

THE DOLLAR GAME ON A GRAPH (WITH 9 SURPRISES!)

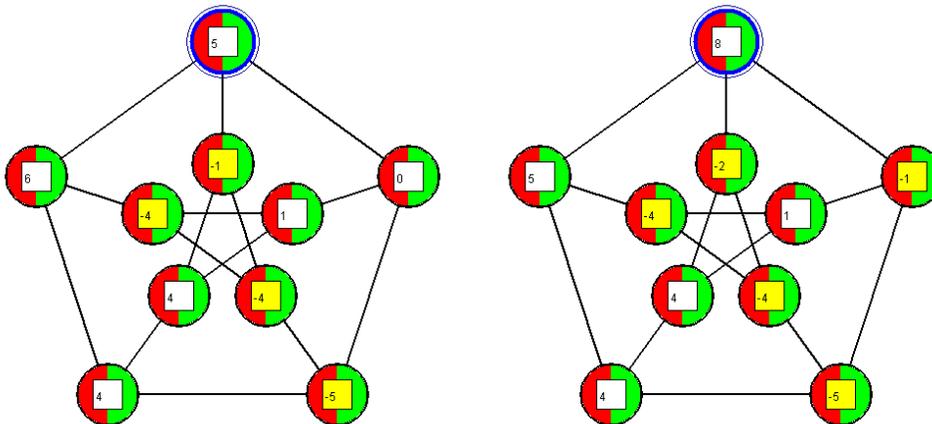
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To Martin Gardner on the occasion of the g4g9 conference

1. THE GAME

The *Dollar Game* is a game of solitaire played on a graph; the underlying mathematical principles are unexpectedly rich and beautiful (see the surprises below). To play the game, select a graph G as the game board.¹ Then select an initial configuration of dollars on G . (A *configuration* is an assignment of some integer number of dollars – which could be positive, negative, or zero – to each vertex of G .) A vertex with a negative number of dollars is said to be in *debt*.

There are two kinds of legal moves in the game. A *lending move* consists of selecting a vertex v and then moving one dollar across each edge adjacent to v in such a way that the money flows from v to its neighbors. A *borrowing move* is similar, except that the money flows in the other direction. For example, here is an illustration of a borrowing move performed on the famous “Petersen graph”. The picture on the left shows the starting configuration, and the picture on the right shows the new configuration after the top vertex performs a borrowing move:



The goal of the dollar game is to get all the vertices of G out of debt by a sequence of legal moves.

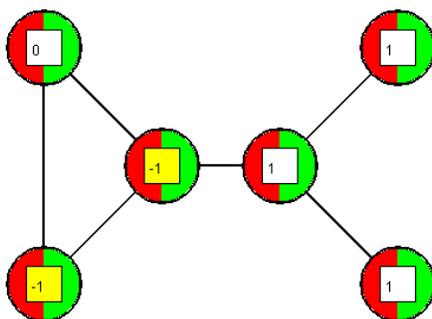
Date: March 9, 2010.

Thanks to Adam Tart for drawing the figures in this paper.

¹A *graph* is a finite set of vertices together with a (possibly empty) finite set of edges between each distinct pair of vertices. We assume that our graphs are connected, meaning that any two vertices can be connected by a sequence of edges.

Since the total amount N of money in the game never changes, a necessary condition for the existence of a winning strategy is that N is nonnegative. However, this is usually not sufficient.

The game depicted below is winnable: can you find a winning strategy?



Take some time to experiment with the game using different game boards and different initial configurations. There is an online simulation of the game, written by Adam Tart, available at <http://people.math.gatech.edu/~mbaker/GraphGame/GraphGame.html>. Once you get a feeling for how the game works, you should proceed to read the following 9 surprises. As a challenge, you might want to try proving some of the assertions made below!

2. 9 SURPRISES

Surprise #1: If the game is winnable, then it can always be won using only borrowing moves. More precisely, any winnable game can be won by simply using the “borrowing binge strategy”: any time there are vertices in debt, pick one of them and do a borrowing move. Repeat until everyone is out of debt!

Surprise #2: The total number of borrowing moves required to win the game when playing the “borrowing binge strategy” is independent of which borrowing moves you do in which order!

Note, however, that it is usually possible to win in fewer moves by employing lending moves in combination with borrowing moves. The optimal strategy when one uses both kinds of moves is not yet understood. (See [2, §5.5] and the references therein for proofs of the facts mentioned in Surprises 1 and 2.)

Surprise #3: There is a number associated to any graph G , called the *Euler number*, which plays a key role in analyzing whether or not the game is winnable. The Euler number g of a graph is defined as $\#edges - \#vertices + 1$. Let N denote the total number of dollars in the game at any time. Then (see [2, Theorem 1.9]):

- If N is at least g , then the game is always winnable.
- If N is less than g , then the game may or may not be winnable. (See Surprise #6 below for a stronger and more precise result.)

For example, in the Petersen graph game depicted on page 1, we have $g = N = 6$ so we conclude that the game depicted there is winnable. (As an exercise, check

that the “credit binge strategy” works in this example, and see if you can find a shorter sequence of winning moves by combining borrowing and lending.)

Surprise #4: If two configurations are *equivalent* (meaning that one can get from one to the other by a sequence of legal moves), then they obviously have the same *degree* (the total number of dollars in the configuration). The converse is false: there are usually many inequivalent configurations of the same degree. This can be quantified in the following surprising way: For any fixed integer N , the maximal number of inequivalent configurations of degree N is equal to the number of *spanning trees*² in G . (This is closely related to Kirchhoff’s celebrated “Matrix-Tree Theorem”.)

Indeed, if one fixes a reference vertex v together with an ordering of the edges of G , it is possible to give an explicit one-to-one correspondence between equivalence classes of degree N configurations and spanning trees in G (this involves the notions of “ G -parking functions” and “Dhar’s burning algorithm”; see [5] for details).

Surprise #5: For each positive integer k , there exist distinct equivalent debt-free configurations D and E of degree k if and only if G can be disconnected by removing at most k edges. Thus the dollar game encodes the *edge connectivity* of the graph G (see [2, Theorem 1.8]).

Surprise #6: Fix an arbitrary labeling v_1, \dots, v_n of the vertices of G . Define the associated *order configuration* M by

$$M(v_i) = (\#\text{edges connecting } v_i \text{ to a vertex } v_j \text{ with } j > i) - 1.$$

Then:

- $\deg(M) = g - 1$ (i.e., the total number of dollars in M is $g - 1$).
- The game with initial configuration M is not winnable.

Moreover (note that it makes sense to add and subtract configurations):

- (\star) For every configuration D , exactly one of the following holds: (a) The game starting with D is winnable; or (b) The game starting with $M - D$ is winnable for some order configuration M .

See [2, Lemma 3.2 and Theorem 3.3] for proofs of these facts. Property (\star) is one of the most important theoretical facts about the dollar game. It implies the facts mentioned in Surprise #3 above.

Surprise #7: If D is a configuration of degree $g - 1$, then the game with initial configuration D is winnable if and only if the game with initial configuration $K - D$ is winnable, where K is the *canonical configuration* given by

$$K(v) = (\#\text{edges adjacent to } v) - 2.$$

This can be deduced from (\star) using the fact that the order configurations associated to a vertex ordering and its reverse ordering always sum to K .

Surprise #8: The dollar game is closely related to the famous *Riemann-Roch theorem* in algebraic geometry. Specifically, for each configuration D define its *robustness* $r(D)$ to be -1 if the game starting with D is not winnable, and to be at least k if the game starting with $D - E$ is winnable for any debt-free configuration

²A *spanning tree* is a subgraph of G which contains all the vertices of G and whose edges do not form any cycles.

E of degree k . In other words, $r(D)$ is at least k if the game is still winnable after subtracting k dollars from D in an arbitrary way.

Riemann-Roch Theorem for Graphs (Baker-Norine [2, Theorem 1.12]): For any configuration D on any graph G ,

$$r(D) - r(K - D) = \deg(D) + 1 - g.$$

A remark for the algorithmically inclined: it is known (see [5]) that one can efficiently determine if an initial configuration is winnable (i.e., there is a polynomial time algorithm for this problem), but it is an open problem whether or not one can compute $r(D)$ in polynomial-time.

Surprise #9: Many other classical results from algebraic geometry appear to have graph-theoretic analogs. For example:

Brill-Noether Conjecture for Graphs: Given positive integers g , r , and d , every graph with Euler number g has a configuration of degree d and robustness at least r if and only if $g - (r + 1)(g - d + r)$ is nonnegative.

The “only if” direction of this conjecture was recently proved by Cools, Draisma, Payne, and Robeva [4] using a clever combinatorial argument. Surprisingly, their result implies the corresponding result in classical algebraic geometry (a difficult theorem first proved by Griffiths and Harris in 1980). The “if” direction of the conjecture is still an open problem (but see [1] for a *metric graph* version of this result).

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