The Secret Life of Graphs

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I will attempt to retrace the rather non-linear development of my mathematical career, which I would characterize as a deterministic simulation of a random walk.
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In my thesis, I studied effective versions of the Manin–Mumford conjecture for modular curves. Specifically, I computed the intersection of \( X_0(p)(\mathbb{Q}) \) with \( J_0(p)(\mathbb{Q})^{\text{tors}} \).

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Barry Mazur suggested looking at points of small canonical height on $X_0(p)$.

Mazur’s suggestion was motivated by the Bogomolov conjecture, proved by Ullmo and Zhang:

**Theorem (Ullmo, 1998)**

Let $K$ be a number field and let $X/K$ be a smooth proper algebraic curve of genus at least 2 embedded in its Jacobian $J$. Let $\hat{h} : J(K) \to \mathbb{R}_{\geq 0}$ be the canonical height associated to the theta divisor. Then there exists $\varepsilon > 0$ such that

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Equidistribution of small points

The proof of the Bogomolov conjecture is based on an equidistribution theorem for small points whose proof relies on Arakelov intersection theory.

- Let $A/K$ be an abelian variety over a number field $K$, and let $\{P_n\}$ be a sequence of points in $A(\bar{K})$.
- We say $\{P_n\}$ is small if $\hat{h}(P_n) \to 0$ and generic if no subsequence is contained in a proper subvariety of $A$.
- Let $\delta_n$ denote the discrete probability measure on $A(\mathbb{C})$ supported equally on the Galois conjugates of $P_n$.

**Theorem (Szpiro-Ullmo-Zhang, 1997)**

*If the sequence $\{P_n\}$ is generic and small, then $\delta_n$ converges weakly to the unit Haar measure on $A(\mathbb{C})$.***
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- Tonghai Yang and I read the Gross–Zagier paper together.
- I read various papers related to the Bogomolov conjecture, including Shouwu Zhang’s paper ”Admissible Pairing on a Curve”.
- I taught a topics course on heights (local decomposition of global heights, equidistribution theorems, Lehmer’s conjecture, heights over abelian extensions, Call–Silverman dynamical heights, Elkies’ abc implies Mordell, Mumford’s gap principle, . . . ).
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A smattering of Arakelov theory

- In Arakelov intersection theory for algebraic curves over a number field, one uses algebraic intersection theory on regular models at finite places and potential theory on Riemann surfaces at infinite places.

- The Archimedean component of the intersection pairing is a normalized Arakelov-Green function $g(x, y)$. It satisfies the differential equation

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\Delta_y g(x, y) = \delta_x - \mu
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for some fixed volume form $\mu$ on $X$, together with the normalization condition

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Liang-Chung Hsia, a former student of Joe Silverman’s working on $p$-adic dynamics, was visiting Harvard that same year and we started to work together on proving a Szpiro-Ullmo-Zhang type result for polynomial dynamical systems.
One of our motivations was the following classical result of Brolin:

**Theorem (Brolin, 1965)**

Let \( \phi \in \mathbb{C}[z] \) be a polynomial of degree \( d \geq 2 \). Then there is a canonical probability measure \( \mu_\phi \) on \( \mathbb{P}^1(\mathbb{C}) \) with the following property. For any \( z_0 \in \mathbb{C} \) with infinite backward orbit under \( \phi \), let \( \delta_n \) be the probability measure supported equally on the points of \( \phi^{-n}(z_0) \). Then \( \delta_n \) converges weakly to \( \mu_\phi \).

This was later generalized by Lyubich (and independently [FLM]) to rational functions.

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Let

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and define

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Remarks on the equidistribution theorem

- The main tool in our proof was the transfinite diameter.
- We also proved a complicated non-Archimedean equidistribution theorem motivated by a result of Bombieri and Zannier.
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In her thesis, she introduced the notion of homogeneous transfinite diameter for a subset of $\mathbb{C}^2$, and applied this to the dynamics of rational functions in one variable over $\mathbb{C}$.

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Laura’s work motivated the following definition:

Let $K$ be a number field, and let $v$ be a place of $K$. Let $\phi \in K(z)$ be a rational map of degree $d \geq 2$. Dehomogenize $\phi : \mathbb{P}^1 \to \mathbb{P}^1$ as $F : \mathbb{A}^2 \to \mathbb{A}^2$, writing $F(x_1, x_2) = (F_1(x_1, x_2), F_2(x_1, x_2))$ with each $F_i$ homogeneous of degree $d$.

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$$\hat{H}_{F,v}(x) = \lim_{d \to \infty} \frac{1}{d^n} \log \max\{|F_1^n(x)|_v, |F_2^n(x)|_v\}.$$ 

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$$g_{\phi,v}((x_1 : x_2), (y_1 : y_2)) = -\log |x_1y_2 - x_2y_1|_v + \hat{H}_{F,v}(x) + \hat{H}_{F,v}(y)$$

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- For $\nu$ Archimedean, $g_{\phi,\nu}$ is a normalized Arakelov-Green function for the canonical measure $\mu_\phi$ on $\mathbb{P}^1(\mathbb{C})$.

- By the product formula and the explicit formula for $g_{\phi,\nu}(x, y)$,

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Let $k$ be a complete and algebraically closed non-Archimedean valued field.

If $V$ is an irreducible algebraic variety over $k$, the Berkovich analytic space associated to $V$ is a path-connected, locally compact Hausdorff space $V^{\text{an}}$ containing $V(k)$ as a dense subspace.

The construction $V \mapsto V^{\text{an}}$ is functorial.

For an open affine subscheme $U = \text{Spec}(A)$ of $V$, $U^{\text{an}}$ is the space of all bounded multiplicative seminorms on $A$ extending the given absolute value on $k$ (endowed with the topology of pointwise convergence).
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Example: The Berkovich projective line
Example: A Berkovich elliptic curve
Example: A Berkovich K3 surface
Potential theory and dynamics on the Berkovich projective line

- Rumely and I showed that potential theory on trees could be used to do non-Archimedean potential theory on \((\mathbb{P}^1)^{an}\), with results that closely parallel the classical theory of harmonic and subharmonic functions on \(\mathbb{P}^1(\mathbb{C})\) (Poisson formula, Harnack’s principle, Poincaré-Lelong formula, Frostman’s theorem, . . .)

- In particular, the Laplacian on metric graphs can be used to define a Laplacian operator on \((\mathbb{P}^1)^{an}\). If one thinks of a metric graph as a resistive electrical network, non-Archimedean potential theory is intimately related to Kirchhoff’s laws.

- We defined the non-Archimedean canonical measure \(\mu_{\phi,v}\) attached to a rational map \(\phi\) over \(\mathbb{C}_v\) by the formula

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Adelic equidistribution for rational functions


Let $K$ be a number field, and let $\phi \in K(z)$ be a rational function of degree $d \geq 2$. For each place $v$ of $K$, there is a canonical probability measure $\mu_{\phi,v}$ supported on $\mathbb{P}^1_v$ with the following property. Let $P_n \in \mathbb{P}^1(\overline{K})$ be an infinite sequence with $\hat{h}_\phi(P_n) \to 0$. Let $\delta_n$ be the probability measure supported equally on the Galois conjugates of $P_n$. Then $\delta_n$ converges weakly to the canonical measure $\mu_{\phi,v}$ on $\mathbb{P}^1_v$ for all places $v$ of $K$.

The proof uses the product formula

$$\sum_v g_{\phi,v}(x, y) = \hat{h}_\phi(x) + \hat{h}_\phi(y)$$

together with non-Archimedean versions of various classical results on transfinite diameter, capacities, Laplacians, and subharmonic functions.

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A sample application

The adelic equidistribution theorem and potential theory on the Berkovich projective line have found many applications in recent years. For example:

**Theorem (B.-DeMarco, 2011)**

Let $a, b \in \mathbb{C}$ with $a \neq \pm b$. Then the set of $c \in \mathbb{C}$ such that both $a$ and $b$ have finite orbit under $z^2 + c$ is finite.

When $a, b$ are algebraic, we apply the equidistribution theorem in the number field setting. When they are transcendental, we apply it to the function field $\bar{\mathbb{Q}}(a, b)$. In particular, we really need Berkovich spaces to handle the transcendental case!
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“Dynamical André–Oort” theorems

A rational map is **post-critically finite** if the forward orbit of every critical point is finite.

- (B.–DeMarco, 2013) For each complex number \( \lambda \), let \( \text{Per}_1(\lambda) \) be the curve (inside the moduli space of cubic polynomials up to conjugacy) parametrizing polynomials with a fixed point of multiplier \( \lambda \). Then \( \text{Per}_1(\lambda) \) contains an infinite number of post-critically finite polynomials if and only if \( \lambda = 0 \).

- (Ghioca–Krieger–Nguyen–Ye, 2015) Let \( C \) be an irreducible plane curve over \( \mathbb{C} \). If there are infinitely many points \((a, b) \in C\) such that both \( z^2 + a \) and \( z^2 + b \) are post-critically finite, then \( C \) is either a horizontal, vertical, or diagonal line.
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OK, so what about curves of higher genus?

Amaury Thuillier, a student of Chambert-Loir, developed (independently and at the same time) non-Archimedean potential theory for arbitrary Berkovich curves.

He applies this theory to give a symmetrical version of Arakelov intersection theory for curves in which one is doing potential theory at all places.

Slogan: Intersection theory is non-Archimedean potential theory.

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A simple but illustrative example

As a concrete example, the non-Archimedean Arakelov-Green function on \((\mathbb{P}^1)^{an} \times (\mathbb{P}^1)^{an}\) with respect to a point mass at the Gauss point is

\[
g_v(x, y) = -\log |x_1y_2 - x_2y_1|_v + \log \max(|x_1|_v, |x_2|_v) + \log \max(|y_1|_v, |y_2|_v).
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The non-Archimedean Poincaré-Lelong formula

**Theorem (Thuillier)**

Let $f$ be a rational function on a curve $X$ over a complete non-Archimedean field $k$. Then

$$\Delta \log |f| = \delta_{\text{div}(f)}.$$
Reinterpretation in the language of tropical geometry

- For a finite metric subgraph $\Gamma$ of $X^\text{an}$ containing the skeleton, let $\text{Trop}(f)$ denote the restriction of $\log |f|$ to $\Gamma$. This is a piecewise-linear function with integer slopes, i.e., a “tropical rational function” on $\Gamma$.
- For $D \in \text{Div}(X)$, let $\text{trop}(D)$ denote the retraction of $D$ to $\Gamma$. This is a “divisor” on $\Gamma$.
- If $F$ is a tropical rational function on $\Gamma$, define the associated principal divisor on $\Gamma$ to be the Laplacian of $F$, i.e.,
\[
\text{div}(F) := \sum_{p \in \Gamma} \Delta_p(F)(p),
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where $\Delta_p(F)$ is the sum of the incoming slopes of $F$ at $p$.
- Thuillier’s Poincare-Lelong formula is equivalent to the statement that for every such $\Gamma$, we have
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Example: retraction of divisors
The divisor of a tropical rational function

\[ f \]

\[ P_1 \]

\[ P_2 \]

\[ P_3 \]

\[ P_4 \]

\[ P_5 \]
In Shouwu Zhang’s Inventiones paper “Admissible pairing on a curve” (an early attempt to prove the Bogomolov conjecture), he encounters (without naming it this) the canonical divisor on a graph:

$$K_\Gamma = \sum_{p \in \Gamma} (\text{valence}(p) - 2)(v).$$

The degree of $K_\Gamma$ is $2g - 2$, where $g$ is the genus of $\Gamma$, i.e. the dimension of $H_1(\Gamma, \mathbb{R})$. 
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Let $\Gamma$ be a metric graph.

For $D \in \text{Div}(\Gamma)$, define $r(D)$ to be the largest integer $k$ such that $D - E$ is equivalent to an effective divisor for all effective divisors $E$ of degree $k$.


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Shortly after establishing the Riemann-Roch theorem, I noticed that the combinatorial rank $r(D)$ has the following semicontinuity property:

**Lemma (B.)**

Let $X$ be an algebraic curve over a complete non-Archimedean field $k$. For every finite subgraph $\Gamma$ of $X^{an}$,

$$r_{\Gamma}(\text{trop}(D)) \geq r_X(D).$$

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**Theorem (Griffiths-Harris, Eisenbud-Harris, Lazarsfeld,...)**

Given nonnegative integers $g, r, d$, define $\rho := g - (r + 1)(g - d + r)$. If $X$ is a nonsingular projective curve of genus $g$, define $W^r_d(X)$ to be the variety parametrizing line bundles $L$ of degree $d$ on $X$ with $h^0(L) \geq r + 1$. Then for a general nonsingular projective curve $X$ of genus $g$, $W^r_d(X)$ has dimension $\rho$ if $\rho \geq 0$, and is empty if $\rho < 0$.

The proof, which Joe Harris explained in his class, uses a brilliant idea that goes back to Castelnuovo: Since $\dim W^r_d(X)$ is upper semicontinuous on $\overline{M}_g$, to show the statement for a general smooth curve of genus $g$ it suffices to prove it for a single stable curve of genus $g$. 
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**Theorem (Griffiths-Harris, Eisenbud-Harris, Lazarsfeld,...)**

Given nonnegative integers $g$, $r$, $d$, define $\rho := g - (r + 1)(g - d + r)$. If $X$ is a nonsingular projective curve of genus $g$, define $W_d^r(X)$ to be the variety parametrizing line bundles $L$ of degree $d$ on $X$ with $h^0(L) \geq r + 1$. Then for a general nonsingular projective curve $X$ of genus $g$, $W_d^r(X)$ has dimension $\rho$ if $\rho \geq 0$, and is empty if $\rho < 0$.

The proof, which Joe Harris explained in his class, uses a brilliant idea that goes back to Castelnuovo: Since $\dim W_d^r(X)$ is upper semicontinuous on $\overline{M}_g$, to show the statement for a general smooth curve of genus $g$ it suffices to prove it for a single stable curve of genus $g$. 
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Then for a general nonsingular projective curve \( X \) of genus \( g \), \( W_d^r(\mathcal{X}) \) has dimension \( \rho \) if \( \rho \geq 0 \), and is empty if \( \rho < 0 \).

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A rational backbone with $g$ elliptic tails
A tropical approach to degenerating linear series

- In 2008 I conjectured a tropical analogue of the Brill-Noether theorem. The conjecture was motivated by extensive computational evidence from my summer REU student Adam Tart.

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**Theorem (CDPR, 2012)**

If $\rho := g - (r + 1)(g - d + r) < 0$, then for the metric graph $\Gamma$ consisting of a chain of $g$ loops with general edge lengths, there is no divisor of degree $d$ and rank at least $r$ on $\Gamma$. 
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Other applications of the combinatorics of chains of loops

In just the last year, generic chains of loops have been used to prove the following:

- (Jensen–Payne, 2015) [Maximal Rank Conjecture for Quadrics] If $X$ is a general curve of genus $g$ and $\mathcal{L}$ is a general line bundle of degree $d$ and rank $r$ on $X$, then the natural map $\text{Sym}^2 H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}^\otimes 2)$ has maximal rank, i.e., it is either injective or surjective.

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Linear series on metric graphs also play a key role in the proofs of the following two recent breakthroughs in number theory:

**Theorem (Katz–Zureick-Brown, 2013)**

Let $X$ be a curve of genus $g$ over $\mathbb{Q}$ and suppose that the Mordell–Weil rank $r$ of $J(\mathbb{Q})$ is less than $g$. Then for every prime $p > 2r + 2$, we have

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where $\mathfrak{X}$ denotes the minimal proper regular model of $X$ over $\mathbb{Z}_p$.

**Theorem (Katz–Rabinoff–Zureick-Brown, 2015)**

There is an explicit bound $M(g) = 76g^2 - 82g + 22$ such that if $X/\mathbb{Q}$ is a curve of genus $g$ with Mordell-Weil rank at most $g - 3$, then

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Coming full-circle: Applications to number theory

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