Hyperfields, Ordered Blueprints, and Moduli Spaces of Matroids

Matt Baker

Georgia Institute of Technology

30 Years of Berkovich Spaces
July 2018
Hyperfields
Matroids over hyperfields
Ordered blueprints and ordered blue schemes
Moduli spaces of matroids
Applications to matroid representations
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The notion of an algebraic structure in which addition is allowed to be multi-valued goes back to Frédéric Marty, who introduced hypergroups in 1934.

In the mid-1950’s, Marc Krasner developed the theory of hyperrings and hyperfields in the context of approximating non-Archimedean fields.
Hyperstructures

The notion of an algebraic structure in which addition is allowed to be multi-valued goes back to Frédéric Marty, who introduced hypergroups in 1934.

In the mid-1950’s, Marc Krasner developed the theory of hyperrings and hyperfields in the context of approximating non-Archimedean fields.
On mathematical taboos

Oleg Viro: “Probably, the main obstacle for hyperfields to become a mainstream notion is that a multivalued operation does not fit to the tradition of set-theoretic terminology, which forces to avoid multivalued maps at any cost. I believe the taboo on multivalued maps has no real ground, and eventually will be removed. I believe hyperfields are to displace the tropical semifield in the tropical geometry. They suit the role better. In particular, with hyperfields the varieties are defined by equations, as in other branches of algebraic geometry.”
A hypergroup is a tuple \((G, \boxplus, 0)\), where \(\boxplus\) is an associative hyperoperation on \(G\) such that:

\[(H0) \ 0 \boxplus x = \{x\} \text{ for all } x \in G.\]

\[(H1) \text{ For every } x \in G \text{ there is a unique element of } G \text{ (denoted } -x \text{ and called the hyperinverse of } x) \text{ such that } 0 \in x \boxplus -x.\]

\[(H2) \ x \in y \boxplus z \text{ if and only if } z \in x \boxplus (-y).\]
A hypergroup is a tuple \((G, \boxplus, 0)\), where \(\boxplus\) is an associative hyperoperation on \(G\) such that:

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A hyperring is a tuple $(R, \cdot, \boxplus, 1, 0)$ such that:

$(R, \cdot, 1)$ is a commutative monoid.

$(R, \boxplus, 0)$ is a commutative hypergroup.

(Absorption rule) $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$.

(Distributive Law) $a \cdot (x \boxplus y) = (a \cdot x) \boxplus (a \cdot y)$ for all $a, x, y \in R$. 
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A hyperring $F$ is called a **hyperfield** if $0 \neq 1$ and every non-zero element of $F$ has a multiplicative inverse.

**Example**

Every field is trivially a hyperfield by setting $x \boxplus y := \{x + y\}$.

**Example**

If $K$ is a field and $G$ is a subgroup of $K^\times$, the set $K/G$ of orbits for the action of $G$ on $K$ by multiplication has a natural hyperfield structure. If $\phi : K \rightarrow K/G$ is the natural map, then $z \in x \boxplus y$ iff there exist $x', y', z' \in K$ with $x' + y' = z'$ and $\phi(x') = x, \phi(y') = y, \phi(z') = z$. 

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**The Krasner hyperfield**

**Example**

(Krasner hyperfield) Let $\mathbb{K} = \{0, 1\}$ with the usual multiplication rule, but with hyperaddition defined by $0 \boxplus x = x \boxplus 0 = \{x\}$ for $x = 0, 1$ and $1 \boxplus 1 = \{0, 1\}$.

This is the hyperfield structure on $\{0, 1\}$ induced by the field structure on $K$, for any field $K$, with respect to the trivial valuation $v : K \to \{0, 1\}$.
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The tropical hyperfield

Example

(Tropical hyperfield) Let $\mathbb{T}_+ := \mathbb{R} \cup \{\infty\}$, and for $a, b \in \mathbb{T}_+$ define $a \cdot b = a + b$ (with $-\infty$ as an absorbing element).

The hyperaddition law is defined by setting $a \boxplus b = \{\min(a, b)\}$ if $a \neq b$ and $a \boxplus b = \{c \in \mathbb{T}_+ \mid c \geq a\}$ if $a = b$.

The additive hyperidentity is $\infty$ and the multiplicative identity is $0$.

One often works instead with the isomorphic hyperfield $\mathbb{T} := \mathbb{R}_{\geq 0}$ in which $0, 1 \in \mathbb{R}$ are the additive (resp. multiplicative) identity elements and multiplication is the usual multiplication. Hyperaddition is defined so that the map $\mathbb{T}_+ \to \mathbb{T}$ given by $t \mapsto e^{-t}$ is an isomorphism of hyperfields.
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(Tropical hyperfield) Let $T_+ := \mathbb{R} \cup \{\infty\}$, and for $a, b \in T_+$ define $a \cdot b = a + b$ (with $-\infty$ as an absorbing element).

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One often works instead with the isomorphic hyperfield $T := \mathbb{R}_{\geq 0}$ in which 0, 1 $\in \mathbb{R}$ are the additive (resp. multiplicative) identity elements and multiplication is the usual multiplication. Hyperaddition is defined so that the map $T_+ \rightarrow T$ given by $t \mapsto e^{-t}$ is an isomorphism of hyperfields.
The hyperfield of signs and the phase hyperfield

Example

(Hyperfield of signs) Let $S := \{0, 1, -1\}$ with the usual multiplication law, and hyperaddition defined by $1 \boxplus 1 = \{1\}$, $-1 \boxplus -1 = \{-1\}$, $x \boxplus 0 = 0 \boxplus x = \{x\}$, and $1 \boxplus -1 = -1 \boxplus 1 = \{0, 1, -1\}$.

The hyperfield structure on $\{0, 1, -1\}$ is induced from that on $\mathbb{R}$ by the map $\sigma : \mathbb{R} \to \{0, 1, -1\}$ taking 0 to 0 and a nonzero real number to its sign.

Example

(Phase hyperfield) Let $P := S^1 \cup \{0\}$, where $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is the complex unit circle. The map $p : \mathbb{C} \to S^1 \cup \{0\}$ taking 0 to 0 and a nonzero complex number $z$ to its phase $z/|z| \in S^1$ induces a hyperfield structure on $S^1 \cup \{0\}$. 
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The hyperfield of signs and the phase hyperfield

**Example**

(Hyperfield of signs) Let \( \mathbb{S} := \{0, 1, -1\} \) with the usual multiplication law, and hyperaddition defined by \( 1 \oplus 1 = \{1\}, \ -1 \oplus -1 = \{-1\}, \ x \oplus 0 = 0 \oplus x = \{x\}, \) and \( 1 \oplus -1 = -1 \oplus 1 = \{0, 1, -1\} \).

The hyperfield structure on \( \{0, 1, -1\} \) is induced from that on \( \mathbb{R} \) by the map \( \sigma : \mathbb{R} \rightarrow \{0, 1, -1\} \) taking 0 to 0 and a nonzero real number to its sign.

**Example**

(Phase hyperfield) Let \( \mathbb{P} := S^1 \cup \{0\} \), where \( S^1 = \{z \in \mathbb{C} \mid |z| = 1\} \) is the complex unit circle.

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A hyperring homomorphism is a map $f : R \to S$ such that $f(0) = 0$, $f(1) = 1$, $f(x \cdot y) = f(x) \cdot f(y)$, and $f(x \boxplus y) \subseteq f(x) \boxplus f(y)$ for all $x, y \in R$.

**Example**

A homomorphism from a commutative ring $R$ to $\mathbb{K}$ is the same thing as a prime ideal of $R$. 
A hyperring homomorphism is a map $f : R \to S$ such that $f(0) = 0$, $f(1) = 1$, $f(x \cdot y) = f(x) \cdot f(y)$, and $f(x \boxplus y) \subseteq f(x) \boxplus f(y)$ for all $x, y \in R$.

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A homomorphism from a commutative ring $R$ to $\mathbb{K}$ is the same thing as a prime ideal of $R$. 
Valuations as homomorphisms

Example

A homomorphism from a field $K$ to $\mathbb{T}$ is the same thing as a real valuation on $K$.

More generally, a homomorphism from a commutative ring $R$ to the tropical hyperfield $\mathbb{T}$ is the same thing as a prime ideal $p$ of $R$ together with a real valuation on the residue field of $p$. 
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A homomorphism from a field $K$ to $\mathbb{T}$ is the same thing as a real valuation on $K$.
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Orderings as homomorphisms

Example

A homomorphism from a commutative ring $R$ to the hyperfield of signs $\mathbb{S}$ is the same thing as a prime ideal $p$ together with an ordering (in the sense of ordered field theory) on the residue field of $p$. 
Linear equations over hyperfields

If \( x_1, \ldots, x_k \in \mathbb{K} \), then 0 \( \in \) \( x_1 \oplus \cdots \oplus x_k \) if and only if all \( x_i = 0 \) or \( \#\{i \mid x_i = 1\} \geq 2 \).

If \( x_1, \ldots, x_k \in \mathbb{S} \), then 0 \( \in \) \( x_1 \oplus \cdots \oplus x_k \) if and only if all \( x_i = 0 \) or both \( \{i \mid x_i = 1\} \) and \( \{i \mid x_i = -1\} \) are non-empty.

If \( x_1, \ldots, x_k \in \mathbb{T}_+ \), then \( \infty \in x_1 \oplus \cdots \oplus x_k \) if and only if the minimum of the \( x_i \) occurs (at least) twice.
Linear equations over hyperfields

If $x_1, \ldots, x_k \in K$, then $0 \in x_1 \boxplus \cdots \boxplus x_k$ if and only if all $x_i = 0$ or $\# \{ i \mid x_i = 1 \} \geq 2$.

If $x_1, \ldots, x_k \in S$, then $0 \in x_1 \boxplus \cdots \boxplus x_k$ if and only if all $x_i = 0$ or both $\{ i \mid x_i = 1 \}$ and $\{ i \mid x_i = -1 \}$ are non-empty.

If $x_1, \ldots, x_k \in T_+$, then $\infty \in x_1 \boxplus \cdots \boxplus x_k$ if and only if the minimum of the $x_i$ occurs (at least) twice.
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Let $F$ be a hyperfield and let $E = \{1, \ldots, m\}$.

The inner product of $X = (x_1, \ldots, x_m)$ and $Y = (y_1, \ldots, y_m)$ in $F^m$ is defined to be

$$X \cdot Y := (x_1 \cdot y_1) \Box \cdots \Box (x_m \cdot y_m).$$

We say that $X$, $Y$ are orthogonal, denoted $X \perp Y$, if $0 \in X \cdot Y$. 
Orthogonality

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Orthogonality

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Applications to matroid representations
Matroids were introduced by Hassler Whitney as a combinatorial abstraction of the notion of linear independence of vectors. There are many different ("cryptomorphic") ways to present the axioms for matroids.
Giancarlo Rota: Like many other great ideas of this century, matroid theory was invented by one of the foremost American pioneers, Hassler Whitney. His paper conspicuously reveals the unique peculiarity of this field, namely, the exceptionally large variety of cryptomorphic definitions of a matroid, each one embarrassingly unrelated to every other. It is as if one were to condense all trends of present day mathematics onto a single structure, a feat that anyone would a priori deem impossible, were it not for the fact that matroids do exist.
A basis is the matroid-theoretic generalization of a maximal independent set of vectors.

**Definition**

(Basis Axioms) A matroid $M$ is a finite set $E$ together with a collection $B$ of subsets of $E$, called the bases of the matroid, such that:

(B1) $B$ is non-empty.

(B2) (Basis Exchange) Given $B, B' \in B$ and $b \in B \setminus B'$, there exists $b' \in B' \setminus B$ such that $(B \cup \{b'\}) \setminus \{b\} \in B$.

By (B2), any two bases have the same cardinality, called the rank of $M$. 
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Definition

(Circuit Axioms) A matroid $M$ is a finite set $E$ together with a collection $C$ of subsets of $E$, called the circuits of the matroid, such that:

(C1) Every circuit is non-empty.

(C2) No proper subset of a circuit is a circuit.

(C3) (Circuit Elimination) If $C_1, C_2$ are distinct circuits and $e \in C_1 \cap C_2$, then $(C_1 \cup C_2) \setminus \{e\}$ contains a circuit.
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Circuit axioms

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(C2) No proper subset of a circuit is a circuit.

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A **circuit** is the matroid-theoretic generalization of a **minimal dependent set** of vectors.

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Representable matroids

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Let $V$ be a vector space over a field $k$, and let $E$ be a finite subset of $V$. Define $B$ to be the collection of maximal linearly independent subsets of $E$ and $C$ to be the collection of minimal linearly dependent subsets of $E$. Then $B$ (resp. $C$) satisfies (B1)-(B2) (resp. (C1)-(C3)) and therefore defines a matroid.

Matroids of this form are called **representable** over $k$.

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Let $G$ be a connected finite graph, let $E$ be the set of edges of $G$, and let $B$ (resp. $C$) be the collection of all subsets of $E$ which form a spanning tree (resp. a simple cycle).

These data define a matroid $M(G)$ which is representable over every field.

By a theorem of Whitney, if $G$ is 3-connected then $M(G)$ determines the graph up to isomorphism.
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Let $E$ be the 8 vertices of the cuboid below. Define $B$ to be the collection of subsets of $E$ of size 4 which are not one of the five square faces in the picture.

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Duality

If $B$ is the set of bases of a matroid $M$ of rank $r$ on $E$, then $E \setminus B$ is the set of bases of a matroid $M^*$ of rank $|E| - r$ on $E$, called the **dual matroid**.

The circuits of $M^*$ are called the **cocircuits** of $M$ (and vice-versa).

If $C$ is a circuit of $M$ and $D$ is a cocircuit of $M$, we always have $|C \cap D| \neq 1$.

If $M = M(G)$ is graphic, the cocircuits of $M$ correspond to **minimal cuts** in $G$. 
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Matroids over hyperfields

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Let $E$ be a finite totally ordered set, let $F$ be a hyperfield, and let $r \in \mathbb{N}$. We denote by $\binom{E}{r}$ the collection of $r$-element subsets of $E$.

A Grassmann-Plücker function of rank $r$ on $E$ with coefficients in $F$ is a function $\varphi : \binom{E}{r} \to B$ such that:

(GP1) $\varphi$ is not identically zero.

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$$0 \in \bigoplus_{k=0}^{r} (-1)^k \varphi(I \setminus i_k) \varphi(J \cup i_k)$$

whenever $J \in \binom{E}{r-1}$ and $I = \{i_0, \ldots, i_r\} \in \binom{E}{r+1}$ with $i_0 < \cdots < i_r$.

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An $F$-matroid of rank $r$ on $E$ is an equivalence class of Grassmann-Plücker functions.

When $F = \mathbb{K}$ is the Krasner hyperfield, (GP2) is just the Basis Exchange axiom (B2). So a $\mathbb{K}$-matroid is just a matroid.
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If $F = K$ is a field and $A$ is an $r \times m$ matrix of rank $r$ with entries in $K$ and columns indexed by $E$, the function $\varphi_A$ taking an $r$-element subset of $E$ to the determinant of the corresponding $r \times r$ minor of $A$ is a Grassmann-Plücker function.

Up to a non-zero scalar multiple, the function $\varphi_A$ depends only on the row space of $A$, and conversely the row space of $A$ is uniquely determined by the function $\varphi_A$. This is equivalent to the well-known fact that the Plücker relations cut out the Grassmannian $G(r, m)$ as a projective variety.

Therefore a $K$-matroid is the same thing as a $K$-point of $G(r, m)$. 
Matroids over a field are just linear subspaces

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A matroid over the hyperfield of signs $\mathbb{S}$ is the same thing as an oriented matroid. Oriented matroids come up naturally in the theory of (real) hyperplane arrangements, linear programming, and convex polytopes.

A matroid over the tropical hyperfield $\mathbb{T}$ is the same thing as a valuated matroid.

In tropical geometry, valuated matroids are in one-to-one correspondence with tropical linear spaces, which can be characterized as balanced polyhedral complexes of degree one.
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Let $F$ be a hyperfield and let $M$ be a (classical) matroid of rank $r$ on $E$ whose set of circuits is $C$.

An $F$-signature of $M$ is a subset $C$ of $F^{E}$ such that:

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We say that \((C, D)\) is a **dual pair of \(F\)-signatures of \(M\)** if:

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**Theorem (B.–Bowler)**

There is a canonical bijection between dual pairs of \(F\)-signatures of rank \(r\) matroids on \(E\) and equivalence classes of Grassmann-Plücker functions \(\varphi : E^r \to F\). Thus we can identify a dual pair of \(F\)-signatures with an \(F\)-matroid \(M\).

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By interchanging the roles of $C$ and $D$, we obtain the dual $F$-matroid of an $F$-matroid $M$.

In terms of Grassmann-Plücker functions, $M^*$ can be defined by the formula

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We would like to construct a “moduli space of matroids” $\mathbb{G}(r, E)$ with the property that for every hyperfield $F$, $\mathbb{G}(r, E)(F)$ is the set of $F$-matroids of rank $r$ on $E$.

In particular, if $K$ is a field and $|E| = m$ then we should have $\mathbb{G}(r, E)(K) = G(r, m)(K)$.

One could attempt to construct the space $\mathbb{G}(r, E)$ as a hyperring scheme in the sense of Jaiung Jun, but this is (potentially) problematic. For example:

- The category of hyperring schemes does not appear to admit fiber products.

- The structure sheaf of a hyperring scheme behaves a bit strangely; for example, the hyperring of global sections of the structure sheaf on $\text{Spec}(R)$ is not always equal to $R$. 
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The category of hyperring schemes does not appear to admit fiber products.

The structure sheaf of a hyperring scheme behaves a bit strangely; for example, the hyperring of global sections of the structure sheaf on $\text{Spec}(R)$ is not always equal to $R$. 
Motivation

We would like to construct a “moduli space of matroids” $\mathbb{G}(r, E)$ with the property that for every hyperfield $F$, $\mathbb{G}(r, E)(F)$ is the set of $F$-matroids of rank $r$ on $E$.

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The structure sheaf of a hyperring scheme behaves a bit strangely; for example, the hyperring of global sections of the structure sheaf on \( \text{Spec}(R) \) is not always equal to \( R \).
We will solve these and other problems by utilizing Oliver Lorscheid’s theory of ordered blueprints and ordered blue schemes.

By expanding our point of view from hyperrings to ordered blueprints, we will also be able to fit the theory of matroids over partial fields into our theory.
Our proposed solution

We will solve these and other problems by utilizing Oliver Lorscheid’s theory of ordered blueprints and ordered blue schemes.

By expanding our point of view from hyperrings to ordered blueprints, we will also be able to fit the theory of matroids over partial fields into our theory.
A semiring is a set $R$ together with binary operations $+$ and $\cdot$ and elements $0, 1 \in R$ such that:

(SR1) $(R, +, 0)$ is a monoid;
(SR2) $(R, \cdot, 1)$ is a monoid;
(SR3) $0 \cdot a = 0$ for all $a \in R$;
(SR4) $a(b + c) = ab + ac$ for all $a, b, c \in R$.

A morphism of semirings is a map $f : R_1 \to R_2$ between semirings $R_1$ and $R_2$ such that

$f(0) = 0, f(1) = 1, f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$

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Matt Baker
Ordered blueprints

An ordered semiring is a semiring $R$ together with a partial order $\leq$ that is compatible with multiplication and addition.

An ordered blueprint is a triple $B = (B^\bullet, B^+, \leq)$ where $(B^+, \leq)$ is an ordered semiring and $B^\bullet$ is a multiplicative subset of $B^+$ which generates $B^+$ as a semiring and contains 0 and 1.

A morphism of ordered blueprints $(B_1^\bullet, B_1^+, \leq_1)$ and $(B_2^\bullet, B_2^+, \leq_2)$ is an order-preserving morphism $f : B_1^+ \to B_2^+$ of semirings with $f(B_1^\bullet) \subset B_2^\bullet$.

We denote the category of ordered blueprints by $\text{OBlpr}$. 
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A morphism of ordered blueprints $(B^\bullet_1, B^+_1, \leq_1)$ and $(B^\bullet_2, B^+_2, \leq_2)$ is an order-preserving morphism $f : B^+_1 \to B^+_2$ of semirings with $f(B^\bullet_1) \subset B^\bullet_2$.

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Ordered blue fields

The unit group $B^\times$ of an ordered blueprint is the abelian group consisting of all multiplicatively invertible elements of $B^\bullet$.

An ordered blue field is an ordered blueprint $B$ with $B^\bullet = B^\times \cup \{0\}$. 
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Examples

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(Monoids with zero) A **monoid with zero** is a commutative semigroup $A$ with neutral element 1 and absorbing element 0. Every monoid with zero defines an ordered blueprint $(A, \mathbb{N}[A], =)$. For example, the monoid usually denoted $\mathbb{F}_1$ corresponds to the ordered blueprint $(\{0, 1\}, \mathbb{N}, =)$, which is the *initial object* in the category of ordered blueprints.
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Given morphisms $B \to C$ and $B \to D$ of ordered blueprints, one can form their tensor product $C \otimes_B D$, which satisfies the universal property of a fiber coproduct.

The semiring $(C \otimes_B D)^+$ is the usual tensor product $C^+ \otimes_B^+ D^+$ of commutative semirings, whose elements are equivalence classes of finite sums $\sum c_i \otimes d_i$ with respect to the usual identifications.

The monoid $(C \otimes_B D)^*$ is defined as the subset of all pure tensors of $(C \otimes_B D)^+$.

The partial order on $(C \otimes_B D)^+$ is defined as the smallest partial order that is closed under addition and multiplication and that contains all relations of the forms

$$\sum a_i \otimes 1 \leq \sum c_k \otimes 1$$
and

$$\sum 1 \otimes b_j \leq \sum 1 \otimes d_l$$

for which $\sum a_i \leq \sum c_k$ in $C$ and $\sum b_j \leq \sum d_l$ in $D$, respectively.
Tensor products

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If $F$ is a hyperfield, we can consider $F$ as an ordered blueprint (in fact an ordered blue field) $F^{oblpr}$:

The associated semiring $(F^{oblpr})^+$ is the free semiring $\mathbb{N}[F^\times]$ over the multiplicative group $F^\times$.

The underlying monoid $(F^{oblpr})^\bullet$ is $(F, \cdot)$.

The partial order $\leq$ of $(F^{oblpr})^+$ is generated by the relations $0 \leq \sum a_i$ whenever $0 \in \boxplus a_i$.

More generally, one can embed the category of hyperrings as a full subcategory of the category of ordered blueprints.
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More generally, one can embed the category of hyperrings as a full subcategory of the category of ordered blueprints.
The map we just defined actually embeds hyperfields into a subcategory of ordered blueprints which will be important for our construction of moduli spaces of matroids.

An ordered blueprint $B$ is called pasteurized if for every $x \in B$ (by which we always mean $x \in B^\bullet$), there exists a unique $-x \in B$ such that $0 \leq x + (-x)$.

We denote the category of pasteurized ordered blueprints by $\text{OBlpr}^\pm$. The inclusion functor $\text{OBlpr}^\pm \to \text{OBlpr}$ has a left adjoint $(-)^\pm : \text{OBlpr} \to \text{OBlpr}^\pm$, called pasteurization, which makes $\text{OBlpr}^\pm$ into a reflective subcategory of $\text{OBlpr}$.

The initial object of $\text{OBlpr}^\pm$ is $\mathbb{F}_1^\pm$, which corresponds to the submonoid $\{0, 1, -1\}$ of $\mathbb{Z}$ together with the partial order generated by $0 \leq 1 + (-1)$. 
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A *pasture* is a pasteurized ordered blue field whose partial order is generated by elements of the form $0 \leq \sum a_i$.

Hyperfields are a full subcategory of the category of pastures. Partial fields are another important full subcategory.

(For the experts, pastures are the same thing as “tracts whose nullset is an ideal” in the terminology of my paper with Nathan Bowler.)
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A partial field $P$ is a (certain equivalence class of) pairs $(G, R)$ consisting of a commutative ring $R$ with 1 and a subgroup $G \leq R^\times$ containing $-1$.

We can identify a partial field $P$ with a (pasteurized) ordered blueprint $P^{oblpr}$ by setting $(P^{oblpr})_+ = \mathbb{N}[G], (P^{oblpr})_0 = G \cup \{0\}$, and letting $\leq$ be the partial order generated by the relations $0 \leq \sum a_i$ whenever $a_i \in G$ satisfy $\sum a_i = 0$ in $R$.

This embeds the category of partial fields as a full subcategory of $OBlpr^\pm$.

The initial object $\mathbb{F}^\pm_1$ of $OBlpr^\pm$ coincides with the ordered blueprint associated to the regular partial field $\mathbb{U}_0 := (\{\pm1\}, \mathbb{Z})$. 
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The initial object $\mathbb{F}_1^\pm$ of $OBlpr^\pm$ coincides with the ordered blueprint associated to the regular partial field $\mathbb{U}_0 := (\{\pm 1\}, \mathbb{Z})$. 
We now turn to the construction of ordered blue schemes, closely following the usual construction of schemes. We begin with the following remarks:

One can form the localization $S^{-1}B$ of an ordered blueprint $B$ with respect to any multiplicative subset $S$. It has the usual universal property.

A monoid ideal, or simply ideal, of $B$ is a subset $I$ of $B$ such that $0 \in I$ and $IB = I$.

A prime ideal of $B$ is an ideal whose complement is a multiplicative subset.
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Ordered blueprinted spaces

An ordered blueprinted space is a topological space $X$ together with a sheaf $\mathcal{O}_X$ in $\text{OBilpr}$.

For every point $x \in X$, the stalk $\mathcal{O}_{X,x}$ is the direct limit of $\mathcal{O}_X(U)$ over all open neighbourhoods $U$ of $x$.

A morphism of ordered blueprinted spaces is a continuous map $\varphi : X \to Y$ between the underlying topological spaces, together with a morphism $\varphi^\# : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves on $X$ which is local in the sense that for every $x \in X$ and $y = \varphi(y)$, the induced morphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ sends non-units to non-units. This defines the category $\text{OBilprSp}$ of ordered blueprinted spaces.
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Spectrum of an ordered blueprint and affine ordered blue schemes

Let $B$ be an ordered blueprint. The spectrum $\text{Spec } B$ of $B$ is constructed as follows:

The topological space of $X = \text{Spec } B$ consists of the prime ideals of $B$, and comes with the topology generated by the principal opens

$$U_h = \left\{ p \in \text{Spec } B \mid h \notin p \right\}_{h \in B}.$$

The structure sheaf $\mathcal{O}_X$ is the unique sheaf with the property that $\mathcal{O}_X(U_h) = B[h^{-1}]$ for all $h \in B$. The stalk of $\mathcal{O}_X$ at a point $x \in X$ corresponding to $p$ is $B_p$.

As usual, a morphism $f : B \to C$ of ordered blueprints defines a morphism $f^* : \text{Spec } C \to \text{Spec } B$ of $\text{OBlpr}$-spaces. This defines the contravariant functor

$$\text{Spec} : \text{OBlpr} \to \text{OBlprSp},$$

whose essential image is the category of affine ordered blue schemes.
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An ordered blue scheme is an $\text{ObIpr}$-space that has an open covering by affine ordered blue schemes $U_i$. A morphism of ordered blue schemes is a morphism of $\text{ObIpr}$-spaces. We denote the category of ordered blue schemes by $\text{OBSch}$.

Some familiar properties from the world of schemes (which fail for hyperring schemes):

The global section functor $\Gamma : \text{OBSch} \rightarrow \text{ObIpr}$ defined by $\Gamma(X, \mathcal{O}_X) := \mathcal{O}_X(X)$ is a left inverse to $\text{Spec}$. In particular, $B \cong \Gamma(\text{Spec } B)$.

The category $\text{OBSch}$ contains fibre products, and in the affine case $\text{Spec}(B) \times_{\text{Spec}(D)} \text{Spec}(C) \cong \text{Spec}(B \otimes_D C)$. 
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If $B$ is a topological blueprint, the set $X(B)$ inherits a topology from $B$. 
Points of ordered blue schemes

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If $B$ is a **topological blueprint**, the set $X(B)$ inherits a topology from $B$. 
If $R$ is a commutative ring, we can identify $R$ with an ordered blueprint $B = R^{oblpr}$ by setting $B^+ = R, B^\bullet = (R, \cdot)$, and $c \leq a + b$ iff $c = a + b$ in $R$.

This gives rise to a functor $(-)^{oblsc}$ from schemes to ordered blue schemes.

**Theorem (Lorscheid)**

Let $k$ be a field which is complete with respect to a valuation $v : k \to T$. Let $X/k$ be a $k$-scheme of finite type, and let $X^{an}$ be the Berkovich analytification of $X$. Then there is a canonical homeomorphism

$$X^{an} \cong (X^{oblsc} \times_k T)(T).$$
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$$X^{an} \simeq (X^{oblsch} \times_k \mathbb{T})(\mathbb{T}).$$
Let $k$ be a field. Given an affine $k$-scheme $X = \text{Spec}(R)$, together with an embedding $X \hookrightarrow Y(\Delta)$ into a toric variety with dense open torus $T = \mathbb{G}_m^n$, we can define an associated ordered blue scheme $X^\text{trop}$ by restricting the underlying monoid $B^\bullet = (R, \cdot)$ of $R^{\text{oblpr}}$ to the monoid generated by the pullbacks of the coordinate functions on $\mathbb{G}_m^n$, and then taking the spectrum.

This construction globalizes to non-affine schemes $X \hookrightarrow Y(\Delta)$.

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Let $k$ be a field which is complete with respect to a valuation $v : k \rightarrow \mathbb{T}$ and let $X \hookrightarrow Y(\Delta)$ be an embedding of a $k$-scheme $X$ into a toric variety as above. Let $\text{Trop}(X) \subset \mathbb{R}^n$ be the usual (Kajiwara–Payne) tropicalization of $X$. Then there is a canonical homeomorphism

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Tropicalization and ordered blue schemes

Let $k$ be a field. Given an affine $k$-scheme $X = \text{Spec}(R)$, together with an embedding $X \hookrightarrow Y(\Delta)$ into a toric variety with dense open torus $T = \mathbb{G}_m^n$, we can define an associated ordered blue scheme $X^{\text{trop}}$ by restricting the underlying monoid $B^\bullet = (R, \cdot)$ of $R^{\text{oblpr}}$ to the monoid generated by the pullbacks of the coordinate functions on $\mathbb{G}_m^n$, and then taking the spectrum.

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Lorscheid has also given interpretations and generalizations of Thuiller analytification, Giansiracusa–Giansiracusa tropical schemes, and other things in terms of ordered blue schemes.

We now turn to an application of ordered blue schemes to the construction of moduli spaces of matroids.
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We now turn to an application of ordered blue schemes to the construction of moduli spaces of matroids.
Grassmann-Plücker functions with coefficients in pasteurized ordered blueprints

To construct moduli spaces, we need to extend the notion of matroids over a hyperfield to matroids over arbitrary pasteurized ordered blue schemes.

We begin with the affine case. Let $B$ be a pasteurized ordered blueprint, let $E$ be a finite totally ordered set, and let $r \in \mathbb{N}$.

A Grassmann-Plücker function of rank $r$ on $E$ with coefficients in $B$ is a function $\varphi : \binom{E}{r} \to B$ such that:

- $\varphi(I) \in B^\times$ for some $I \in \binom{E}{r}$.
- $\varphi$ satisfies the Plücker relations:

$$0 \leq \sum_{k=0}^{r} (-1)^k \varphi(I \setminus i_k) \varphi(J \cup i_k)$$

whenever $J \in \binom{E}{r-1}$ and $I = \{i_0, \ldots, i_r\} \in \binom{E}{r+1}$ with $i_0 < \cdots < i_r$. 
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Hyperfields, Ordered Blueprints, and Moduli Spaces of Matroids
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We say that two Grassmann-Plücker functions $\varphi, \varphi' : \binom{E}{r} \to B$ are equivalent if $\varphi = a\varphi'$ for some $a \in B^\times$.

A $B$-matroid of rank $r$ on $E$ is an equivalence class of Grassmann-Plücker functions.

We denote by $\text{Mat}_B(r, E)$ the set of all $B$-matroids of rank $r$ on $E$.

(Functoriality) If $f : B \to C$ is a morphism of pastures (or more generally pasteurized ordered blueprints), there is an induced map $f_* : \text{Mat}_B(r, E) \to \text{Mat}_C(r, E)$. 
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Applications to matroid representations
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If $F$ is a pasture, we can characterize $F$-matroids of rank $r$ on $E$ in several cryptomorphic ways. For example:

**Theorem (B.–Bowler, B.–Lorscheid)**

If $F$ is a pasture, there is a canonical bijection between dual pairs of $F$-signatures of rank $r$ matroids on $E$ and equivalence classes of Grassmann-Plücker functions $\varphi : E^r \to F$.

The definition of orthogonality and dual pairs is the same as for hyperfields, replacing all occurrences of “$0 \in a_i$” with “$0 \leq \sum a_i$”.

Matroids over pastures generalize **matroids over fuzzy rings** in the sense of Dress and Wenzel.
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Matroids over pastures generalize **matroids over fuzzy rings** in the sense of Dress and Wenzel.
The Krasner hyperfield $\mathbb{K}$ is the final object in the category of pastures: if $F$ is a pasture, there is a unique morphism $\psi : F \rightarrow \mathbb{K}$.

If $M$ is an $F$-matroid, the push-forward $\psi_*(M)$ is called the underlying matroid of $M$.

We say that a matroid $M$ is representable over a pasture $F$ if there is an $F$-matroid $M'$ such that $\psi_*(M') = M$.

When $F = K^{oblpr}$ for some field $K$, this coincides with the usual notion of representability for matroids over fields.
Representability over pastures

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When $F = K^{\text{oblp}}$ for some field $K$, this coincides with the usual notion of representability for matroids over fields.
Let’s return to the problem of constructing moduli spaces of matroids. To do so, we should define “families of matroids” parametrized by an arbitrary (not necessarily affine) pasteurized ordered blue schemes $X$.

For this, we first define invertible sheaves in the context of ordered blue schemes.

An invertible sheaf on an ordered blue scheme $X$ is a sheaf which is locally isomorphic to the structure sheaf $\mathcal{O}_X$ of $X$.

(A more formal definition would require first introducing sheaves of ordered blue modules... )
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Picard groups of ordered blue schemes

One can define the **tensor product** of two invertible sheaves, which is again an invertible sheaf. The dual $L^\vee = \text{Hom}(L, \mathcal{O}_X)$ of an invertible sheaf is also an invertible sheaf, and $L \otimes L^\vee \simeq \mathcal{O}_X$. Thus the set $\text{Pic} X$ of isomorphism classes of invertible sheaves on $X$ is naturally an abelian group.

If $\varphi : X \rightarrow Y$ is a morphism of ordered blue schemes and $L$ an invertible sheaf on $Y$, one can define the pullback $\varphi^*(L)$, which is again an invertible sheaf. Pullback commutes with tensor products, so there is a natural homomorphism $\varphi^* : \text{Pic} Y \rightarrow \text{Pic} X$. 
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Let $X$ be a pasteurized ordered blue scheme. A Grassmann–Plücker function of rank $r$ on $E$ over $X$ is an invertible sheaf $\mathcal{L}$ on $X$ together with a map $\varphi : \binom{E}{r} \to \Gamma(X, \mathcal{L})$ such that $\{\varphi(I)\}_{I \in \binom{E}{r}}$ generate $\mathcal{L}$ and the $\varphi(I)$ satisfy the Plücker relations in $\Gamma(X, \mathcal{L} \otimes^2)$.

Two such functions $(\mathcal{L}, \varphi)$ and $(\mathcal{L}', \varphi')$ are isomorphic if there is an isomorphism from $\mathcal{L}$ to $\mathcal{L}'$ taking $\varphi$ to $\varphi'$.

A matroid bundle (of rank $r$ on the set $E$) over $X$ is an isomorphism class of Grassmann–Plücker functions.

One can define the pullback $f^* \mathcal{M}$ of a matroid bundle $\mathcal{M}$ along a morphism $f : X \to Y$ of pasteurized ordered blue schemes.

If $X = \text{Spec}(B)$ is an affine pasteurized ordered blue scheme then a matroid bundle over $X$ is the same thing as a $B$-matroid.
Matroid bundles

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The moduli functor of matroids

Let $E$ be a finite ordered set and $r$ a natural number. We can extend the (covariant) functor taking a pasteurized ordered blueprint $B$ to $\text{Mat}_B(r, E)$ to a (contravariant) functor $\text{Mat}(r, E) : \text{OBSch}^\pm \to \text{Sets}$ taking $X$ to the set of matroid bundles of rank $r$ on $E$ over $X$.

**Theorem (B.–Lorscheid)**

This moduli functor $\text{Mat}(r, E)$ is representable by a pasteurized ordered blue scheme $G(r, E)$. Furthermore, for every pasteurized ordered blue scheme $X$ there is a natural bijection

$$\text{Hom}_{\text{OBSch}^\pm}(X, G(r, E)) \sim \text{Mat}(r, E)(X).$$

Thus $G(r, E)$ is a **fine moduli space** of matroid bundles. In particular, for any pasture (e.g. a hyperfield) $F$ we have

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The universal pasture

A classical matroid $M$ corresponds to a morphism $\text{Spec}(K) \to \mathbb{G}(r, E)$.

The image point $x_M \in \mathbb{G}(r, E)$ of this morphism has an associated (nonzero) residue pasture $k_M$ which we call the universal pasture of $M$.

Given a pasture $F$, the realization space $\chi_M(F)$ is the set of isomorphism classes of $F$-matroids with underlying matroid $M$.

**Corollary (B.–Lorscheid)**

There is a canonical (and functorial) bijection

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Regular matroids

A matroid $M$ is called regular if it is representable over every field.

This is known to be equivalent to saying that $M$ can be represented (over $\mathbb{R}$, say) by a totally unimodular integer matrix. (A matrix is called totally unimodular if every square submatrix has determinant 0, 1, or $-1$.)

Regularity is also known to be equivalent to saying that $M$ is representable over $\mathbb{F}_2$ and $\mathbb{F}_p$ for some odd prime $p$, or that $M$ is both binary (representable over $\mathbb{F}_2$) and orientable (representable over $\mathbb{S}$).
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Define a pasture $F$ to be odd if $-1 \neq 1$ in $F$.

**Theorem (B.–Lorscheid)**

The following are equivalent for a matroid $M$:

1. $M$ is regular.
2. $M$ is representable over the initial object $\mathbb{F}_{1}^{\pm} = U_0$ in the category of pastures.
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If we take $F = \mathbb{S}$ in (4), we obtain a “conceptual” proof of the fact that a matroid is regular iff it is both binary and orientable.
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Thank you!