Unlikely intersections in complex dynamics

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(Joint work with L. DeMarco)

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The Masser-Zannier theorem

Theorem (Masser–Zannier)

Let P_λ, Q_λ be linearly independent points in $E_\lambda(\overline{\mathbb{C}(\lambda)})$, where $E_\lambda : y^2 = x(x-1)(x-\lambda)$ is the Legendre family of elliptic curves. Then

$$\{\lambda \in \mathbb{C} \mid P_\lambda \text{ and } Q_\lambda \text{ are both torsion } \}$$

is finite.

Zannier's question

Motivated by his joint work with Masser, and by the analogy between torsion points on elliptic curves and preperiodic points for dynamical systems, Zannier posed the following question in 2008:

Question: What can be said about the set of $c \in \mathbb{C}$ such that both 0 and 1 are **preperiodic** for the map $z \mapsto z^2 + c$?

- $c = 0$: $0 \mapsto 0, 1 \mapsto 1$
- $c = -1$: $1 \mapsto 0 \mapsto -1 \mapsto 0$
- $c = -2$: $0 \mapsto -2 \mapsto 2 \mapsto 2, 1 \mapsto -1 \mapsto -1.$

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Main theorem of [BDM1]

Theorem (B.-DeMarco)

Fix $a \neq b$ in $\mathbb{C}$ with $a \neq \pm b$. Then

$$\{c \in \mathbb{C} \mid a \text{ and } b \text{ are both preperiodic for } z^2 + c\}$$

is finite.

Postcritically finite maps

A rational map $f \in \mathbb{C}(t)$ is called **postcritically finite** (PCF) if the critical points of f are all preperiodic.

PCF maps are very important in complex dynamics, roughly speaking because of the

Slogan: The dynamics of a rational map f under iteration is governed by what happens to the critical points.

Here is a plot of some values of $c \in \mathbb{C}$ for which $z^2 + c$ is PCF:

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PCF maps as special points

Let $\mathrm{Rat}_d \subset \mathbb{P}^{2d+1}$ be the space of rational maps of degree d, and let $M_d = \mathrm{Rat}_d/\mathrm{PSL}_2(\mathbb{C})$. Let P_d be the subset of M_d consisting of (conjugacy classes of) **polynomial** maps.

Fact: The PCF maps form a countable, Zariski dense subset of P_d. They form a Zariski dense subset of Rat_d which is countable outside of the **Lattès locus** Lat_d (which is either empty, if d is not a square, or an algebraic curve).

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Justifying the analogy

There are many reasons to subscribe to the idea that PCF maps are analogous to CM points; for example:

- The set of PCF points in $\mathrm{Rat}_d \backslash \mathrm{Lat}_d$ has bounded height and thus there are only finitely many non-Lattès PCF maps defined over number fields of bounded degree (analogue of Gauss's class number problem). [Benedetto-Ingram-Jones-Levy]
- The *arboreal Galois representation* attached to a PCF map tends to have much smaller image than in the non-PCF case. [Jones et. al.]

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Dynamical André-Oort

By analogy with the André-Oort conjecture, it is natural to ask which algebraic subvarieties of M_d contain a dense set of "special" (PCF) points. Before formulating a conjectural answer, let's look at some examples.

Example 1: Consider the parametrized curve in P_3 given by

$$f_t(z) = z^3 - 3t^2 z + i.$$

The (finite) critical points are $c_1(t) = t$ and $c_2(t) = -t$, which are "dynamically independent" in a sense to be made precise shortly. It follows from the main theorem of [BDM2] that there are only a **finite** number of PCF maps in this family of cubic polynomials.

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The Lattès curves

Example 2: For the curve in M_4 defined by the Lattès family

$$f_\lambda(z) = \frac{(z^2-\lambda)^2}{4z(z-1)(z-\lambda)},$$

every point is PCF. By a (difficult) theorem of Thurston, Lattès curves are the **only** examples of positive-dimensional subvarieties of M_d consisting entirely of PCF points (which one might call **very special**).

Example of a special subvariety

Example 3: Consider the parametrized curve in P_5 given by

$$f_t(z) = z^2(z^3 - t^3).$$

One can show that this curve is **special**, i.e., the above family contains infinitely many PCF maps. To see this, let $\beta = \sqrt[3]{2/5}$ and let ω be a primitive cube root of unity. Then the critical points are $c_j(t) = \omega^j \beta t$ for $j = 1, 2, 3$ and $c_4(t) = 0$.
The critical point c_4 is **persistently periodic**, since it's fixed in every fiber. And for $i, j \in \{1, 2, 3\}$ and $t_0 \in \mathbb{C}$, we have

$$c_i(t_0) \text{ is preperiodic if and only if } c_j(t_0) \text{ is preperiodic.}$$

Indeed, these three points satisfy the **critical orbit relations**

$$f_t^{(2)}(c_j(t)) = \omega \cdot f^{(2)}(c_{j-1}(t))$$

and in addition we have $f_t^{(2)}(\omega z) = \omega f_t^{(2)}(z)$.

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Dynamically dependent orbits

Let V be a irreducible complex algebraic variety of dimension at least 1, and let

$$f = \{f_t \ : \ t \in V\}$$

be a family of rational maps of degree at least 2 over $\mathbb{C}$.

The **dimension** of the family is the dimension of the image of V in $M_d = \mathrm{Rat}_d / \mathrm{PSL}_2(\mathbb{C})$ under the morphism $V \to \mathrm{Rat}_d$ defined by f.

Let $a_1(t), \ldots, a_m(t) \in \mathbb{P}^1(\overline{\mathbb{C}(V)})$ be marked points.
We say that $a_{i_1}, \ldots, a_{i_n}$ have **dynamically dependent orbits** if the point $(a_{i_1}, \ldots, a_{i_n}) \in \mathbb{A}^n(\overline{\mathbb{C}(V)})$ is contained in an algebraic hypersurface which is invariant under the action of $(f, f, \ldots, f)$.

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The general conjecture of [BDM2]

Conjecture (B.-DeMarco): Let $f = \{f_t \ : \ t \in V\}$ be an N-dimensional family of rational maps of degree at least 2 over $\mathbb{C}$, and let $a_1(t), \ldots, a_m(t) \in \mathbb{P}^1(\overline{\mathbb{C}(V)})$ be marked points. Then the set of $t \in V$ such that $a_1(t), \ldots, a_m(t)$ are simultaneously preperiodic is Zariski dense in V **if and only if** at most N of the marked points $a_1, \ldots, a_m$ have dynamically independent orbits.

We call the special case where $a_1(t), \ldots, a_m(t)$ are the critical points of f the **dynamical André-Oort conjecture**, since it gives a conjectural characterization of the special subvarieties.

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A concrete special case

As a concrete special case, one has:

Conjecture: Let $C \subset \mathbb{A}^2_{\mathbb{C}}$ be an algebraic curve containing infinitely many points (a, b) such that both $z^2 + a$ and $z^2 + b$ are PCF. Then C is either horizontal, vertical, or diagonal.

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The one-parameter polynomial case

We proved the following stronger version of the main conjecture for 1-**parameter families** of **polynomial maps** assuming that the $a_i(t)$ are also **polynomials**:

Theorem (B.-DeMarco)

Let $f = \{f_t\}$ be a one-parameter family of complex polynomials of degree at least 2, and let $a_1(t), \ldots, a_m(t) \in \mathbb{C}[t]$ be polynomials. Then the set of $t \in V$ such that $a_1(t), \ldots, a_m(t)$ are simultaneously preperiodic is infinite if and only if every pair of critical points which are not persistently periodic satisfies a critical orbit relation of the form

$$f^{(m)}(c_i) = h \circ f^{(n)}(c_j)$$

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Ingredients in the proof

The proof involves several different kinds of ingredients, notably:

- An adelic equidistribution theorem for Galois orbits of points of small dynamical height
- Potential theory on both $\mathbb{P}^1(\mathbb{C})$ and its non-Archimedean Berkovich space analogue
- Complex analysis, especially univalent function theory and the theory of normal families
- The recent work of Medvedev-Scanlon, who use the classical methods of Ritt to classify the invariant subvarieties for a certain class of polynomial dynamical systems.

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The Milnor curves

For some non-polynomial examples, consider the curve $\mathrm{Per}_1(\lambda)$ in P_3 (resp. M_2) consisting of (conjugacy classes of) rational maps with a marked fixed point of multiplier λ.

Theorem (B.-DeMarco for P_3, DeMarco-Wan-Ye for M_2)

The curve $\mathrm{Per}_1(\lambda)$ contains infinitely many PCF points if and only if $\lambda = 0$.

Note that even the case of P_3 is not a consequence of the previous theorem, since the critical points in this family are not parametrized by polynomials.

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The Masser-Zannier theorem revisited

Recently, DeMarco, Wan, and Ye have given a new proof of the Masser-Zannier theorem using the methods of proof from [BDM1] and [BDM2], as applied to the degree-four Lattès family. The method of proof in fact yields a Bogomolov-type strengthening of the Masser-Zannier theorem which applies to points of small height and not just torsion points:

Theorem (DeMarco-Wan-Ye)

For $a \neq b$ in $\overline{\mathbb{Q}} \backslash \{0, 1\}$, there exists $\epsilon > 0$ such that the set of $\lambda \in \overline{\mathbb{Q}} \backslash \{0, 1\}$ for which the points $P_\lambda = (a, \sqrt{a(a-1)(a-\lambda)})$ and $Q_\lambda = (b, \sqrt{b(b-1)(b-\lambda)})$ both have Néron-Tate canonical height less than ϵ is **finite**.

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A theorem of Ghioca-Krieger-Nguyen

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Let $f(z) \in \mathbb{C}[z]$ be a non-constant polynomial. There are infinitely many $t \in \mathbb{C}$ such that both $z^d + t$ and $z^d + f(t)$ are PCF if and only if $f(z) = \zeta z$ for some $(d-1)^{\mathrm{st}}$ root of unity ζ.

For $d = 2$, this is the special case $y = f(x)$ of the conjecture than an algebraic curve in $\mathbb{A}^2$ containing infinitely many points (a, b) such that both $z^2 + a$ and $z^2 + b$ are PCF if and only if C is either horizontal, vertical, or diagonal.

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Thank you!

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