A Note on Orthogonal Dirichlet Polynomials with Rational Weight

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Communicated by S. De Marchi

Abstract

Let \( \lambda_j \) be a strictly increasing sequence of positive numbers with \( \lambda_1 > 0 \). We find an explicit formula for the orthogonal Dirichlet polynomials \( \{ \phi_n \} \) formed from linear combinations of \( \{ \lambda_j^{-it} \} \), associated with rational weights

\[
w(t) = \sum_{j=1}^{L} \frac{c_j}{\pi \left( 1 + (b_j t)^2 \right)},
\]

where \( 0 < b_1 < b_2 < \ldots \), and the \( \{ c_j \} \) are appropriately chosen. Only \( \lambda_j^{-it} \) appear in the formula. In the case \( L = 2 \), we show that the weight can always be taken positive in \( \mathbb{R} \).

Keywords: Dirichlet polynomials, orthogonal polynomials.

AMS Subject Class 2010: 42C05

1 Introduction

Throughout, let

\[
0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots \quad (1)
\]

Let \( L_n \) denote the set of Dirichlet polynomials

\[
\sum_{j=1}^{n} c_j \lambda_j^{-it}
\]

with complex coefficients \( \{ c_j \} \).

In a 2014 paper [5], we showed that

\[
\phi_n(t) = \frac{\lambda_n^{-1-it} - \lambda_{n-1}^{-1-it}}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} = \frac{-1}{\sqrt{\lambda_n^2 - \lambda_{n-1}^2}} \det \begin{bmatrix} \lambda_{n-1}^{-1-it} & \lambda_n^{-1-it} \\ \lambda_n^{-2-it} & \lambda_{n-1}^{-2-it} \end{bmatrix}
\]

is the \( n \)th orthogonal Dirichlet polynomial for the arctan density, that is

\[
\int_{-\infty}^{\infty} \phi_n(t) \phi_m(t) \frac{dt}{\pi(1 + t^2)} = \delta_{mn}, \quad m, n \geq 1. \quad (2)
\]

We also estimated the Christoffel functions, convergence of associated orthonormal expansions, and universality limits. These orthonormal polynomials have been applied and provided in a variety of questions by Weber and Dimitrov as well as the author [4], [6], [8], [10], [11], [12]. In a follow up paper [7], the author considered orthogonal Dirichlet polynomials for the Laguerre weight, though it turned out that much of the material there was already subsumed by Müntz orthogonal polynomials [3].

In this note, we consider rational densities

\[
w(t) = \sum_{m=1}^{L} \frac{c_m}{\pi \left( 1 + (b_m t)^2 \right)} \quad (3)
\]

with appropriately chosen \( \{ c_j \} \). Here \( L \geq 1 \), and

\[
1 = b_1 < b_2 < \ldots < b_L. \quad (4)
\]

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Define, for $n \geq L$,
\[
\psi_n(t) = \det \begin{bmatrix}
\lambda_n^{-1} & \lambda_{n-1}^{-1} & \cdots & \lambda_1^{-1} & \lambda_0^{-1} \\
\lambda_n^{-2} & \lambda_{n-1}^{-2} & \cdots & \lambda_1^{-2} & \lambda_0^{-2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\lambda_n^{-L} & \lambda_{n-1}^{-L} & \cdots & \lambda_1^{-L} & \lambda_0^{-L} \\
\lambda_n^{-L+1} & \lambda_{n-1}^{-L+1} & \cdots & \lambda_1^{-L+1} & \lambda_0^{-L+1}
\end{bmatrix}.
\] (5)

Observe that $\psi_n(t)$ is a linear combination of only $\lambda_j^{-it}$ for $j \leq n$. Also define for a given fixed $n$, and $j \geq 1$, $1 \leq m \leq L$,
\[
d_{jm} = \int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda_j^{it}}{\pi \left(1 + (b_m t)^2\right)} \, dt
\] (6)

and let $B$ be the $(L-1) \times L$ matrix
\[
B = \begin{bmatrix}
d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\
d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\
\vdots & \vdots & \ddots & \vdots \\
d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L}
\end{bmatrix}
\] (7)

and
\[
D = \det \begin{bmatrix}
d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\
d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\
\vdots & \vdots & \ddots & \vdots \\
d_n,1 & d_n,2 & \cdots & d_n,L
\end{bmatrix}
\] (8)

**Theorem 1**

Let $n \geq L \geq 1$. Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n$ and $\psi_n$ be given by (5).

(a) Let $c = [c_1 c_2 \ldots c_L]^T$ be taken as any non-trivial solution of $Bc = 0$. Let
\[
w(t) = \sum_{m=1}^{L} c_m \frac{\psi_n(t)}{\pi \left(1 + (b_m t)^2\right)}.
\] (9)

Then for $1 \leq j \leq n-1$,
\[
\int_{-\infty}^{\infty} \psi_n(t) \lambda_j^{it} w(t) \, dt = 0.
\] (10)

(b) If $D$ defined by (8) is non-zero, then we can take
\[
w(t) = A \det \begin{bmatrix}
d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\
d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1 & \alpha_2 & \cdots & \alpha_L
\end{bmatrix},
\] (11)

for any $A \neq 0$, while
\[
\int_{-\infty}^{\infty} \psi_n(t) \lambda_n^{it} w(t) \, dt = AD.
\] (12)

(c)
\[
\psi_n(t) = \sum_{j=1}^{n} a_j \lambda_j^{it}
\] (13)

where for $n-L \leq j \leq n$,
\[
a_j (-1)^{n+1} > 0.
\] (14)

**Remarks**

(a) Note that as $\left\{ \frac{1}{\pi \left(1 + (b_m t)^2\right)} \right\}_{m=1}^{L}$ are linearly independent, $w$ above is not identically 0. As an even rational function with numerator degree at most $2L - 2$ and denominator degree $2L$, $w$ has at most $L-1$ sign changes in $(0, \infty)$. It seems to be an interesting problem to investigate the positivity of $w$.

(b) In addition to the orthogonality relation above, we note that for any $1 \leq m \leq L$, and $0 < \lambda < \lambda_{n-L}$,
\[
\int_{-\infty}^{\infty} \psi_n(t) \frac{\lambda^{it}}{\pi \left(1 + (b_m t)^2\right)} \, dt = 0.
\]
This does not require anything of the \( \{ c_i \} \) above.

In the case \( L = 2 \), we can prove positivity of the weight:

**Theorem 2**

Assume the notation of Theorem 1 with \( L = 2 \). Then we can choose \( c_1 < 0 < c_2 \) such that if

\[
w(t) = \sum_{k=1}^{2} c_k \frac{1}{\pi(1 + (bt)^2)}
\]

then

\[
w(t) > 0, \ t \in \mathbb{R},
\]

and \( w \) is given by the determinant (11), with

\[
A = \frac{c_2}{d_{n-1,1}} < 0.
\]

**Remark**

In the proof of Theorem 2, we show that one can take

\[
c_1 = -c_2 \frac{g \left( \frac{1}{c_1} \right)}{g \left( \frac{1}{c_2} \right)}
\]

where

\[
g(s) = s \left[ \left( \frac{\lambda_{n-2}}{\lambda_{n-1}} \right)^s - \left( \frac{\lambda_{n-2}}{\lambda_{n-1}} \right)^{-s} \right].
\]

We prove the theorems in the next section.

## 2 Proofs

**Proof of Theorem 1**

(a) We use the following simple consequence of the residue theorem: for real \( \mu \),

\[
\int_{-\infty}^{\infty} e^{\mu t} \frac{1}{\pi(1 + t^2)} \ dt = e^{-|\mu|}.
\]

(15)

Then if \( 0 < \lambda \leq \lambda_{n-L} \), and \( n - L \leq k \leq n \),

\[
\int_{-\infty}^{\infty} \frac{(\lambda/\lambda_k)^s}{\pi(1 + (b_n t)^2)} \ dt = \frac{1}{b_n} \int_{-\infty}^{\infty} e^{(\lambda/\lambda_k) \log(\lambda/\lambda_k)} \ dt = \frac{1}{b_n} \left( \frac{\lambda}{\lambda_k} \right)^{1/2n}.
\]

Then for such \( \lambda \),

\[
\int_{-\infty}^{\infty} \psi_n(t) \left( \frac{\lambda}{\pi(1 + (b_n t)^2)} \right)^s \ dt = 0,
\]

by taking \( \frac{1}{b_n} \lambda^{1/2n} \) times row \( m + 1 \) from the first row. So we have the orthogonality relation (10) for \( \lambda = \lambda_j \), all \( j \leq n - L \). Next, the equations

\[
\int_{-\infty}^{\infty} \psi_n(t) \lambda^{1/2n} w(t) \ dt = 0, 1 \leq j \leq L - 1
\]

are equivalent to (recall (3) and (6))

\[
\sum_{n=1}^{L} c_n d_{n-L+j,n} = \sum_{n=1}^{L} c_n \int_{-\infty}^{\infty} \psi_n(t) \left( \frac{\lambda^{1/2n}}{\pi(1 + (b_n t)^2)} \right) \ dt = 0, 1 \leq j \leq L - 1
\]
which in turn is equivalent to $Bc = 0$, recall (7). This is a system of $L - 1$ homogeneous linear equations in $L$ variables, so there is a non-trivial solution for $c$.

(b) First observe that $w$ defined by (11) is indeed a linear combination of $\{\frac{1}{\pi(1 + (b_k t)^2)}\}_{j=1}^L$. Next, we see from (11) that

$$
\int_{-\infty}^{\infty} \psi_n(t) \lambda_n^i w(t) \, dt = A \det \begin{bmatrix}
    d_{n-L+1,1} & d_{n-L+1,2} & \cdots & d_{n-L+1,L} \\
    d_{n-L+2,1} & d_{n-L+2,2} & \cdots & d_{n-L+2,L} \\
    \vdots & \vdots & \ddots & \vdots \\
    d_{n-1,1} & d_{n-1,2} & \cdots & d_{n-1,L} \\
    d_{k,1} & d_{k,2} & \cdots & d_{k,L}
\end{bmatrix} = 0,
$$

if $n - L + 1 \leq k \leq n - 1$. If $k = n$, we instead obtain the non-0 number $A$. It also then follows that $w$ cannot be the zero function.

(c) Let $E$ be the $L \times (L + 1)$ matrix

$$
E = \begin{bmatrix}
    \lambda_n^{-1/b_1} & \lambda_n^{-1/b_2} & \cdots & \lambda_n^{-1/b_L} & \lambda_n^{-1/b_1} \\
    \vdots & \vdots & \ddots & \vdots & \vdots \\
    \lambda_n^{-1/b_{L-1}} & \lambda_n^{-1/b_L} & \cdots & \lambda_n^{-1/b_1} & \lambda_n^{-1/b_1} \\
    \lambda_n^{-1/b_1} & \lambda_n^{-1/b_2} & \cdots & \lambda_n^{-1/b_L} & \lambda_n^{-1/b_1} \\
    \lambda_n^{-1/b_1} & \lambda_n^{-1/b_2} & \cdots & \lambda_n^{-1/b_L} & \lambda_n^{-1/b_1} \\
\end{bmatrix}.
$$

Thus $E$ consists of the last $L$ rows of the matrix used to define $\psi_n$. For $1 \leq k \leq L + 1$, let $E(k)$ denote the $L \times L$ matrix obtained from $E$ by deleting its $k$th column. Then with the notation (13), we see that

$$
\alpha_k = (-1)^{j-n+1} \det(E(j-n+L+1)).
$$

To show that each $\det(E(k)) > 0$, we use the fact that the kernel $K(s, t) = e^{it}$ is totally positive for $s, t \in \mathbb{R}$ [1, p. 212] or [9]. If we set $s_j = -\frac{1}{b_j}$, while $t_i = \log \lambda_{n-L+i-1}$, then $s_1 < s_2 < \ldots < s_L$ and $t_1 < t_2 < \ldots < t_L$, then

$$
\det(E(k)) = \det \begin{bmatrix}
    K(s_1, t_1) & \cdots & K(s_1, t_{k+1}) & K(s_1, t_{k+1}) & \cdots & K(s_1, t_{L+1}) \\
    K(s_2, t_1) & \cdots & K(s_2, t_{k+1}) & K(s_2, t_{k+1}) & \cdots & K(s_2, t_{L+1}) \\
    \vdots & \ddots & \cdots & \vdots & \cdots & \vdots \\
    K(s_L, t_1) & \cdots & K(s_L, t_{k+1}) & K(s_L, t_{k+1}) & \cdots & K(s_L, t_{L+1}) \\
\end{bmatrix} > 0.
$$

**Proof of Theorem 2**

From (5) for $L = 2$,

$$
\psi_n(t) = \det \begin{bmatrix}
    \lambda_n^{-it} & \lambda_n^{-it} \\
    \lambda_n^{-1/b_2} & \lambda_n^{-1/b_2} \\
    \lambda_n^{-1/b_1} & \lambda_n^{-1/b_1} \\
\end{bmatrix}.
$$

Let

$$
w(t) = \sum_{k=1}^{2} \frac{c_k}{\pi(1 + (b_k t)^2)}
$$

where for the moment we do not specify the choice of $c_1, c_2$. Then we already have for $k = 1, 2, \ldots, n - 2$,

$$
\int_{-\infty}^{\infty} \psi_n(t) \lambda_n^i w(t) \, dt = 0
$$

no matter what is the choice of $c_1, c_2$ - as follows from the proof of Theorem 1(a). So let us investigate the remaining condition in (10), namely

$$
\int_{-\infty}^{\infty} \psi_n(t) \lambda_n^{-it} w(t) \, dt = 0.
$$

This is equivalent to

$$
0 = \sum_{k=1}^{2} c_k \int_{-\infty}^{\infty} \psi_n(t) \lambda_n^{-it} \frac{dt}{\pi(1 + (b_k t)^2)} = c_1 d_{n-1,1} + c_2 d_{n-1,2}.
$$

(17)
Now for $k = 1, 2$, we see from the determinant expression (16) and then from (15) that

$$d_{n-1,k} = \frac{1}{b_k} \det \left[ \begin{array}{ccc} \frac{1}{b_k} & \frac{1}{b_k} & \frac{1}{b_k} \\ \frac{1}{b_k} & \frac{1}{b_k} & \frac{1}{b_k} \\ \frac{1}{b_k} & \frac{1}{b_k} & \frac{1}{b_k} \end{array} \right]$$

$$= \frac{1}{b_k} \lambda_{n-1}^{-1/b_k} \lambda_n^{-1/b_k} \lambda_{n-2}^{-1/b_k}$$

$$= \frac{1}{b_k} \lambda_{n-1}^{-1/b_k} \lambda_n^{-1/b_k} \lambda_{n-2}^{-1/b_k}$$

$$= \frac{1}{b_k} \lambda_{n-1}^{-1/b_k} \lambda_n^{-1/b_k} \lambda_{n-2}^{-1/b_k}$$

$$= \frac{1}{b_k} \lambda_{n-1}^{-1/b_k} \lambda_n^{-1/b_k} \lambda_{n-2}^{-1/b_k}$$

$$= \frac{1}{b_k} \left( \frac{\lambda_{n-1}}{\lambda_n} \right)^{1/b_k} \left( \frac{\lambda_{n-2}}{\lambda_n} \right)^{1/b_k} \left( \frac{\lambda_{n-1}}{\lambda_n} \right)^{1/b_k}$$

as $\frac{\lambda_{n-1}}{\lambda_n} \in (0, 1)$, $\frac{1}{b_k} - \frac{1}{b_k} > 0$, and

$$= \lambda_{n-1}^{-1/b_k} \lambda_n^{-1/b_k} \left[ 1 - \left( \frac{\lambda_{n-1}}{\lambda_n} \right)^{1/b_k} \right] > 0.$$

In summary,

$$d_{n-1,k} < 0, k = 1, 2.$$  (19)

Next, let $r = \frac{\lambda_{n-1}}{\lambda_n} \in (0, 1)$, and

$$g(s) = s [r^s - r^{-s}].$$

From (18) and (17) and cancelling a common factor of $\lambda_{n-1}^{-1/b_k} \lambda_n^{-1/b_k} - \lambda_{n-1}^{-1/b_k} \lambda_{n-2}^{-1/b_k}$, we have

$$c_1 g \left( \frac{1}{b_1} \right) + c_2 g \left( \frac{1}{b_2} \right) = 0.$$  (20)

Here

$$g'(s) = (r^s - r^{-s}) + (s \ln r)(r^s + r^{-s}) < 0,$$

as $r = \frac{\lambda_{n-1}}{\lambda_n} < 1$ so $\ln r < 0$. Then $g$ is decreasing and negative, and

$$0 > g \left( \frac{1}{b_2} \right) > g \left( \frac{1}{b_1} \right)$$

so (20) gives

$$c_1 = -c_2 \frac{g \left( \frac{1}{b_2} \right)}{g \left( \frac{1}{b_1} \right)} \text{ and } |c_1| < |c_2|. $$  (21)

To ensure that $w(0) = \frac{1}{n} (c_1 + c_2) > 0$, we then need to choose $c_1 < 0 < c_2$. To ensure that $w(t) > 0$ for all $t$, we need for all such $t$,

$$|c_1| \leq c_2 \frac{1 + (b_1 t)^2}{1 + (b_2 t)^2}.$$  

As

$$\min_{t \in \mathbb{R}} \frac{1 + (b_1 t)^2}{1 + (b_2 t)^2} = \left( \frac{b_1}{b_2} \right)^2,$$

this is equivalent to

$$g \left( \frac{1}{b_2} \right) \leq \left( \frac{b_1}{b_2} \right)^2.$$
that is, (recall $g < 0$),

$$b_2 \left[ r^{-1/b_2} - r^{1/b_2} \right] \leq b_1 \left[ r^{-1/b_1} - r^{1/b_1} \right].$$

Now let

$$h(s) = \frac{1}{s} \left[ r^s - r^t \right],$$

so that we want

$$h \left( \frac{1}{b_2} \right) \leq h \left( \frac{1}{b_1} \right).$$

This would be true if $h$ is increasing over the range $\left[ \frac{1}{b_2}, \frac{1}{b_1} \right]$. Now

$$h'(s) = -\frac{1}{s^2} \left[ r^s - r^t \right] - \frac{1}{s} \left( \ln r \right) \left[ r^s + r^t \right]$$

$$= -\frac{r^s}{s} \left[ 1 - r^{2s} + \frac{1}{2} \left( \ln r \right) \left[ 1 + r^2 \right] \right] = -\frac{r^s}{s} G(x)$$

where

$$x(s) = r^{2s} \in (0, 1)$$

decreases as $s$ increases

and

$$G(x) = 1 - x + \frac{1}{2} \left( \ln x \right) (1 + x).$$

Here $G(0+) = -\infty$ and $G(1) = 0$ while for $x \in (0, 1)$,

$$G'(x) = -\frac{1}{2} + \frac{1}{2x} + \frac{1}{2} \ln x$$

$$\Rightarrow \ \ G''(x) = \frac{1}{2x} \left( 1 - \frac{1}{x} \right) < 0.$$

Thus $G$ is concave in $(0, 1)$ and $G'$ is a decreasing function of $x$ with $G'(0+) = \infty$ and $G'(1) = 0 = G(1)$. It follows that $G'(x) > 0$ for $x \in (0, 1)$, so

$$G(x) < G(1) = 0 \ \text{for} \ x \in (0, 1).$$

So, indeed,

$$h'(s) = -\frac{r^s}{s^2} G(x) > 0 \ \text{for} \ s > 0,$$

and as desired, we have (22). Then with $c_1$ and $c_2$ given by (21), and $c_2 > 0$, we do have

$$w(t) > 0, \ t \in (-\infty, \infty).$$

It remains to show that this $w$ is also given by (11) with $L = 2$. We know that $c_1, c_2$ are non-0 so

\[
\det \begin{bmatrix}
d_{n-1,1} & d_{n-1,2} \\
d_{n-1,1} & d_{n-1,2} + \frac{1}{c_2} w(t)
\end{bmatrix}
= \det \begin{bmatrix}
d_{n-1,1} & d_{n-1,2} \\
d_{n-1,1} & d_{n-1,2} + \frac{1}{c_2} \frac{1}{1+t^2} w(t)
\end{bmatrix}
= \det \begin{bmatrix}
d_{n-1,1} & 0 \\
d_{n-1,1} & \frac{1}{c_2} w(t)
\end{bmatrix}
= \frac{d_{n-1,1}}{c_2} w(t).
\]

Thus the determinant is of one sign. Choosing $A = -\frac{c_2}{d_{n-1,1}} < 0$ gives the result. ■

References