# When does the $K_4$-free process stop?

Lutz Warnke
University of Oxford

(Pre-)Doc-Course: Probabilistic & Enumerative Combinatorics

# A related Process

**Random graph process**

(a) Start with empty graph on $n$ vertices

(b) Add edges, one at a time, chosen uniformly at random from all remaining pairs

**Well-understood:**

- Gives uniform distribution (after inserting $i$ edges)

# The Problem

**Random $H$-free graph process**

(a) Start with empty graph on $n$ vertices

(b) Add edges, one at a time, chosen uniformly at random from all remaining pairs that do not complete a copy of $H$

**Observation:**

- ends with maximal $H$-free graph on $n$ vertices

**Basic question: Erdős-Suen-Winkler (1995)**

Typical final number of edges?

# Motivation for the $H$-free process

## Motivation 1: Random graphs with structural constraints

- understanding typical properties of such random objects
  - e.g. the degrees, or the number of small subgraphs

## Motivation 2: Extremal Combinatorics

- analysis of $H$-free process gives new results
  - e.g. lower bounds for the off-diagonal Ramsey numbers $R(s, t)$

## Motivation 3: Developement of tools/methods

- many standard tools/methods require ‘lots of independence’

# Previous Results: 1/2

**Final number of edges of the $H$-free process**

**For 'many' $H$ up to log-factors:**

- Osthus-Taraz, 2001
- Bohman and Keevash, 2009

**Exact order of magnitude only for $H = K_{1,d+1}$ and $H = K_3$:**

- Ruciński and Wormald, 1992
- Bohman, 2008

# Previous Results: 2/2

## Final number of edges for $H = K_4$

**At least $\Omega(n^{8/5}\sqrt[5]{\log n})$ and at most $O(n^{8/5}\sqrt{\log n})$ edges:**

- Bollobás-Riordan, 2000
- Osthus-Taraz, 2001
- Bohman, 2008

## Conjecture (Bohman-Keevash, 2009)

**For special case $H = K_4$ their conjecture implies:**

- final number of edges: $\Theta(n^{8/5}\sqrt[5]{\log n})$
- maximum degree: $O(n^{3/5}\sqrt[5]{\log n})$

# Our Result: maxdegree in $K_4$-free process

**Main result (W., 2010+)**

There exists $C > 0$ such that whp the maximum-degree is at most $Cn^{3/5}\sqrt[5]{\log n}$ in the $K_4$-free process

**Best possible up to the constant:**

- the minimum degree is at least $cn^{3/5}\sqrt[5]{\log n}$ whp (Bohman-Keevash)

**Contribution:**

(a) proves a conjecture of Bohman and Keevash for $H = K_4$

(b) allows us to answer a question of Erdős-Suen-Winkler for $H = K_4$:

- whp final number of edges is $\Theta(n^{8/5}\sqrt[5]{\log n})$

# Summary: the $K_4$-free process

**Final number of edges of the $H$-free process (Previous results)**

- order of magnitude was only known for $H = K_{1,d+1}$ and $H = K_3$

**Typical structural properties of the $K_4$-free proces**

- final number of edges: $\Theta(n^{8/5}\sqrt[5]{\log n})$
- maximum degree: $O(n^{3/5}\sqrt[5]{\log n})$