Note on Sunflowers

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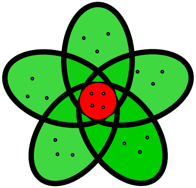
Based on 2020 REU at Georgia Tech:
https://arxiv.org/abs/2009.09327

- Let $\mathcal{F}$ be a k-**uniform** family of subsets of X, i.e., $|S| = k$ and $S \subseteq X$ for all $S \in \mathcal{F}$
- $\mathcal{F}$ is a **sunflower with p petals** if $|\mathcal{F}| = p$ and there exists $Y \subseteq X$ with $Y = S_i \cap S_j$ for all distinct $S_i, S_j \in \mathcal{F}$
- Y is the **core** and $S_i \setminus Y$ are the **petals**
- Note that p disjoint sets forms a sunflower with p petals and empty core.



Sunflower with $k = 7$ and $p = 5$

Applications

Sunflowers have many uses in computer science:

- Fast algorithms for matrix multiplication
- Cryptography
- Pseudorandomness
- Lower bounds on circuitry
- Data structure efficiency
- Random approximations

Research Question

What is the smallest $r = r(p, k)$ such that every k-uniform family with at least r^k sets must contain a sunflower with p petals?

Erdős–Rado (1960)

(a) $r = pk$ is **sufficient** to guarantee a sunflower:
every family with more than $(pk)^k > k!(p-1)^k$ sets contains a sunflower

(b) $r > p - 1$ is **necessary** to guarantee a sunflower:
there is a family of $(p-1)^k$ sets without a sunflower

- Erdős conjectured $r = r(p)$ is sufficient (no k dependency), offered $1000 reward
- Until 2018, best known upper bound on r was still $k^{1-o(1)}$ with respect to k

"[The sunflower problem] has fascinated me greatly – I really do not see why this question is so difficult."
–Paul Erdős (1981)

Recent Exciting Developments

- Erdős conjectured $r = r(p)$ is sufficient (no k dependency)
- Until 2018, best known upper bound on r was still $k^{1-o(1)}$ with respect to k

Alweiss–Lovett–Wu–Zhang (Breakthrough Aug 2019)

$r = p^3(\log k)^{1+o(1)}$ is sufficient to guarantee a sunflower

New papers built off their breakthrough ideas:

- Sep 2019: *Rao* used Shannon's coding theorem for a cleaner proof and slightly better bound
- Oct 2019: *Frankston–Kahn–Narayanan–Park* improved a key lemma of ALWZ, enabling them to prove a conjecture of Talagrand regarding thresholds functions
- Jan 2020: *Rao* improved to $r = O(p \log(pk))$ by incorporating ideas from FKNP
- July 2020: *Tao* matched Rao's bound with shorter proof using Shannon entropy

Our Results (REU 2020)

Rao (Jan 2020)

$r = O(p\log(pk))$ is sufficient to guarantee a sunflower

Bell–Chueluecha–Warnke (September 2020)

$r = O(p\log k)$ is sufficient to guarantee a sunflower

Further REU 2020 results:

- *Rao/Tao methods not needed for this result:*
  2019 Frankston–Kahn–Narayanan–Park result suffices with our proof variant
- *Main Technical Lemma is asymptotically sharp:*
  Bound cannot be improved further without change of proof strategy

Strategy: Reduction to r-spread Families

- *Key Definition*: $\mathcal{F}$ is **r-spread** if $|\mathcal{F}| \geq r^k$ and for every nonempty $S \subseteq X$ the number of sets in $\mathcal{F}$ which contain S is at most $r^{k-|S|}$

The Inductive Reduction

If every r-spread family contains p disjoint sets, then r^k sets guarantees a sunflower.

Proof. Induction on k.

Question: How to *find* p disjoint sets in an r-spread family?

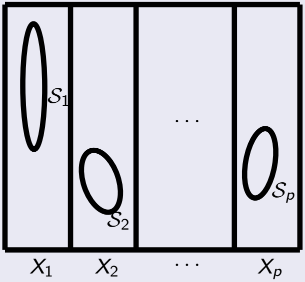
- We now review the common proof framework of previous work.

Strategy: Reduction to Main Technical Lemma

Question: How to find p disjoint sets?

The Probabilistic Method

1. Consider a random partition of X to $X_1, X_2, \ldots, X_p$ ($x \in X$ goes in random X_i)
2. Use **probabilistic method**
   - Show $\mathbb{P}(\nexists\ S_i \in \mathcal{F}$ such that $S_i \subseteq X_i) < \frac{1}{p}$
   - *Union bound*: $\mathbb{P}(\exists$ i where X_i has no $S_i) < p \cdot \frac{1}{p} = 1$
   - There is partition where each X_i has S_i
   - Then $S_1, \ldots, S_p$ are disjoint sets in $\mathcal{F}$



Main Technical Lemma (Rao 2020)

Let X_p be set where $\forall\ x \in X$, $x \in X_p$ w.p. $\frac{1}{p}$ independently. $\exists\ C > 1$ s.t. for $r \geq Cp\log(pk)$, $\mathbb{P}$(There <u>does not</u> exist $S_i \in \mathcal{F}$ such that $S_i \subseteq X_p) < \frac{1}{p}$

Our Probabilistic Improvement

Main Technical Lemma (Rao 2020)

Let X_a be set where $\forall\ x \in X$, $x \in X_a$ w.p. $\frac{1}{a}$ independently. $\exists\ C > 1$ s.t. for $r \geq Ca \log(bk)$, $\mathbb{P}$(There <u>does not</u> exist $S_i \in \mathcal{F}$ such that $S_i \subseteq X_a$) $< \frac{1}{b}$

Bell–Chueluecha–Warnke (September 2020)

$r = O(p \log k)$ is sufficient to guarantee a sunflower

Proof Sketch (improve union bound via linearity of expectation):

- Partition $X_1, \cdots, X_{2p}$ instead of $X_1, \cdots, X_p$.
- To get p disjoint sets, half of our sets need to contain a set in $\mathcal{F}$
- *Linearity of expectation*: if each X_i has less than half chance of failure, there is some partition where at least half succeed
- Apply main lemma with $a = 2p$, $b = 2$.
- $r = 2Cp \log(2k) = O(p \log k)$ suffices!

Summary

$\mathcal{F}$, a k-uniform family of subsets of X, is a **sunflower with p petals** if $|\mathcal{F}| = p$ and there exists $Y \subseteq X$ with $Y = S_i \cap S_j$ for all distinct $S_i, S_j \in \mathcal{F}$.

Research Question

What is the smallest $r = r(p, k)$ such that every k-uniform family with at least r^k sets must contain a sunflower with p petals?

- *Erdős–Rado* (1960): $r = pk$ is sufficient and $r > p - 1$ is necessary
- *Erdős* (1981): Conjectured $r = r(p)$ sufficient
- Alweiss–Lovett–Wu–Zhang (2019): Breakthrough that $r = p^3(\log k)^{1+o(1)}$ suffices
- *Rao* (2020): By Shannon's Coding Theorem, $r = O(p \log(pk))$ suffices

Bell–Chueluecha–Warnke (2020)

- $r = O(p \log k)$ suffices by minor variant of existing probabilistic arguments
- This bound cannot be improved without change of strategy

References

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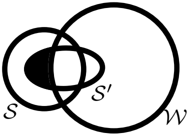
- Bell–Chueluecha–Warnke (2020). *Note on Sunflowers.* Discrete Mathematics. Preprint at arXiv:2009.09327

Main Technical Lemma

$\mathbb{P}$(There <u>does not</u> exist $S_i \in \mathcal{F}$ such that $S_i \subseteq X_a$) $< \frac{1}{b}$

<u>Proof.</u> Partition X_i to V_1, V_2 with equal size, so $|V_1| = |V_2| = |X|/(2a)$.

- *Key Definition*: Given $S \in \mathcal{F}$ and $W \subseteq X$, (S, W) is **m-good** if there exists $S' \in \mathcal{F}$ such that $S' \subseteq W \cup S$ and $|S' \setminus W| \leqslant m$

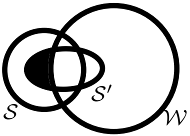


<u>Iteration</u>: $\mathbb{P}$(Less than half of sets in $\mathcal{F}$ are m-good with respect to V_1) $\leqslant \frac{1}{2b}$

<u>Final Step</u>: $\mathbb{P}(V_1 \cup V_2$ does not contain a set in $\mathcal{F}$ | successful iteration$) < \frac{1}{2b}$

Strategy: Iteration + Janson

- *Key Definition*: Given $S \in \mathcal{F}$ and $W \subseteq X$, (S, W) is **m-good** if there exists $S' \in \mathcal{F}$ such that $S' \subseteq W \cup S$ and $|S' \setminus W| \leqslant m$



Iteration: $\mathbb{P}$(Less than half of sets in $\mathcal{F}$ are m-good with respect to V_1) $\leqslant \frac{1}{2b}$

- Partition V_1 to $W_1, W_2, \ldots, W_x$ with equal size
- Iteratively replace each good $(S, \bigcup_{1 \le i \le j} W_i)$ pair with the guaranteed S'
- Bound the number of bad pairs by a key counting lemma & Markov's inequality
- Moving from S to S' reduces the set sizes at each step as $\bigcup_{1 \le i \le j} W_i$ expands

Final Step: $\mathbb{P}(V_1 \cup V_2$ does not contain a set in $\mathcal{F}$ | successful iteration$) < \frac{1}{2b}$

- Construct an m-uniform $\mathcal{F}'$ from sets in $\mathcal{F}$ which are m-good with respect to V_1
- Apply Janson's Inequality with V_2 and $\mathcal{F}'$ to bound $\mathbb{P}(\exists\ S \in \mathcal{F}'$ s.t. $S \subseteq V_2)$