The evolution of subcritical Achlioptas processes

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Joint work with Oliver Riordan
Achlioptas processes

- Start with an empty graph on $n$ vertices
- In each step: pick two random edges, add one of them to the graph (using some rule)

Remarks

- Yields family of random graph processes
- Contains ‘classical’ Erdős–Rényi process

Motivation

- Improve our understanding of phase transition phenomena
- Test / develop methods for analyzing processes with dependencies
Key example (suggested by Achlioptas)

Fraction of vertices in largest component after $tn$ steps: $L_1(tn)/n$

Goal of this talk

Understand how these evolve over time.
Widely studied Achlioptas rules

Size rules

Decision (which edge to add) depends *only* on component sizes $c_1, \ldots, c_4$

- Product rule (Bollobás)

Bounded-size rules (most of previous work)

All component sizes larger than some constant $B$ are treated the same

- Erdős–Rényi ($B = 0$)
- Bohman–Frieze ($B = 1$)
Previous work

Bounded-size rules (Spencer–Wormald, Bohman–Kravitz, Riordan–W., . . . )

For any bounded-size rule $\mathcal{R}$ the key statistics are convergent:

- **Small components**: $N_k(tn) \sim \varphi_k^\mathcal{R}(t)n$
- **Largest component**: $L_1(tn) \sim \varphi^\mathcal{R}(t)n$

Proofs use Wormald’s differential equation method

- $\varphi_k^\mathcal{R}$ determined by an associated system of DEs

Size rules (Riordan–W.)

For any ‘sensible’ size rule $\mathcal{R}$ the key statistics are convergent
IF an associated system of differential equations has a *unique* solution.

- ‘Direct’ proof
- Caveat: unique solution
  - Well-known for ‘nice’ size rules (e.g., bounded-size rules)
  - Open problem for general size rules (e.g., product rule)
**New Result for Size Rules**

**Susceptibility** \( \chi(G) = \frac{1}{n} \sum_{k \geq 1} kN_k(G) \)
- Expected size of component containing randomly selected vertex

**Riordan–W. (simplified)**

Any size rule \( \mathcal{R} \) is ‘well-behaved’ until the critical time \( t_c = t_c^\mathcal{R} \), where the susceptibility \( \chi \) diverges. For \( t < t_c \) whp

- **Small components**: \( N_k(tn) \sim \varphi_k^\mathcal{R}(t)n \)
- **Exponential tails**: \( N_k(tn) \leq Ae^{-ak}n \)
- **Largest component**: \( L_1(tn) \leq B \log n \)

**Remarks**
- Removes earlier uniqueness assumption up to \( t_c \)
- \( t_c \) is important time in evolution
Critical time $t_c$

Conjecture for size rules (simplified)

For $t > t_c$ we have a giant component: $L_1(tn) = \Omega(n)$

- Was known for bounded-size rules (Spencer–Wormald)
- True for certain classes of size rules (e.g., maximum product rule)

Convergence up to $t_c$ seems best possible

Beyond $t_c$ some rules look *nonconvergent* in simulations
Inductively establish concentration

Need: evolution starting from initial graph $F$
- Assumption: initial graph $F$ is ‘nice’
- Conclusion: graph after $\sigma n$ steps is again ‘nice’ (if $\sigma$ small enough)

In comparison to bounded size rules
- We track key statistics \textit{without} using differential equations
- We \textit{investigate dependencies} among choices in more detail
For size rules, decisions can only propagate inside clusters.

Here we ignore order of pairs.

Inside each cluster:

- Order of the pairs uniquely determines decisions of any size rule.
**Glimpse of the proof**

Determine component size $|C_v|$ via two-step exposure

- Reveal all *pairs* of edges offered
  - Determine relevant *cluster* for $v$ \( \approx \) Branching process
- Reveal *order* of all (relevant) *pairs*
  - Apply size rule $R$ inside *cluster*

**Why do we need susceptibility $\chi < \infty$?**

- Branching process must be ‘sub-critical’ (need $\sigma \leq c\chi^{-1}$)
- Only ‘few’ edges/components influence $|C_v|$ \( \rightarrow \) Concentration
First rigorous result for size rules (Riordan–W.)

Key statistics are ‘well-behaved’ until the susceptibility $\chi$ diverges.

Open problem

How can we analyze the later evolution of size rules?