Large Sieve Inequalities via Subharmonic Methods and the Mahler Measure of the Fekete Polynomials

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Abstract

We investigate large sieve inequalities such as

\[ \frac{1}{m} \sum_{j=1}^{m} \psi \left( \log |P(e^{i\tau_j})| \right) \leq \frac{C}{2\pi} \int_{0}^{2\pi} \psi \left( \log |e^{i\tau}| \right) d\tau, \]

where \( \psi \) is convex and increasing, \( P \) is a polynomial or an exponential of a potential, and the constant \( C \) depends on the degree of \( P \), and the distribution of the points \( 0 \leq \tau_1 < \tau_2 < \cdots < \tau_m \leq 2\pi \). The method allows greater generality and is in some ways simpler than earlier ones. We apply our results to estimate the Mahler measure of Fekete polynomials.

1 Results

The large sieve of number theory [14, p. 559] asserts that if

\[ P(z) = \sum_{k=-n}^{n} a_k z^k \]

is a trigonometric polynomial of degree \( \leq n \), and

\[ 0 \leq \tau_1 < \tau_2 < \cdots < \tau_m \leq 2\pi, \]

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and

$$\delta = \min \{ \tau_2 - \tau_1, \tau_3 - \tau_2, \ldots, \tau_m - \tau_{m-1}, 2\pi - (\tau_m - \tau_1) \},$$

then

$$\sum_{j=1}^{m} |P(e^{i\tau_j})|^2 \leq \left( \frac{n}{2\pi} + \delta^{-1} \right) \int_{0}^{2\pi} |P(e^{i\tau})|^2 \, d\tau. \quad (1)$$

There are numerous extensions of this to $L_p$ norms, or involving $\psi \left( |P(e^{i\tau})|^p \right)$, where $\psi$ is a convex function, and $p > 0$ [8], [12]. There are versions of this that estimate Riemann sums, for example,

$$\sum_{j=1}^{m} |P(e^{i\tau_j})|^2 (\tau_j - \tau_{j-1}) \leq C \frac{1}{2\pi} \int_{0}^{2\pi} |P(e^{i\tau})|^2 \, d\tau, \quad (2)$$

with $C$ independent of $n$, $P$, $\{\tau_1, \tau_2, \ldots, \tau_m\}$. These are often called forward Marcinkiewicz-Zygmund inequalities. Converse Marcinkiewicz-Zygmund Inequalities provide estimates for the integrals above in terms of the sums on the left-hand side [11], [13], [16].

A particularly interesting case is that of the $L_0$ norm. A result of the first author asserts that if $\{z_1, z_2, \ldots, z_n\}$ are the $n$th roots of unity, and $P$ is a polynomial of degree $\leq n$,

$$\prod_{j=1}^{n} |P(z_j)|^{1/n} \leq 2M_0(P), \quad (3)$$

where

$$M_0(P) := \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \log |P(e^{it})| \, dt \right)$$

is the Mahler measure of $P$.

The focus of this paper is to show that methods of subharmonic function theory provide a simple and direct way to generalize previous results. We also extend (3) to points other than the roots of unity. Given $c \geq 0$, $\kappa \in [0, \infty)$, and a positive measure $\nu$ of compact support and total mass at most
\( \kappa \geq 0 \) on the plane, we define the associated exponential of its potential by

\[
P(z) = c \exp \left( \int \log |z - t| d\nu(t) \right).
\]

We say that this is an exponential of a potential of mass \( \leq \kappa \), and that its degree is \( \leq \kappa \). The set of all such functions is denoted by \( \mathbb{P}_\kappa \). Note that if \( P \) is a polynomial of degree \( \leq n \), then

\[
|P| \in \mathbb{P}_n.
\]

More generally, the generalized polynomials studied by several authors [3], [7] also lie in \( \mathbb{P}_\kappa \), for an appropriate \( \kappa \). We prove:

**Theorem 1.1** Let \( \psi : \mathbb{R} \rightarrow [0, \infty) \) be nondecreasing and convex. Let \( m \geq 1 \), \( \kappa > 0 \), \( \alpha > 0 \), and

\[
0 < \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \leq 2\pi.
\]

Let \( w_j \geq 0 \), \( 1 \leq j \leq m \) with

\[
\sum_{j=1}^{m} w_j = 1.
\]

Let \( \mu_m \) denote the corresponding Riemann-Stieltjes measure, defined for \( \theta \in [0, 2\pi] \) by

\[
\mu_m ([0, \theta]) := \sum_{j : \tau_j \leq \theta} w_j.
\]

Let

\[
\Delta := \sup \left\{ \left| \mu_m ([0, \theta]) - \frac{\theta}{2\pi} \right| : \theta \in [0, 2\pi] \right\}
\]

(4)

denote the discrepancy of \( \mu_m \). Then for \( P \in \mathbb{P}_\kappa \),

\[
\sum_{j=1}^{m} w_j \psi \left( \log P(e^{i\tau_j}) \right) \leq \left( 1 + \frac{8}{\alpha \kappa \Delta} \right) \frac{1}{2\pi} \int_{0}^{2\pi} \psi \left( \log \left[ e^{\alpha P(e^{i\theta})} \right] \right) d\theta.
\]

(5)
Example 1 Let us choose all equal weights,

\[ w_j = \frac{1}{m}, \quad 1 \leq j \leq m. \]

Then \( \mu_m \) is counting measure,

\[ \mu_m ([0, \theta]) = \frac{1}{m} \# \{ j : \tau_j \in [0, \theta] \}. \]

If we take \( \psi(t) = \max \{0, t\} \), and \( \alpha = 1 \), and use the notation \( \log^+ t = \max \{0, \log t\} \), we obtain

\[
\frac{1}{m} \sum_{j=1}^{m} \log^+ P (e^{i\tau_j}) \leq (1 + 8\kappa \Delta) \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left[ e P \left( e^{i\theta} \right) \right] d\theta. \quad (6)
\]

This result is new. Previous inequalities have been limited to sums involving \( \psi (P (e^{i\tau_j})^p) \), some \( p > 0 \). If we let \( p > 0 \), \( \psi(t) = e^{pt} \), and \( \alpha = \frac{1}{p} \), (5) becomes

\[
\frac{1}{m} \sum_{j=1}^{m} P (e^{i\tau_j})^p \leq (1 + 8p\kappa \Delta) \frac{e}{2\pi} \int_0^{2\pi} P \left( e^{i\theta} \right)^p d\theta. \quad (7)
\]

This choice of \( \alpha \) is not optimal. The optimal choice is

\[
\alpha = 4\kappa \Delta \left[ -1 + \sqrt{1 + \frac{1}{2p\kappa \Delta}} \right]
\]

but one needs further information on the size of \( p\kappa \Delta \) to exploit this. For example, if \( p\kappa \Delta \leq 1 \), the optimal choice is of order \( \sqrt{\frac{\kappa \Delta}{p}} \), and choosing this \( \alpha \) in (5), we obtain

\[
\frac{1}{m} \sum_{j=1}^{m} P (e^{i\tau_j})^p \leq \left( 1 + C \sqrt{p\kappa \Delta} \right) \frac{1}{2\pi} \int_0^{2\pi} P \left( e^{i\theta} \right)^p d\theta, \quad (8)
\]

where \( C \) is independent of \( p, \kappa, \Delta, P \).

For well distributed \( \{ \tau_1, \tau_2, \ldots, \tau_m \} \), \( \Delta \) is of order \( \frac{1}{m} \). In particular, when these points are equally spaced and include \( 2\pi \), but not 0, so that

\[ \tau_j = \frac{2j\pi}{m}, \quad 1 \leq j \leq m, \]

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we have
\[ \Delta = \frac{2\pi}{m}, \]
and (7) becomes
\[ \frac{1}{m} \sum_{j=1}^{m} P(e^{i\gamma_j})^p \leq \left( 1 + \frac{16\pi \rho \kappa}{m} \right) \frac{e}{2\pi} \int_0^{2\pi} P(e^{i\theta})^p \, d\theta. \quad (9) \]

**Example 2** Another important choice of the weights \( w_j \) is
\[ w_j = \frac{\tau_j - \tau_{j-1}}{2\pi}, \quad 1 \leq j \leq m, \]
where now we assume \( \tau_0 = 0 \) and \( \tau_m = 2\pi \). For this case (5) becomes an estimate for Riemann sums,
\[ \frac{1}{2\pi} \sum_{j=1}^{m} (\tau_j - \tau_{j-1}) \psi \left( \log P(e^{i\gamma_j}) \right) \leq \left( 1 + \frac{8}{\alpha \kappa \Delta} \right) \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \log \left[ e^\alpha P(e^{i\theta}) \right] \right) \, d\theta. \quad (10) \]
The discrepancy \( \Delta \) in this case is
\[ \Delta = \sup_j \frac{\tau_j - \tau_{j-1}}{2\pi}. \]

**Remarks**

(a) In many ways, the approach of this paper is simpler than that in [12] where Dirichlet kernels were used, or that of [8], where Carleson measures were used. The main idea is to use the Poisson integral inequality for subharmonic functions.

(b) We can reformulate (5) as
\[ \int_0^{2\pi} \psi \left( \log \left| P(e^{i\tau}) \right| \right) d\mu_m(\tau) \leq \left( 1 + \frac{8}{\alpha \kappa \Delta} \right) \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \log \left[ e^\alpha P(e^{i\theta}) \right] \right) \, d\theta. \]
In fact this estimate holds for any probability measure \( \mu_m \) on \([0,2\pi]\), not just the pure jump measures above.

(c) The one severe restriction above is that \( \psi \) is nonnegative. In particular, this excludes \( \psi(x) = x \). For this case, we prove 2 different results:

**Theorem 1.2** Assume that \( m, \kappa, \{\tau_1, \tau_2, \ldots, \tau_m\} \) and \( \{w_1, w_2, \ldots, w_m\} \) are as in Theorem 1.1. Let

\[
Q(z) = \prod_{j=1}^{m} |z - e^{i\tau_j}|^{w_j}.
\]

Then for \( P \in \mathbb{P}_\kappa \),

\[
\sum_{j=1}^{m} w_j \log P(e^{i\tau_j}) \leq \frac{1}{2\pi} \int_{0}^{2\pi} \log P(e^{i\theta}) \, d\theta + \kappa \log \|Q\|_{L_\infty(\{z\} = 1)}.
\]

**Remarks**

If we choose all \( w_j = \frac{1}{m} \), this yields

\[
\prod_{j=1}^{m} P(e^{i\tau_j})^{1/m} \leq \|Q\|_{L_\infty(\{z\} = 1)}^{\frac{\kappa}{m}} \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \log P(e^{i\theta}) \, d\theta \right).
\]

If we take \( \{e^{i\pi}, e^{2i\pi}, \ldots, e^{i\pi n}\} \) to be the \( n \)th roots of unity, then

\[
Q(z) = |z^n - 1|^{1/m}
\]

and (13) becomes

\[
\prod_{j=1}^{m} P(e^{i\tau_j})^{1/m} \leq 2^{\kappa/m} \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \log P(e^{i\theta}) \, d\theta \right).
\]

In the case \( \kappa = m = n \), this gives the first author's inequality (3). In general however, it is not easy to bound \( \|Q\|_{L_\infty(\{z\} = 1)} \). Using an alternative method, we can avoid the term involving \( Q \), when the spacing between successive \( \tau_j \) is \( O(\kappa^{-1}) \).
Theorem 1.3 Assume that $m, \kappa$ and $\{\tau_1, \tau_2, \ldots, \tau_m\}$ are as in Theorem 1.1. Let $\tau_0 := \tau_m - 2\pi$ and $\tau_{m+1} := \tau_1 + 2\pi$. Let

$$\delta := \max \{\tau_1 - \tau_0, \tau_2 - \tau_1, \ldots, \tau_m - \tau_{m-1}\}.$$ 

Let $A > 0$. There exists $B > 0$ such that whenever $\kappa \geq 1$ and

$$\delta \leq A\kappa^{-1},$$

then for all $P \in \mathbb{P}_\kappa$,

$$\sum_{j=1}^{m} \frac{\tau_{j+1} - \tau_j - 1}{2} \log P (e^{i\tau_j}) \leq \int_{0}^{2\pi} \log P (e^{i\theta}) \, d\theta + B. \quad (15)$$

One application of Theorem 1.2 is to estimation of Mahler measure. Recall that for a bounded measurable function $Q$ on $[0, 2\pi]$, its Mahler measure is

$$M_0 (Q) = \exp \left( \frac{1}{2\pi} \int_{0}^{2\pi} \log |Q (e^{i\theta})| \, d\theta \right).$$

It is well known that

$$M_0 (Q) = \lim_{p \to 0^+} M_p (Q),$$

where for $p > 0$,

$$M_p (Q) := \|Q\|_p := \left( \frac{1}{2\pi} \int_{0}^{2\pi} |Q (e^{i\theta})|^p \, d\theta \right)^{1/p}.$$ 

It is a simple consequence of Jensen’s formula that if

$$Q (z) = c \prod_{k=1}^{n} (z - z_k)$$

is a polynomial, then

$$M_0 (Q) = |c| \prod_{k=1}^{n} \max \{1, |z_k| \}.$$
The construction of polynomials with suitably restricted coefficients and maximal Mahler measure has interested many authors. The Littlewood polynomials,

\[ L_n := \left\{ p : p(z) = \sum_{k=0}^{n} \alpha_k z^k, \quad \alpha_k \in \{-1, 1\} \right\}, \]

which have coefficients \( \pm 1 \), and the unimodular polynomials,

\[ K_n := \left\{ p : p(z) = \sum_{k=0}^{n} \alpha_k z^k, \quad |\alpha_k| = 1 \right\} \]

are two of the most important classes considered. Beller and Newman [1] constructed unimodular polynomials of degree \( n \) whose Mahler measure is at least \( \sqrt{n} - c/\log n \). Here we show that for Littlewood polynomials, we can achieve almost \( \frac{1}{2} \sqrt{n} \), by considering the Fekete polynomials.

For a prime number \( p \), the \( p \)th Fekete polynomial is

\[ f_p(z) = \sum_{k=1}^{p-1} \left( \frac{k}{p} \right) z^k, \]

where

\[ \left( \frac{k}{p} \right) = \begin{cases} 1, & \text{if } x^2 \equiv k \pmod{p} \text{ has a non-zero solution } x \\ 0, & \text{if } p \text{ divides } k \\ -1, & \text{otherwise}. \end{cases} \]

Since \( f_p \) has constant coefficient 0 it is not a Littlewood polynomial, but

\[ g_p(z) = f_p(z)/z \]

is a Littlewood polynomial, and has the same Mahler measure as \( f_p \). Fekete polynomials are examined in detail in [2, pp. 37–42].

**Theorem 1.4** Let \( \varepsilon > 0 \). For large enough prime \( p \), we have

\[ M_0(f_p) = M_0(g_p) \geq \left( \frac{1}{2} - \varepsilon \right) \sqrt{p}. \] (16)
Remarks

From Jensen’s inequality,

\[ M_0 (f_p) \leq \| f_p \|_2 = \sqrt{p - 1}. \]

However \( \frac{1}{2} - \varepsilon \) in Theorem 1.4 cannot be replaced by \( 1 - \varepsilon \). Indeed if \( p \) is prime, and we write \( p = 4m + 1 \), then \( g_p \) is self-reciprocal, that is,

\[ z^{p-1} g_p \left( \frac{1}{z} \right) = g_p (z), \]

and hence

\[ g_p (e^{2it}) = e^{i(p-2)t} \sum_{k=0}^{(p-3)/2} a_k \cos ((2k + 1) t), \quad a_k \in \{-2, 2\}. \]

A result of Littlewood [10, Theorem 2] implies that

\[ M_0 (f_p) = M_0 (g_p) \leq \frac{1}{2\pi} \int_0^{2\pi} |g_p (e^{2it})| \, dt \leq (1 - \varepsilon_0) \sqrt{p - 1}, \]

for some absolute constant \( \varepsilon_0 > 0 \). It is an interesting question whether there is a sequence of Littlewood polynomials \( (f_n) \) such that for an arbitrary \( \varepsilon > 0 \), and \( n \) large enough,

\[ M_0 (f_n) \geq (1 - \varepsilon) \sqrt{n}. \]

The results are proved in the next section.

2 Proofs

We assume the notation of Theorem 1.1. We let

\[ \tau = 1 + \frac{\alpha}{\kappa}, \tag{17} \]

and define the Poisson kernel for the ball \( |z| \leq r \) (cf. [15, p. 8]),

\[ P_r \left( se^{i\theta}, re^{it} \right) = \frac{r^2 - s^2}{r^2 - 2rs \cos (t - \theta) + s^2}. \]
where $0 \leq s < r$ and $t, \theta \in \mathbb{R}$.

**Proof of Theorem 1.1**

**Step 1 The Basic Inequality**

Let $P \in \mathbb{P}_n \setminus \{0\}$, so that for some $c > 0$ and some measure $\nu$ with total mass $\leq \kappa$ and compact support,

$$
\log P(z) = \log c + \int \log |z - t| \, d\nu(t).
$$

As $\log P$ is subharmonic, and as $\psi$ is convex and increasing, $\psi(\log P)$ is subharmonic [15, Theorem 2.6.3, p. 43]. Then we have for $|z| < r$, the inequality [15, Theorem 2.4.1, p. 35]

$$
\psi(\log P(z)) \leq \frac{1}{2\pi} \int_0^{2\pi} \psi(\log P(re^{it})) \mathcal{P}_r(z, re^{it}) \, dt.
$$

Choosing $z = e^{i\tau j}$, multiplying by $w_j$, and adding over $j$ gives

$$
\sum_{j=1}^m w_j \psi(\log P(e^{i\tau j})) - \frac{1}{2\pi} \int_0^{2\pi} \psi(\log P(re^{it})) \, dt
$$

$$
\leq \frac{1}{2\pi} \int_0^{2\pi} \psi(\log P(re^{it})) \mathcal{H}(t) \, dt
$$

(18)

where

$$
\mathcal{H}(t) := \sum_{j=1}^m w_j \mathcal{P}_r(e^{i\tau j}, re^{it}) - 1
$$

$$
= \int_0^{2\pi} \mathcal{P}_r(e^{i\tau}, re^{it}) \, d\left(\mu_{m}(\tau) - \frac{\tau}{2\pi}\right).
$$

Here we have used the elementary property of the Poisson kernel, that it integrates to 1 over any circle center 0 inside its ball of definition.

**Step 2 Estimating $\mathcal{H}$**

We integrate this relation by parts, and note that both $\mu_m[0,0] = 0$ and
\[ \mu_m [0, 2\pi] = 1. \] This gives
\[ \mathcal{H}(t) = -\int_0^{2\pi} \left( \frac{\partial}{\partial \tau} \mathcal{P}_r (e^{i\tau}, re^{it}) \right) \left( \mu_m ([0, \tau]) - \frac{\tau}{2\pi} \right) d\tau \]
and hence
\[ |\mathcal{H}(t)| \leq \Delta \int_0^{2\pi} \left| \frac{\partial}{\partial \tau} \mathcal{P}_r (e^{i\tau}, re^{it}) \right| d\tau. \] \hspace{1cm} (19)

Now
\[ \frac{\partial}{\partial \tau} \mathcal{P}_r (e^{i\tau}, re^{it}) = \frac{(r^2 - 1) 2r \sin (t - \tau)}{(r^2 - 2r \cos (t - \tau) + 1)^2} \]
so a substitution \( s = t - \tau \) and \( 2\pi \)-periodicity give
\[ \int_0^{2\pi} \left| \frac{\partial}{\partial \tau} \mathcal{P}_r (e^{i\tau}, re^{it}) \right| d\tau = \int_{-\pi}^{\pi} \left| \frac{\partial}{\partial s} \mathcal{P}_r (e^{is}, r) \right| ds \]
\[ = -2 \int_0^\pi \frac{\partial}{\partial s} \mathcal{P}_r (e^{is}, r) ds \]
\[ = -2 \left[ \mathcal{P}_r (e^{i\pi}, r) - \mathcal{P}_r (1, r) \right] = \frac{8 \pi}{r^2 - 1}. \] \hspace{1cm} (20)

Combining (18)–(20), gives
\[ \sum_{j=1}^n w_j \psi (\log \mathcal{P} (e^{i\gamma})) \leq \left( 1 + \Delta \frac{8 \pi}{r^2 - 1} \right) \frac{1}{2\pi} \int_0^{2\pi} \psi (\log \mathcal{P} (re^{it})) dt. \] \hspace{1cm} (21)

**Step 3 Return to the unit circle**

Next, we estimate the integral on the right-hand side in terms of an integral over the unit circle. Let us assume that \( \nu \) has total mass \( \lambda (\leq \kappa) \). Let
\[ S(z) = |z|^\lambda \mathcal{P} \left( \frac{r}{z} \right) \]
so that
\[ \log S(z) = \log c + \int \log |r - tz| d\nu (t), \]
a function subharmonic in \( C \). Then the same is true of \( \psi (\log S) \), so its integrals over circles centre 0 increase with the radius [15, Theorem 2.6.8, p. 46]. In particular
\[ \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \log S (e^{i\theta}) \right) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \log S (re^{i\theta}) \right) d\theta \]
and a substitution $\theta \to -\theta$ gives
\[
\frac{1}{2\pi} \int_0^{2\pi} \psi \left( \log P \left( r e^{i\theta} \right) \right) d\theta \leq \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \lambda \log r + \log P \left( e^{i\theta} \right) \right) d\theta \\
\leq \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \kappa \log r + \log P \left( e^{i\theta} \right) \right) d\theta \\
\leq \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \alpha + \log P \left( e^{i\theta} \right) \right) d\theta,
\]
recall our choice (17) of $r$. Then (21) becomes
\[
\sum_{j=1}^{m} w_j \psi \left( \log P \left( e^{i\tau_j} \right) \right) \\
\leq \left( 1 + \Delta \frac{8\kappa}{r^2 - 1} \right) \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \log \left[ e^\alpha P \left( e^{i\theta} \right) \right] \right) d\theta \\
\leq \left( 1 + 8\Delta \frac{\kappa}{\alpha} \right) \frac{1}{2\pi} \int_0^{2\pi} \psi \left( \log \left[ e^\alpha P \left( e^{i\theta} \right) \right] \right) d\theta.
\]

\[\square\]

Proof of Theorem 1.2

Write
\[\log P \left( z \right) = \log c + \int \log |z - t| \, d\nu \left( t \right)\]
so
\[
\sum_{j=1}^{m} w_j \log P \left( e^{i\tau_j} \right) = \log c + \int \left( \sum_{j=1}^{m} w_j \log |e^{i\tau_j} - t| \right) \, d\nu \left( t \right) \\
= \log c + \int \log Q \left( t \right) \, d\nu \left( t \right), \tag{22}
\]
recall (11). Now as all zeros of $Q$ are on the unit circle,
\[g \left( u \right) := \log Q \left( u \right) - \log \|Q\|_{L_\infty \left( |z|=1 \right)} - \log |u|\]
is harmonic in the exterior $\{u : |u| > 1\}$ of the unit ball, with limit 0 at $\infty$, and with $g \left( u \right) \leq 0$ for $|u| = 1$. By the maximum principle for subharmonic functions,
\[g \left( u \right) \leq 0, \quad |u| > 1.\]
We deduce that for $|u| > 1$,

$$
\log Q(u) \leq \log \|Q\|_{L^\infty(\{|z|=1\})} + \log^+ |u|.
$$

Moreover, inside the unit ball, we can regard $Q$ as the absolute value of a function analytic there (with any choice of branches). So the last inequality holds for all $u \in \mathbb{C}$. Then assuming (as above) that $\nu$ has total mass $\lambda \leq \kappa$,

$$
\int \log Q(t) \, d\nu(t) \leq \lambda \log \|Q\|_{L^\infty(\{|z|=1\})} + \int \log^+ |t| \, d\nu(t)
$$

$$
= \lambda \log \|Q\|_{L^\infty(\{|z|=1\})} + \int \left( \frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\theta} - t| \, d\theta \right) \, d\nu(t)
$$

$$
\leq \kappa \log \|Q\|_{L^\infty(\{|z|=1\})} + \frac{1}{2\pi} \int_0^{2\pi} \left( \int \log |e^{i\theta} - t| \, d\nu(t) \right) \, d\theta.
$$

(23)

In the second last line we used a well known identity [15, Exercise 2.2, p. 29], and in the last line we used the fact that the sup norm of $Q$ on the unit circle is larger than 1. This is true because

$$
\frac{1}{2\pi} \int_0^{2\pi} \log Q(e^{i\theta}) \, d\theta = \sum_{j=1}^m w_j \frac{1}{2\pi} \int_0^{2\pi} \log |e^{i\tau_j} - e^{i\theta}| \, d\theta = 0,
$$

while $\log Q < 0$ in a neighborhood of each $\tau_j$, so that $\log Q(e^{i\theta}) > 0$ on a set of $\theta$ of positive measure. Substituting (23) into (22) gives

$$
\sum_{j=1}^m w_j \log P(e^{i\tau_j}) \leq \kappa \log \|Q\|_{L^\infty(\{|z|=1\})} + \frac{1}{2\pi} \int_0^{2\pi} \log \left| P(e^{i\theta}) \right| \, d\theta.
$$

□

**Proof of Theorem 1.3**

Note first that our choice of $\tau_0, \tau_{m+1}$ give

$$
\sum_{j=1}^m \frac{\tau_{j+1} - \tau_j - 1}{2} = 2\pi.
$$

It suffices to prove that for every $a \in \mathbb{C}$,

$$
\sum_{j=1}^m \frac{\tau_{j+1} - \tau_j - 1}{2} \log |e^{i\tau_j} - a| \leq \int_0^{2\pi} \log |e^{it} - a| \, dt + B\kappa^{-1}
$$

$$
= 2\pi \log^+ |a| + B\kappa^{-1}.
$$

(24)

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For, we can integrate this against the measure \( d\nu(a) \) that appears in the representation of \( P \in \mathbb{P}_\kappa \). Since
\[
\log |e^{ir} - a| = \log |e^{ir} - a^{-1}| + \log |a|
\]
for \( r \in \mathbb{R} \) and \( |a| < 1 \), we can assume that \(|a| \geq 1\). Moreover it is sufficient to prove (24) in the case \(|a| \geq 1 + \kappa^{-1}\). Indeed the case \(|a| \in [1, 1 + \kappa^{-1}]\) follows easily from the case \(|a| = 1 + \kappa^{-1}\), and the fact that the left-hand and right-hand sides in (24) increase as we increase \(|a|\), while keeping \( \arg(a) \) fixed. We may also assume that \( a \in [1 + \kappa^{-1}, \infty) \), simply rotate the unit circle. To prove (24), we use the integral form of the error for the trapezoidal rule [6, p. 288, (4.3.16)]: if \( f'' \) exists and is integrable in \([\alpha, \beta]\),
\[
\int_{\alpha}^{\beta} f(t) \, dt - \frac{\beta - \alpha}{2} (f(\alpha) + f(\beta)) = \frac{1}{2} \int_{\alpha}^{\beta} f''(t) (\alpha - t) (\beta - t) \, dt.
\]
From this we deduce that if \( f'' \) does not change sign on \([\alpha, \beta]\),
\[
\left| \int_{\alpha}^{\beta} f(t) \, dt - \frac{\beta - \alpha}{2} (f(\alpha) + f(\beta)) \right| \leq \frac{(\beta - \alpha)^2}{2} |f'(\beta) - f'(\alpha)|. \tag{25}
\]
Moreover, if \( f'' \) changes sign at most twice, then
\[
\left| \int_{\alpha}^{\beta} f(t) \, dt - \frac{\beta - \alpha}{2} (f(\alpha) + f(\beta)) \right| \leq 3 (\beta - \alpha)^2 \max_{t \in [\alpha, \beta]} |f'(t)|. \tag{26}
\]
Now let
\[
f(t) := \log |e^{it} - a|.
\]
Then
\[
f'(t) = \frac{a \sin t}{1 + a^2 - 2a \cos t} \quad \text{and} \quad f''(t) = \frac{-2a^2 + (1 + a^2) a \cos t}{(1 + a^2 - 2a \cos t)^2}.
\]
Elementary calculus shows that \(|f'| \) achieves its maximum on \([0, 2\pi]\) when \( \cos t = \frac{2a}{1 + a^2} \). Then \(|\sin t| = \frac{a^2 - 1}{a^2 + 1} \). Hence, as \( a \geq 1 + \kappa^{-1} \), and \( \kappa \geq 1 \),
\[
|f'(t)| \leq (a - a^{-1})^{-1} \leq \kappa \quad \text{for all } t \in \mathbb{R}. \tag{27}
\]
Also, since $f''$ has at most two zeros in the period, the total variation $V_0^{2\pi} f'$ on $[0,2\pi]$ satisfies
\[ V_0^{2\pi} f' \leq 6 \max_{[0,2\pi]} |f'| \leq 6\kappa. \] (28)

Now we apply (25) to (28) to the interval $[\alpha, \beta] = [\tau_{j-1}, \tau_j]$ and add over $j$. We also use our conventions on $\tau_{m+1}$ and $\tau_m$. Then
\[
\left| \int_0^{2\pi} f(t) \, dt - \sum_{j=1}^m \frac{\tau_{j+1} - \tau_{j-1}}{2} f(\tau_j) \right|
\leq \sum_{j=1}^m \left( \int_{\tau_{j-1}}^{\tau_j} f(t) \, dt - \frac{\tau_j - \tau_{j-1}}{2} [f(\tau_{j-1}) + f(\tau_j)] \right)
\leq \frac{1}{2} \delta^2 V_0^{2\pi} f'' + 6\delta^2 \kappa \leq 9A^2 \kappa^{-1}.
\]

So we have (24) with $B = 9A^2$. □

Proof of Theorem 1.4

We begin by recalling two facts about zeros of Littlewood and unimodular polynomials:

(I) $\exists c > 0$ such that every unimodular polynomial of degree $\leq n$ has at most $c\sqrt{n}$ real zeros [4].

(II) $\exists c > 0$ such that every Littlewood polynomial of degree $\leq n$ has at most $c \log^2 n / \log \log n$ zeros at 1 [5].

Now suppose that 1 is a zero of $f_p$ with multiplicity $m = m(p)$. By (I) or (II), $m = O(p^{1/2})$. Let
\[ h_m(z) = (z - 1)^m \]
and
\[ F_p(z) = f_p(z) / h_m(z). \]

Note that all coefficients of $F_p$ are integers (as $1/h_m(z)$ has Maclaurin series with integer coefficients), so $F_p(1)$ is a non-zero integer. Also $h_m$ is monic
and has all zeros on the unit circle, so its Mahler measure is 1. Then as Mahler measure is multiplicative,

\[ M_0(f_p) = M_0(F_p) M_0(h_m) = M_0(F_p). \]

Let \( z_p = \exp\left( \frac{2\pi i}{p} \right) \). The special case (3) of Theorem 1.2 gives

\[
M_0(f_p) \geq \frac{1}{2} \left( |F_p(1)| \prod_{k=1}^{p-1} \left| F_p\left( z_p^k \right) \right| \right)^{1/p} \\
\geq \frac{1}{2} \left( 1 \prod_{k=1}^{p-1} \left| \frac{f_p(z_p^k)}{(z_p^k - 1)^m} \right| \right)^{1/p}.
\]

It is known [2, Section 5] that for \( 1 \leq k \leq p - 1 \),

\[
f_p(z_p^k) = \sqrt{\left( \frac{-1}{p} \right)} p.
\]

Then

\[
M_0(f_p) \geq \frac{1}{2} \left( \frac{\sqrt{p^{p-1}}}{p^m} \right)^{1/p} = \frac{1}{2} \sqrt{p^{-(\frac{1}{2}+m)/p}}.
\]

Since \( m = O\left(p^{1/2}\right) \), the bound (16) follows for large \( p \). \( \square \)

References


