. Then the second eigenvalue of

$$H(\kappa) \ := \ -\frac{d^2}{ds^2} \ - \ \kappa^2.$$

is less than or equal to 0*, with equality if and only if $\mathcal{C}$ is a circle (of circumference 1).*

Proof:

In the first step we show that whether or not H has two negative eigenvalues depends on the analysis of a simple functional independent of μ, eq. (1), below.

The Birman-Schwinger operator in the case of a curve is

$$K_\mu \ := \ \kappa \left(-\frac{d^2}{ds^2} \ + \ \mu \right)^{-1} \kappa.$$

As remarked in the introduction, since the operator norm of K_μ tends to 0 as $\mu \to \infty$, we can show the existence of (at least) 2 negative eigenvalues of $H(\kappa)$ by showing that for sufficiently small $\mu > 0$, K_μ has 2 eigenvalues larger than 1, except in the case of the circle.

By the min-max principle as applied to K_μ, if we have two linearly independent functions $f_{1,2}$ such that the 2×2 matrix

$$M \ := \ \langle f_j \ , \ K_\mu f_\kappa \rangle \ - \ \langle f_j \ , \ f_\kappa \rangle$$

is strictly positive for sufficiently small μ, then K_μ has the two desired eigenvalues > 1.

We choose $f_1(s) \ = \ \kappa(s)$, and seek $f_2(s)$ bounded and orthogonal to $\kappa(s)$. For any such function, κf_2 is orthogonal to 1, which is the eigenfunction of $-\frac{d^2}{ds^2}$ with eigenvalue 0.

From the spectral theorem it follows that the operator $R_\mu := \left(-\frac{d^2}{ds^2} + \mu\right)^{-1}$ acts boundedly on κf_2 as $\mu \to 0$. The limit R_0 exists on the set of functions orthogonal to 1 and could be written explicitly as an integral expression (with constants of integration ensuring periodicity and orthogonality to 1).

With these choices of $f_{1,2}$, the matrix M is found to have the form

$$\begin{bmatrix} \frac{1}{\mu}\oint \kappa^2 ds & O(1) \\ O(1) & \langle \kappa f_2,\ R_0\ \kappa f_2 \rangle\ -\ \|f_2\|^2\ +\ O(\mu) \end{bmatrix},$$

which will be positive provided that both its determinant and trace are positive. The trace is clearly positive for sufficiently small μ, while the determinant will also be positive for sufficiently small μ if the functional

$$\Phi(f_2)\ :=\ \frac{\langle \kappa f_2,\ R_0\ \kappa f_2 \rangle}{\|f_2\|^2}\ >\ 1. \tag{1}$$

Synopsis: H has at least two negative eigenvalues if we can find $f_2(s)$ bounded, orthogonal to $\kappa(s)$, and satisfying (1).

Let us therefore define

$$\Lambda\ :=\ \sup\left\{\Phi(\mathrm{f})\ :\ \mathrm{f} \in \mathrm{L}^2,\ \mathrm{f}\ \not\equiv\ 0,\ \oint \mathrm{f(s)}\ \kappa(\mathrm{s})\ \mathrm{ds}\ =\ 0\right\}. \tag{2}$$

Next we show that $\Lambda \geq 1$ by choosing as a test function $f(s)$ any of the coordinates of the normal vector $\mathbf{N}$ to the curve. It will be convenient to recall the Frenet-Serret formulae for space curves: Let $\mathbf{x}$ denote the position of a point on $\mathcal{C}$, embedded in R^3. Then

$$\begin{aligned} \frac{d\mathbf{x}}{ds} &= \mathbf{T} \\ \frac{d\mathbf{T}}{ds} &= \kappa \mathbf{N} \\ \frac{d\mathbf{N}}{ds} &= -\kappa \mathbf{T}\ +\ \tau \mathbf{B} \\ \frac{d\mathbf{B}}{ds} &= -\tau \mathbf{T}\ . \end{aligned} \tag{3}$$

Since $\mathbf{T}$ is periodic, the formula for $d\mathbf{T}/ds$ guarantees that each component of $\mathbf{N}$ is orthogonal to κ, and is thus a suitable choice for (1). Since

$$\frac{d^2\mathbf{x}}{\mathrm{ds}^2} = \kappa\,\mathbf{N}, \tag{4}$$

we calculate:

$$R_0\,\kappa\,\mathrm{N_j} = \mathrm{x_j(s)} - \mathrm{y_j}, \tag{5}$$

where $\mathrm{x_j}(s)$ is the j-th coordinate of $\mathbf{x}$ and $\mathrm{y_j}$ is the constant needed so that $\mathrm{x_j(s)} - \mathrm{y_j}$ is orthogonal to 1. Hence

$$\begin{aligned} \langle N_j\ ,\ \kappa R_0\ \kappa N_j\rangle &= \left\langle \tfrac{dT_j}{ds},\ R_0\ \tfrac{dT_j}{ds} \right\rangle \\ &= -\left\langle \tfrac{dT_j}{ds},\ \mathrm{x_j(s)} - \mathrm{y_j} \right\rangle. \\ &= \langle T_j,\ T_j\rangle\,. \end{aligned}$$

In the final step the boundary term in the integration by parts vanishes because the curve is closed.

Summing on j, we obtain

$$\begin{aligned} \sum_{j=1}^{3} \langle N_j\ ,\ \kappa R_0\ \kappa N_j\rangle &= \int_0^1 |\mathbf{T}|^2 = 1 \\ &= \int_0^1 |\mathbf{N}|^2 = \sum_{j=1}^{3} \langle N_j\ ,\ N_j\rangle\ . \end{aligned}$$

Either $\Phi(N_j) > 1$ for some j, or else $\Phi(N_j) = 1$ for all $j = 1, 2, 3$ (strictly speaking, in the case of a planar curve one of the N_j might vanish identically, and $\Phi(N_j) = 1$ for the other two coordinates). This establishes that $\Lambda \geq 1$.

If $\Lambda > 1$ we are done, so we now assume that $\Lambda = 1$, which means that each N_j which does not vanish identically is an optimizer for the variational problem (2).

We shall now demonstrate that this possibility implies that $\mathcal{C}$ is a circle. To this end we calculate the first variation of Φ, and discover that a necessary condition for maximality is

$$\kappa R_0 \, \kappa \mathbf{N} \;=\; \mathbf{N} \;+\; \mathbf{\Gamma} \; \kappa.$$

Here, $\mathbf{\Gamma}$ is a vector of Lagrange multipliers, and the vectorial notation of this equation indicates that the operator R_0 operates on each Cartesian component. (In case some component N_j vanishes identically, the equation holds trivially.) Using (5), the condition for maximality reads

$$- \, \kappa \left(\mathbf{x}(\mathbf{s}) \;-\; \mathbf{y} \;+\; \mathbf{\Gamma} \;\right) \;=\; \mathbf{N},$$

which implies among other things that κ is bounded away from 0. Divided by κ, the equation becomes

$$-\mathbf{x}(s) \;+\; \mathbf{y} \;-\; \mathbf{\Gamma} \;=\; \frac{\mathbf{N}}{\kappa}, \tag{6}$$

and when we differentiate using the Frenet-Serret equations, we find

$$-\mathbf{T}\,(s) \;\;=\;\; \frac{d}{ds}\left(\frac{\mathbf{N}}{\kappa}\right) \;=\; \mathbf{N}\,(s)\;\; \frac{d}{ds}\left(\frac{1}{\kappa}\right) \;-\; \mathbf{T}\,(s) \;+\; \left(\frac{\tau}{\kappa}\right)\mathbf{B}\,(s)\,,$$

By comparing components we learn that $\tau = 0$ and $\kappa = constant$. This implies that $\mathcal{C}$ is a circle, the formula for which is obtained by taking the magnitude of both sides of (6).

III. An extremal property of S^d

The higher-dimensional Theorem 1 hinges on the generalization of (4), that

$$-\Delta \mathbf{x} = \mathrm{h}\, \mathbf{N}. \tag{7}$$

Here the vector $\mathbf{x}$ is simply the position of a point on Ω as embedded in R^{d+1}. The vector notation in this equation indicates that the Laplace–Beltrami operator Δ acts on each of the $d+1$ components of $\mathbf{x}$ independently as scalar functions – no Christoffel symbols are introduced. The useful identity (7) results from a direct, elementary calculation.

Observe that the unit normal for a hypersurface is conventionally defined as outward, which will lead to some differences of sign from the ones used for space curves, where $\mathbf{N}$ may be inward. We also remark for future purposes that none of the functions $h(x)$ or $N_j(x)$ can vanish identically on a compact hypersurface.

Proof of Theorem 1:

The proof will follow the conceptual outline of the one for space curves rather closely. As before, we look at the Birman–Schwinger operator, which in this case is

$$K_\mu := \frac{1}{d} h \left(-\Delta + \mu\right)^{-1} h.$$

We shall show that K_μ has two eigenvalues ≥ 1 by projecting it onto the two-dimensional space spanned by two trial functions h and f, restricted so that

$$\int_\Omega h(x)\; f(x)\; dVol = 0$$

Precisely the same argument as in section II shows that the original operator H has two negative eigenvalues provided that the functional

$$\Phi(f) \ := \ \frac{\langle hf, \ R_0 \ hf \rangle}{d \, \|f\|^2} \ > \ 1, \tag{8}$$

where the reduced resolvent R_0 is the limit as $\mu \ \downarrow \ 0$ of $(-\Delta \ + \ \mu)^{-1}$. This is well-defined on the space of functions of mean 0. The variational problem now concerns

$$\Lambda \ := \ \sup \left\{ \Phi(\mathrm{f}) \ : \ \mathrm{f} \in \mathrm{L}^2(\Omega), \ \mathrm{f} \ \not\equiv \ 0, \ \int_\Omega \mathrm{h(x)} \ \mathrm{f(x)} \ \mathrm{dVol} \ = \ 0 \right\}. \tag{9}$$

In order to show that $\Lambda \geq 1$, we choose $f(x) = N_j(x)$, and sum over all j, to compute:

$$\begin{aligned} \tfrac{1}{d} \sum_{\mathrm{j=1}}^{\mathrm{d+1}} \langle \mathrm{N_j}, \ h \ R_0 \ h \ N_j \rangle \ &= \ \tfrac{1}{\mathrm{d}} \sum_{\mathrm{j=1}}^{\mathrm{d+1}} \langle -\Delta \mathrm{x_j}, \ \mathrm{R_0} \ (-\Delta \mathrm{x_j}) \rangle \\ &= \ \tfrac{1}{\mathrm{d}} \sum_{\mathrm{j=1}}^{\mathrm{d+1}} \textstyle\int_\Omega |\nabla \mathrm{x_j}|^2 \ \mathrm{dVol} \\ &= \ 1. \end{aligned}$$

Summing the denominators of $\Phi(N_j)$,

$$\sum_{\mathrm{j=1}}^{\mathrm{d+1}} \langle \mathrm{N_j}, \ \mathrm{N_j} \rangle \ = \ 1;$$

we conclude as in section II that either $\Phi(N_j) \ > \ 1$ for some j, or else $\Phi(N_j) \ = \ 1$ for all j. This establishes that $\Lambda \geq \ 1$.

If $\Lambda = 1$, then each N_j is an optimizer, and we next show that this implies that Ω is a sphere. The Euler–Lagrange equation (again using vector notation) now states that

$$\frac{1}{d} \ h \ R_0 \ h \ \mathbf{N} \ = \ \mathbf{N} \ + \ \ \mathbf{\Gamma},$$

where $\mathbf{\Gamma}$ is a $d+1$–tuple of Lagrange multipliers. Using (7), this reads

$$h(\mathbf{x} \ - \ \mathbf{y} \ - \ \mathrm{d} \ \mathbf{\Gamma}) \ = \ \mathrm{d} \ \mathbf{N},$$

which clearly shows that h cannot vanish. Dividing by h:

$$(\mathbf{x} \;-\; \mathbf{y} \;-\; \mathrm{d}\,\mathbf{\Gamma}) \;=\; \frac{\mathrm{d}\,\mathbf{N}}{\mathrm{h}}. \tag{10}$$

If we now differentiate (10) along any curve in Ω, the left side is a tangential vector, so the normal component of the derivative of right side, i.e., the derivative of d/h, is 0. Thus h is constant. Together with (10), the constancy of h implies that Ω is a sphere.

IV. Concluding remarks

Two natural questions have not been fully addressed here. One of them is how a nontrivial topology can increase the number of negative eigenvalues of H beyond 2. This seems to be within reach for the case of a planar curve, where the topology is given by the winding number, and we believe that for winding number n there are at least $2n$ negative eigenvalues, except in the case of a (multiply traversed) circle. For space curves and spheres, however, it is not at all clear how the topology controls the number of negative eigenvalues.

The second question has to do with the larger categories of potentials depending on curvature, as in the operator

$$-\Delta \;-\; \alpha \sum_{\mathrm{j}} \kappa_{\mathrm{j}}^2$$

for $0 \le \alpha < 1$. Such potentials were allowed in two dimensions in [Ha], which thus connects Theorem 1 in two dimensions with the one with $\alpha = 0$ of [He]. Since the second eigenvalue of the Laplace-Beltrami operator is known not to to be maximized by the sphere in certain higher-dimensional settings [Ur], the result of this article will not extend to all $\alpha \ge 0$.

References

[Al] Nicholas D. Alikakos, private communication.

[AlFu] Nicholas D. Alikakos and Giorgio Fusco, The spectrum of the Cahn-Hilliard operator for generic interface in higher space dimensions, Indiana U. Math. J. **4**, 1993, pp. 637–674.

[Bi] M.S. Birman, The spectrum of singular boundary problems, Mat. Sbornik **55**, 1961, pp. 125–174 (Amer. Math. Soc. Trans. **53**, 1966, pp. 23–80.

[Ha] Evans M. Harrell II, On the second eigenvalue of the Laplace operator penalized by curvature, J. Diff. Geom. and Appl. **6**, 1996, pp. 397–400.

[He] Joseph Hersch, Quatre propriétés isopérimetriques de membranes sphériques homogènes, C.R. Acad. Sci. Paris, sér A-B **270**, 1970, pp. A1645–1648.

[Pa] Vassilis G. Papanicolaou, The second periodic eigenvalue and the Alikakos– Fusco conjecture, J. Diff. Eqns. **130**, 1996, pp. 321–332.

[ReSi] Michael Reed and Barry Simon, Methods of modern mathematical physics, II: Fourier analysis, self–adjointness, Academic Press, 1975, p. 162.

[Sc] Julian Schwinger, On the bound states of a given potential, Proc. Nat. Acad. Sci. U.S.A. **47**, 1961, pp.122–129.

[Si] Barry Simon, On the number of bound states of two body Schröedinger operators – a review, in Studies in mathematical physics, (E. H. Lieb, B. Simon, A. S. Wightman eds.) Princeton Univ. Press, Princeton, 1976, pp. 305–326.

[Ur] Hajime Urakawa, On the least positive eigenvalue of the Laplacian for compact group manifolds, J. Math. Soc. Japan **31**, 1979, pp. 209–226.