RIEMANN-ROCH AND ABEL-JACOBI THEORY ON A FINITE GRAPH

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Abstract. It is well-known that a finite graph can be viewed, in many respects, as a discrete analogue of a Riemann surface. In this paper, we pursue this analogy further in the context of linear equivalence of divisors. In particular, we formulate and prove a graph-theoretic analogue of the classical Riemann-Roch theorem. We also prove several results, analogous to classical facts about Riemann surfaces, concerning the Abel-Jacobi map from a graph to its Jacobian. As an application of our results, we characterize the existence or non-existence of a winning strategy for a certain chip-firing game played on the vertices of a graph.

1. Introduction

1.1. Overview. In this paper, we explore some new analogies between finite graphs and Riemann surfaces. Our main result is a graph-theoretic analogue of the classical Riemann-Roch theorem. We also study the Abel-Jacobi map $S$ from a graph $G$ to its Jacobian, as well as the higher symmetric powers $S^{(k)}$ of $S$. We prove, for example, that $S^{(g)}$ is always surjective, and that $S^{(1)}$ is injective when $G$ is 2-edge-connected. These results closely mirror classical facts about the Jacobian of a Riemann surface. As an application of our results, we characterize the existence or non-existence of a winning strategy for a certain chip-firing game played on the vertices of a graph.

The paper is structured as follows. In §1, we provide all of the relevant definitions and state our main results. The proof of the Riemann-Roch theorem for graphs occupies §2-3. In §4, we study the injectivity and surjectivity of $S^{(k)}$ for $k \geq 1$, and explain the connection with the
chip-firing game. Related results and further questions are discussed in §5.

1.2. Notation and Terminology. Throughout this paper, a Riemann surface will mean a compact, connected one-dimensional complex manifold, and a graph will mean a finite, unweighted multigraph having no loop edges. All graphs in this paper are assumed to be connected. We denote by $V(G)$ and $E(G)$, respectively, the set of vertices and edges of $G$. We will simply write $G$ instead of $V(G)$ when there is no danger of confusion. Also, we write $E_v = E_v(G)$ for the set of edges incident to a given vertex $v$.

For $k \geq 2$, a graph $G$ is called $k$-edge-connected if $G-W$ is connected for every set $W$ of at most $k-1$ edges of $G$. (By convention, we consider the trivial graph having one vertex and no edges to be $k$-edge-connected for all $k$.) Alternatively, define a cut to be the set of all edges connecting a vertex in $V_1$ to vertex in $V_2$ for some partition of $V(G)$ into disjoint non-empty subsets $V_1$ and $V_2$. Then $G$ is $k$-edge-connected if and only if every cut has size at least $k$.

1.3. The Jacobian of a finite graph. Let $G$ be a graph, and choose an ordering $\{v_1, \ldots, v_n\}$ of the vertices of $G$. The Laplacian matrix $Q$ associated to $G$ is the $n \times n$ matrix $Q = D - A$, where $D$ is the diagonal matrix whose $(i,i)$ entry is the degree of vertex $v_i$, and $A$ is the adjacency matrix of the graph, whose $(i,j)$ entry is the number of edges connecting $v_i$ and $v_j$. Since loop edges are not allowed, the $(i,i)$ entry of $A$ is zero for all $i$. It is well-known and easy to verify that $Q$ is symmetric, has rank $n - 1$, and that the kernel of $Q$ is spanned by the vector whose entries are all equal to 1 (see [2, 7, 12]).

Let $\text{Div}(G)$ be the free abelian group on the set of vertices of $G$. We think of elements of $\text{Div}(G)$ as formal integer linear combinations of elements of $V(G)$, and write an element $D \in \text{Div}(G)$ as $\sum_{v \in V(G)} a_v(v)$, where each $a_v$ is an integer. By analogy with the Riemann surface case, elements of $\text{Div}(G)$ are called divisors on $G$.

For convenience, we will write $D(v)$ for the coefficient $a_v$ of $(v)$ in $D$.

There is a natural partial order on the group $\text{Div}(G)$: we say that $D \geq D'$ if and only if $D(v) \geq D'(v)$ for all $v \in V(G)$. A divisor $E \in \text{Div}(G)$ is called effective if $E \geq 0$. We write $\text{Div}_+(G)$ for the set of all effective divisors on $G$.

The degree function $\deg : \text{Div}(G) \to \mathbb{Z}$ is defined by $\deg(D) = \sum_{v \in V(G)} D(v)$. 
Remark 1.1. Note that the definitions of the partial order $\geq$, the space $\text{Div}_+(G)$, and the map $\text{deg}$ make sense when $V(G)$ is replaced by an arbitrary set $X$. This observation will be used in §2 when we formulate an abstract “Riemann-Roch Criterion”.

We let $\mathcal{M}(G) = \text{Hom}(V(G), \mathbb{Z})$ be the abelian group consisting of all integer-valued functions on the vertices of $G$. One can think of $\mathcal{M}(G)$ as analogous to the space of meromorphic functions on a Riemann surface $X$ (though it is actually more like the set $\{\log |f| : f \in \mathcal{M}(X)^+\}$, see Remark 1.4).

Using our ordering of the vertices, we obtain isomorphisms between $\text{Div}(G)$, $\mathcal{M}(G)$, and the space of $n \times 1$ column vectors having integer coordinates. We will freely use these identifications when it makes things easier to understand. We write $[D]$ (resp. $[f]$) for the column vector corresponding to $D \in \text{Div}(G)$ (resp. $f \in \mathcal{M}(G)$). The Laplacian operator $\Delta : \mathcal{M}(G) \to \text{Div}(G)$ is given by the formula

$$\Delta(f) = \sum_{v \in V(G)} \Delta_v(f)(v),$$

where

$$\Delta_v(f) = \deg(v)f(v) - \sum_{e=uv \in E_v} f(w) = \sum_{e=uv \in E_v} (f(v) - f(w)).$$

In terms of matrices, it follows from the definitions that

$$[\Delta(f)] = Q[f].$$

We will therefore use $\Delta$ and $Q$ interchangeably to denote the Laplacian operator on $G$.

Remark 1.2. The fact that $Q$ is a symmetric matrix is equivalent to the fact that $\Delta$ is self-adjoint with respect to the bilinear pairing $\langle f, D \rangle = \sum_{v \in V(G)} f(v)D(v)$ on $\mathcal{M}(G) \times \text{Div}(G)$. This is the graph-theoretic analogue of the Weil reciprocity theorem on a Riemann surface (see p. 242 of [13] and Remark 1.4 below).

We define the subgroup $\text{Div}^0(G)$ of $\text{Div}(G)$ consisting of divisors of degree zero to be the kernel of $\text{deg}$, i.e.,

$$\text{Div}^0(G) = \{D \in \text{Div}(G) : \deg(D) = 0\}.$$

More generally, for each $k \in \mathbb{Z}$ we define $\text{Div}^k(G) = \{D \in \text{Div}(G) : \deg(D) = k\}$, and $\text{Div}^+_k(G) = \{D \in \text{Div}(G) : D \geq 0 \text{ and } \deg(D) = k\}$. The set $\text{Div}^+_1(G)$ is canonically isomorphic to $V(G)$. 
We also define the subgroup $\text{Prin}(G)$ of $\text{Div}(G)$ consisting of principal divisors to be the image of $\mathcal{M}(G)$ under the Laplacian operator, i.e.,
\begin{align*}
\text{Prin}(G) := \Delta(\mathcal{M}(G)).
\end{align*}

It is easy to see that every principal divisor has degree zero, so that $\text{Prin}(G)$ is a subgroup of $\text{Div}^0(G)$.

Remark 1.4. The classical motivation for (1.3) is that the divisor of a nonzero meromorphic function $f$ on a Riemann surface $X$ can be recovered from the extended real-valued function $\log |f|$ using the (distributional) Laplacian operator $\Delta$. More precisely if $\Delta(\varphi)$ is defined so that
\begin{align*}
\int_X \psi \Delta(\varphi) = \int_X \varphi \Delta(\psi)
\end{align*}
for all suitably smooth test functions $\psi : X \to \mathbb{R}$, where $\Delta(\psi)$ is given in local coordinates by the formula
\begin{align*}
\Delta(\psi) = \frac{1}{2\pi} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) dx \wedge dy,
\end{align*}
then
\begin{align*}
\Delta(\log |f|) = \sum_{P \in X} \text{ord}_P(f) \delta_P.
\end{align*}
In other words, the divisor of $f$ can be identified with the Laplacian of $\log |f|$.

Following [1] and [24], we define the group $\text{Jac}(G)$, called the Jacobian of $G$, to be the corresponding quotient group:
\begin{align*}
\text{Jac}(G) = \frac{\text{Div}^0(G)}{\text{Prin}(G)}.
\end{align*}

As shown in [1], $\text{Jac}(G)$ is a finite abelian group whose order $\kappa(G)$ is the number of spanning trees in $G$. (This is a direct consequence of Kirchhoff’s famous Matrix-Tree Theorem, see §14 of [3].) The group $\text{Jac}(G)$ is a discrete analogue of the Jacobian of a Riemann surface. We will write $[D]$ for the class in $\text{Jac}(G)$ of a divisor $D \in \text{Div}^0(G)$. (There should not be any confusion between this notation and our similar notation for the column vector associated to a divisor.)

In [1], the group $\text{Jac}(G)$ is called the Picard group, and denoted $\text{Pic}(G)$, and the term Jacobian is reserved for an a priori different group denoted $J(G)$. However, as shown in Proposition 7 of [1], the two groups are canonically isomorphic. The isomorphism $\text{Pic}(G) \cong J(G)$ is the graph-theoretic analogue of Abel’s theorem (see Theorem VIII.2.2 of [23]).
1.4. The Abel-Jacobi map from a graph to its Jacobian. If we fix a base point \( v_0 \in V(G) \), we can define the Abel-Jacobi map \( S_{v_0} : G \to \text{Jac}(G) \) by the formula

\[
S_{v_0}(v) = [(v) - (v_0)] .
\]

We also define, for each natural number \( k \geq 1 \), a map \( S_{v_0}^{(k)} : \text{Div}^k_+(G) \to \text{Jac}(G) \) by

\[
S_{v_0}^{(k)}((v_1) + \cdots + (v_k)) = S_{v_0}(v_1) + S_{v_0}(v_2) + \cdots + S_{v_0}(v_k) .
\]

The map \( S_{v_0} \) can be characterized by the following universal property (see §3 of [1]). A map \( \varphi : G \to A \) from \( V(G) \) to an abelian group \( A \) is called harmonic if for each \( v \in G \), we have

\[
\deg(v) \cdot \varphi(v) = \sum_{e=uv \in E} \varphi(w) .
\]

Then \( S_{v_0} \) is universal among all harmonic maps from \( G \) to abelian groups sending \( v_0 \) to 0, in the sense that if \( \varphi : G \to A \) is any such map, then there is a unique group homomorphism \( \psi : \text{Jac}(G) \to A \) such that \( \varphi = \psi \circ S_{v_0} \).

Let \( g = |E(G)| - |V(G)| + 1 \) be the genus\(^1\) of \( G \), which is the number of linearly independent cycles of \( G \), or equivalently, the dimension of \( H_1(G, \mathbb{R}) \).

We write \( S \) instead of \( S_{v_0} \) when the base point \( v_0 \) is understood. In §4, we will prove:

**Theorem 1.7.** The map \( S^{(k)} \) is surjective if and only if \( k \geq g \).

The surjectivity of \( S^{(g)} \) is the graph-theoretic analogue of a classical result about Riemann surfaces known as Jacobi’s Inversion Theorem (see p. 235 of [13]). For a Riemann surface \( X \), it is clear that \( S^{(g-1)} : X^{(g-1)} \to \text{Jac}(X) \) is not surjective, since \( \dim S^{(g-1)} = g - 1 < \dim \text{Jac}(X) = g \).

As a complement to Theorem 1.7, we will also precisely characterize the values of \( k \) for which \( S^{(k)} \) is injective:

**Theorem 1.8.** The map \( S^{(k)} \) is injective if and only if \( G \) is \((k + 1)\)-edge-connected

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\(^1\)In graph theory, the term “genus” is traditionally used for a different concept, namely, the smallest genus of any surface in which the graph can be embedded, and the integer \( g \) is called the “cyclomatic number” of \( G \). We call \( g \) the genus of \( G \) in order to highlight the analogy with Riemann surfaces.
For 2-edge-connected graphs, Theorem 1.8 is the analogue of the well-known fact that the Abel-Jacobi map from a Riemann surface $X$ to its Jacobian is injective if and only if $X$ has genus at least 1. (See Proposition VIII.5.1 of [23].)

1.5. Chip-firing games on graphs. There have been a number of papers devoted to “chip-firing games” played on the vertices of a graph; see, e.g., [4, 5, 6, 12, 20, 21, 30, 32]. In this paper, as an application of Theorem 1.7, we study a new chip firing game with some rather striking features.

Our chip-firing game, like the one considered by Biggs in [4] (see also §31-32 of [3]), is most conveniently stated using “dollars” rather than chips. Let $G$ be a graph, and consider the following game of “solitaire” played on the vertices of $G$. The initial configuration of the game assigns to each vertex $v$ of $G$ an integer number of dollars. Such a configuration can be identified with a divisor $D \in \text{Div}(G)$. A vertex which has a negative number of dollars assigned to it is said to be in debt. A move consists of a vertex $v$ either borrowing one dollar from each of its neighbors or giving one dollar to each of its neighbors. Note that any move leaves the total number of dollars unchanged. The object of the game is to reach, through a sequence of moves, a configuration in which no vertex is in debt. We will call such a configuration a winning position, and a sequence of moves which achieves such a configuration a winning strategy.

As before, we let $g = |E(G)| - |V(G)| + 1$. In §4.2, we will prove:

**Theorem 1.9.** Let $N = \deg(D)$ be the total number of dollars present at any stage of the game.

1. If $N \geq g$, then there is always a winning strategy.
2. If $N \leq g - 1$, then there is always an initial configuration for which no winning strategy exists.

We will prove this theorem by showing that it is equivalent to Theorem 1.7.

1.6. Linear systems and the Riemann-Roch theorem. We define an equivalence relation $\sim$ on the group $\text{Div}(G)$ by declaring that $D \sim D'$ if and only if $D - D' \in \text{Prin}(G)$. Borrowing again from the theory of Riemann surfaces, we call this relation linear equivalence. Since a principal divisor has degree zero, it follows that linearly equivalent divisors have the same degree. Note that by (1.5), the Jacobian of $G$ is the set of linear equivalence classes of degree zero divisors on $G$. 
For $D \in \text{Div}(G)$, we define the linear system associated to $D$ to be the set $|D|$ of all effective divisors linearly equivalent to $D$:

$$|D| = \{ E \in \text{Div}(G) : E \geq 0, E \sim D \} .$$

As we will see in §4.2, it follows from the definitions that two divisors $D$ and $D'$ on $G$ are linearly equivalent if and only if there is a sequence of moves taking $D$ to $D'$ in the chip firing game described in §1.5. It follows that there is a winning strategy in the chip-firing game whose initial configuration corresponds to $D$ if and only if $|D| \neq \emptyset$.

We define the dimension $r(D)$ of the linear system $|D|$ by setting $r(D)$ equal to $-1$ if $|D| = \emptyset$, and then declaring that for each integer $s \geq 0$, $r(D) \geq s$ if and only if $|D - E| \neq \emptyset$ for all effective divisors $E$ of degree $s$. It is clear that $r(D)$ depends only on the linear equivalence class of $D$.

The canonical divisor on $G$ is the divisor $K$ given by

$$(1.10) \quad K = \sum_{v \in V(G)} (\deg(v) - 2) (v) .$$

Since the sum over all vertices $v$ of $\deg(v)$ equals twice the number of edges in $G$, we have $\deg(K) = 2|E(G)| - 2|V(G)| = 2g - 2$.

We can now state a graph-theoretic analogue of the Riemann-Roch theorem (see Theorem VI.3.11 of [23]). The proof will be given in §3.

**Theorem 1.11** (Riemann-Roch for Graphs). Let $G$ be a graph, and let $D$ be a divisor on $G$. Then

$$r(D) - r(K - D) = \deg(D) + 1 - g .$$

**Remark** 1.12. (i) Our definition of $r(D)$ agrees with the usual definition of $r(D)$ as $\dim L(D) - 1$ in the Riemann surface case (see, e.g., p. 250 of [13] or §III.8.15 of [9]).

(ii) One must be careful, however, not to rely too much on intuition from the Riemann surface case when thinking about the quantity $r(D)$ for divisors on graphs. For example, for Riemann surfaces one has $r(D) = 0$ if and only if $|D|$ contains exactly one element, but neither implication is true in general for graphs. For example, consider the canonical divisor $K$ on a graph $G$ with two vertices $v_1$ and $v_2$ connected by $m$ parallel edges. Then clearly $r(K) \geq m - 2$, and in fact we have $r(K) = m - 2$. (This can be proved directly, or deduced as a consequence of Theorem 1.11.) However, $|K| = \{K\}$ as

$$D \sim K \iff \exists i \in \mathbb{Z} : D = (m - 2 + im)(v_1) + (m - 2 - im)(v_2) .$$
To see that the other implication also fails, consider a graph $G$ with $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_3v_1\}$, and $D = 2(v_4) \in \text{Div}(G)$. Then $(v_3) + (v_5) \in |D|$, but it follows from Lemma 3.1 (or can be verified directly) that $|D - (v_1)| = \emptyset$, and therefore $r(D) = 0$.

(iii) The set $L(D) := \{f \in \mathcal{M}(G) : \Delta(f) \geq -D\}$ is not a vector space, so one cannot just define the number $r(D)$ as dim $L(D) - 1$ as in the classical case. This should not be surprising, since elements of $L(D)$ are analogous to functions of the form $\log |f|$ with $f$ a nonzero meromorphic function on a Riemann surface $X$. On the other hand, if we define $L(D) := \{f \in \mathcal{M}(G) : \Delta(f) \geq -D \cup \{+\infty\}\}$, then $L(D)$ is naturally a finitely generated semimodule over the tropical semiring $(\mathbb{N} \cup \{\infty\}, \min, +)$ (see §2.4 of [11]), and there is a natural notion in this context for the dimension of $L(D)$ (see Corollary 95 in [11]). However, examples like the ones above show that the tropical dimension of $L(D)$ is not the same as $r(D) + 1$, and does not obey Theorem 1.11.

2. A Riemann-Roch criterion

In this section, we formulate an abstract criterion giving necessary and sufficient conditions for the Riemann-Roch formula $r(D) - r(K - D) = \deg(D) + 1 - g$ to hold, where $r(D)$ is defined in terms of an equivalence relation on an arbitrary free abelian group. This result, which is purely combinatorial in nature, will be used in §3 in our proof of the Riemann-Roch theorem for graphs.

The general setup for our result is as follows.

Let $X$ be a set, and let $\text{Div}(X)$ be the free abelian group on $X$. As usual, elements of $\text{Div}(X)$ are called divisors on $X$, divisors $E$ with $E \geq 0$ are called effective, and each integer $d$, we denote by $\text{Div}^d(X)$ the set of effective divisors of degree $d$ on $X$.

Let $\sim$ be an equivalence relation on $\text{Div}(X)$ satisfying the following two properties:

(E1) If $D \sim D'$ then $\deg(D) = \deg(D')$.
(E2) If $D_1 \sim D'_1$ and $D_2 \sim D'_2$, then $D_1 + D'_1 \sim D_2 + D'_2$.

For each $D \in \text{Div}(X)$, define $|D| = \{E \in \text{Div}(G) : E \geq 0, E \sim D\}$, and define the function $\dim : \text{Div}(X) \to \{-1, 0, 1, 2, \ldots\}$ by declaring that for each integer $s \geq 0$,

$r(D) \geq s \iff |D - E| \neq \emptyset \forall E \in \text{Div}(X) : E \geq 0$ and $\deg(E) = s$.

It is easy to see that $r(D) = -1$ if $\deg(D) < 0$, and if $\deg(D) = 0$ then $r(D) = 0$ if $D \sim 0$ and $r(D) = -1$ otherwise.
Lemma 2.1. For all $D, D' \in \text{Div}(X)$ we have $r(D + D') \geq r(D) + r(D')$.

Proof. Let $E_D = (x_1) + \cdots + (x_{r(D)+r(D')})$ be an arbitrary effective divisor of degree $r(D) + r(D')$, and let $E = (x_1) + \cdots + (x_{r(D)})$ and $E' = (x_{r(D)+1}) + \cdots + (x_{r(D)+r(D')})$. Then $|D - E|$ and $|D' - E'|$ are non-empty, so that $D - E \sim F$ and $D' - E' \sim F'$ with $F, F' \geq 0$. It follows that $(D + D') - (E + E') = (D + D') - E \sim F + F' \geq 0$, and thus $r(D + D') \geq r(D) + r(D')$. □

Let $g$ be a nonnegative integer, and define

$$\mathcal{N} = \{D \in \text{Div}(X) : \deg(D) = g - 1 \text{ and } |D| = \emptyset\}.$$ 

Finally, let $K$ be an element of $\text{Div}(X)$ having degree $2g - 2$. The following theorem gives necessary and sufficient conditions for the Riemann-Roch formula to hold for elements of $\text{Div}(X)/\sim$.

Theorem 2.2. Define $\epsilon : \text{Div}(X) \to \mathbb{Z}/2\mathbb{Z}$ by declaring that $\epsilon(D) = 0$ if $|D| \neq \emptyset$ and $\epsilon(D) = 1$ if $|D| = \emptyset$. Then the Riemann-Roch formula

$$r(D) - r(K - D) = \deg(D) + 1 - g$$

holds for all $D \in \text{Div}(X)$ if and only if the following two properties are satisfied:

(RR1) For every $D \in \text{Div}(X)$, there exists $\nu \in \mathcal{N}$ such that $\epsilon(D) + \epsilon(\nu - D) = 1$.

(RR2) For every $D \in \text{Div}(X)$ with $\deg(D) = g - 1$, we have $\epsilon(D) + \epsilon(K - D) = 0$.

Remark 2.4. (i) Property (RR2) is equivalent to the assertion that $r(K) \geq g - 1$. Indeed, if (RR2) holds then for every effective divisor $E$ of degree $g - 1$, we have $|K - E| \neq \emptyset$, which means that $r(D) \geq g - 1$. Conversely, if $r(D) \geq g - 1$ then $\epsilon(K - E) = \epsilon(E) = 0$ for every effective divisor $E$ of degree $g - 1$. Therefore $\epsilon(D) = 0$ implies $\epsilon(K - D) = 0$. By symmetry, we obtain $\epsilon(D) = 0$ if and only if $\epsilon(K - D) = 0$, which is equivalent to (RR2).

(ii) When the Riemann-Roch formula (2.3) holds, we automatically have $r(K) = g - 1$.

Remark 2.5. (i) When $X$ is a Riemann surface and $\sim$ denotes linear equivalence of divisors, then one can show independently of the Riemann-Roch theorem that $r(K) = g - 1$, i.e., that the vector space of holomorphic 1-forms on $X$ is $g$-dimensional. Thus one can prove directly that (RR2) holds. We do not know if there is a direct proof of (RR1) which does not make use of Riemann-Roch, but if so, one could deduce the classical Riemann-Roch theorem from it using Theorem 2.2.
(ii) Divisors of degree \( g - 1 \) on a Riemann surface \( X \) which belong to \( N \) are classically referred to as non-special (which explains our use of the symbol \( N \)).

Before giving the proof of Theorem 2.2, we need a couple of preliminary results. The first is the following simple lemma, whose proof is left to the reader.

**Lemma 2.6.** Suppose \( \psi : A \to A' \) is a bijection between sets, and that \( f : A \to \mathbb{Z} \) and \( f' : A' \to \mathbb{Z} \) are functions which are bounded below. If there exists a constant \( c \in \mathbb{Z} \) such that
\[
f(a) - f'(\psi(a)) = c
\]
for all \( a \in A \), then
\[
\min_{a \in A} f(a) - \min_{a' \in A'} f'(a') = c.
\]

If \( D = \sum a_i(x_i) \in \text{Div}(X) \), we define
\[
\deg^+(D) = \sum_{a_i \geq 0} a_i.
\]

The key observation needed to deduce (2.3) from (RR1) and (RR2) is the following alternate characterization of the quantity \( r(D) \):

**Lemma 2.7.** For every \( D \in \text{Div}(X) \) we have
\[
(2.8) \quad r(D) = \left( \min_{D' \sim D, \nu \in N} \deg^+(D' - \nu) \right) - 1.
\]

**Proof.** Let \( r'(D) \) denote the right-hand side of (2.8). If \( r(D) < r'(D) \), then there exists an effective divisor \( E \) of degree \( r'(D) \) for which \( r(D - E) = -1 \). By (RR1), this means that there exists a divisor \( \nu \in N \) and an effective divisor \( E' \) such that \( \nu - D + E \sim E' \). But then \( D' - \nu = E - E' \) for some divisor \( D' \sim D \), and thus
\[
\deg^+(D' - \nu) - 1 \leq \deg(E) - 1 = r'(D) - 1,
\]
contradicting the definition of \( r'(D) \). It follows that \( r(D) \geq r'(D) \).

Conversely, if we choose divisors \( D' \sim D \) and \( \nu \in N \) achieving the minimum in (2.8), then \( \deg^+(D' - \nu) = r'(D) + 1 \), and therefore there are effective divisors \( E, E' \) with \( \deg(E) = r'(D) + 1 \) such that \( D' - \nu = E - E' \). But then \( D - E \sim \nu - E' \), and since \( \nu - E' \) is not equivalent to any effective divisor, it follows that \( |D - E| = \emptyset \). Therefore \( r(D) \leq r'(D) \). \( \Box \)

We can now give the proof of Theorem 2.2.
Proof of Theorem 2.2. We first prove that (2.3) implies (RR1) and (RR2).

Let $D$ be a divisor on $X$. Property (RR2) is more or less immediate, since (2.3) implies that if $\deg(D) = g - 1$ then $r(D) = r(K - D)$.

We cannot have $\epsilon(D) = \epsilon(\nu - D) = 0$, or else by Lemma 2.1 we would have $r(\nu) \geq 0$, contradicting the definition of $N$. Therefore, to prove (RR1) it suffices to show that if $r(D) = -1$ then $r(\nu - D) \geq 0$ for some $\nu \in N$.

If $r(D + E) \geq 0$ for all $E \in \operatorname{Div}_{\geq 0}^g(X)$, then (2.3) implies that $r(K - D) \geq g - 1 - d$. Another application of (2.3) then yields $r(D) = r(K-D)+d+1-g \geq 0$.

Therefore when $r(D) = -1$, there exists an effective divisor $E$ of degree $g - 1 - d$ such that $r(D + E) = -1$. Since $\deg(D + E) = g - 1$, this means that $D + E \in N$, and therefore $D + E = \nu$ for some $\nu \in N$. For this choice of $\nu$, we have $r(\nu - D) \geq 0$, which proves (RR1).

We now show that (RR1) and (RR2) imply (2.3). Let $D \in \operatorname{Div}(X)$. For every $D' \sim D$ and $\nu \in N$, property (RR2) implies that $\nu := K - \nu$ is also in $N$. Writing

$$\nu - D' = K - D' - \nu,$$

it follows that

$$\deg^+(D' - \nu) - \deg^+((K - D') - \nu) = \deg^+(D' - \nu) - \deg^+(\nu - D')$$

$$= \deg(D' - \nu)$$

$$= \deg(D) + 1 - g.$$

Since the difference $\deg^+(D' - \nu) - \deg^+((K - D') - \nu)$ has the constant value $\deg(D) + 1 - g$ for all $D'$ and $\nu$, and since $\nu = K - \nu$ runs through all possible elements of $N$ as $\nu$ does, it follows from Lemmas 2.6 and 2.7 that $r(D) - r(K - D) = \deg(D) + 1 - g$ as desired. \hfill \Box

3. Riemann-Roch for graphs

3.1. Proof of the Riemann-Roch theorem. By Theorem 2.2, in order to prove the Riemann-Roch theorem for graphs (Theorem 1.11), it suffices to verify properties (RR1) and (RR2) when $X = G$ is a graph and $\sim$ denotes linear equivalence of divisors. This will be accomplished through a careful analysis of a certain family of divisors of degree $g - 1$ on $G$.

For each linear (i.e., total) order $<_P$ on $V(G)$, we define

$$\nu_P = \sum_{v \in V(G)} (|\{e = vw \in E(G) : w <_P v\}| - 1)(v).$$
It is clear that \( \deg(\nu_D) = |E(G)| - |V(G)| = g - 1 \).

**Lemma 3.1.** For every linear order \( \prec \) on \( V(G) \) we have \( \nu_P \in \mathcal{N} \).

**Proof.** Let \( D \in \text{Div}(G) \) be any divisor of the form \( D = \nu_D - \Delta(f) \) for some \( f \in \mathcal{M}(G) \). Let \( V^\max_f \) be the set of vertices \( v \in G \) at which \( f \) achieves its maximum value, and let \( u \) be the minimal element of \( V^\max_f \) with respect to the order \( \prec \). Then \( f(w) \leq f(u) \) for all \( w \in V(G) \), and if \( w \prec u \) then \( f(w) < f(u) \). Thus

\[
D(u) = (|\{e = uw \in E(G) : w \prec u \}| - 1) - \sum_{e = uw \in E(G)} (f(u) - f(w))
\]

\[
= -1 + \sum_{e = uw \in E(G)} (f(u) - f(w)) + \sum_{e = uw \in E(G)} (f(w) - f(u) + 1)
\]

\[
\leq -1,
\]

since each term in these sums is non-positive by the choice of \( u \). It follows that \( \nu_D \) is not equivalent to any effective divisor. \( \square \)

**Theorem 3.2.** For every \( D \in \text{Div}(G) \), exactly one of the following holds

(N1) \( r(D) \geq 0 \); or

(N2) \( r(\nu_D - D) \geq 0 \) for some order \( \prec \) on \( V(G) \).

**Proof.** First, note that if (N1) and (N2) both hold, then \( r(\nu_D) \geq 0 \) by Lemma 2.1, contradicting Lemma 3.1. So (N1) and (N2) cannot occur simultaneously.

Let \( n = |V(G)| \), and let \( \mathbb{Z}_+ \) denote the set of nonnegative integers. Fix a divisor \( D \in \text{Div}(G) \), and for \( i \in \mathbb{Z}_+ \) define a sequence of functions \( g_i : V(G) \rightarrow \mathbb{Z} \) and subsets \( S_i \subseteq V(G) \) as follows. We set \( g_0(v) = D(v) \) for all \( v \in V(G) \), \( S_0 = \{v \in V(G) : D(v) < 0\} \), and for \( i \geq 1 \) we define \( g_i \) and \( S_i \) inductively as follows:

\[
g_i(v) = D(v) - |\{e = vw : w \in \bigcup_{j<i} S_j\}|
\]

\[
S_i = \{v \in V(G) : g_i(v) < 0\} - \bigcup_{j<i} S_j.
\]

We make the following elementary observations about the functions \( g_i \) and the sets \( S_i \), which follow easily from the definitions:

(i) The sets \( S_i \) are mutually disjoint.

(ii) \( g_i(v) < 0 \) for all \( v \in S_i \). Conversely, if \( v \not\in S_i \) then \( g_i(v) \geq 0 \) or \( v \in \bigcup_{k<i} S_k \).

(iii) For each \( k \geq 1 \), we have \( g_k(v) \leq g_{k-1}(v) \) for all \( v \in V(G) \), with strict inequality if and only if \( v \) has a neighbor in \( S_{k-1} \).
(iv) If \( k \geq 1 \) and \( v \in S_k \), then \( v \) has a neighbor in \( S_{k-1} \). (If \( v \in S_k \) then \( v \not\in S_i \) for \( i < k \) by (i), so \( g_i(v) \geq 0 \) for \( i < k \) and \( g_k(v) < 0 \) by (ii). By (iii), \( v \) has a neighbor in \( S_{k-1} \).)

(v) If \( S_k \neq \emptyset \), then \( S_i \neq \emptyset \) for all \( 0 \leq i \leq k \). (This follows inductively from (iv).)

(vi) If \( N > n \) then \( S_N = \emptyset \). (This follows from (i) and (v).)

Define the vector \( \mu(D) \in \mathbb{Z}^{n+1} \) by

\[
\mu(D) = \left( \sum_{v \in S_0} g_0(v), \sum_{v \in S_1} g_1(v), \ldots, \sum_{v \in S_n} g_n(v) \right),
\]

where by convention we set \( \sum_{v \in S_i} g_i(v) = 0 \) if \( S_i = \emptyset \). Since \( r(D) \) and \( r(\nu - D) \) both depend only on the linear equivalence class of \( D \), by replacing \( D \) by an equivalent divisor if necessary, we may assume without loss of generality that \( \mu(D) = \max_{D' \sim D} \mu(D') \), where the maximum is taken in the lexicographic order. (The maximum is certainly achieved, since for every divisor \( D \), all coordinates of \( \mu(D) \) are non-positive.)

Claim: If \( S_0 \neq \emptyset \), then \( \bigcup_{i=0}^{n} S_i = V(G) \).

Assuming the claim, we show how to conclude the proof of the theorem. If \( S_0 = \emptyset \), then \( D \geq 0 \) and (N1) holds. On the other hand, suppose \( \bigcup_{i=0}^{n} S_i = V(G) \), so that the sets \( S_i \) partition \( V(G) \). Let \( <_P \) be any total order on \( V(G) \) such that if \( v \in S_i \) and \( w \in S_j \) with \( i \neq j \), then

\[
v <_P w \iff i < j.
\]

(The vertices within each set \( S_i \) can be ordered arbitrarily.)

Then for every \( i \in \mathbb{Z}_+ \) and every \( v \in S_i \), we have

\[
D(v) = g_i(v) + |\{e = vw : w \in \bigcup_{j<i} S_j\}|
\leq -1 + |\{e = vw : w <_P v\}|.
\]

It follows that \( D \leq \nu_P \), and therefore condition (N2) is satisfied.

It remains to prove the claim, which we do in several steps. Suppose for the sake of contradiction that \( T := V(G) - \bigcup_{i \in \mathbb{Z}_+} S_i \) is non-empty and not equal to \( V(G) \).

Define a function \( f \in \mathcal{M}(G) \) by

\[
f(v) = \begin{cases} 1 & v \in T \\ 0 & v \not\in T \end{cases},
\]

and let \( D' = D - \Delta(f) \), so that \( D' \sim D \). We will write \( g'_i \) (resp. \( S'_i \)) for the functions (resp. sets) defined by (3.3) with \( D \) replaced by \( D' \).

We make the following elementary observations about the relationship between \( D \) and \( D' \):
(i) If \( v \in T \), then \( D(v) \geq D'(v) \).
(ii) If \( v \notin T \), then \( D'(v) \geq D(v) \), with strict inequality if and only if \( v \) has a neighbor in \( T \).
(iii) If \( v \in T \), then \( D'(v) \geq 0 \). (For \( v \in T \) and all sufficiently large \( N \), we have \( D'(v) = D(v) - |\{ e = vw : w \in \cup_i S_i \}| = g_N(v) \).
So if \( D'(v) < 0 \) then \( g_N(v) < 0 \), and consequently \( v \in \cup_{i=0}^N S_i \) by (ii), contradicting the definition of \( T \).

Choose \( j \geq 0 \) minimally subject to the constraint that there exists an edge connecting some vertex in \( S_j \) with some vertex in \( T \). (Such a \( j \) necessarily exists, since we are assuming that \( G \) is connected.) We will prove that

\[
\sum_{v \in S_i} g_i(v) \leq \sum_{v \in S'_i} g'_i(v) \forall i \leq j, \text{ with strict inequality when } i = j.
\]

The claim will then follow, since (3.4) implies that \( \mu(D') > \mu(D) \) in lexicographic order on \( \mathbb{Z}^{n+1} \), contradicting our choice of \( D \).

We first prove by induction on \( i \) that

\[
S'_i = S_i \text{ for } i < j, \text{ and } S'_j \subseteq S_j.
\]

The base case \( i = 0 \) follows from (ii)' and (iii)'. Indeed, if \( v \in S'_0 \), then \( v \notin T \) by (ii)' and therefore \( D(v) \leq D'(v) < 0 \) by (ii)', which implies that \( v \in S_0 \). Moreover, if \( j > 0 \) and \( v \in S_0 \), then \( v \) has no neighbor in \( T \) by our choice of \( j \), and thus \( D'(v) = D(v) \) by (ii)', which implies that \( v \in S'_0 \).

Suppose now that \( k \geq 1 \) and \( S'_i = S_i \) for every \( i < k \). Then for every \( v \in V(G) \), we have

\[
g'_k(v) - g_k(v) = D'(v) - D(v).
\]

If \( v \notin S_k \), then by (ii) either \( g_k(v) \geq 0 \) or

\[
v \in \cup_{i<k} S_i = \cup_{i<k} S'_i.
\]

If moreover \( v \in S'_k \), then \( v \notin \cup_{i<k} S'_i \), so we must have \( g_k(v) \geq 0 \). Therefore

\[
v \in S'_k - S_k \Rightarrow g'_k(v) < 0 \text{ and } g_k(v) \geq 0,
\]
so that \( D'(v) - D(v) = g'_k(v) - g_k(v) < 0 \). By (ii)', this means that \( v \in T \). Therefore \( S'_k - S_k \subseteq T \).

On the other hand, if \( v \in S'_k \), then by (iv), \( v \) has a neighbor in \( S'_{k-1} = S_{k-1} \). It follows that \( S'_k - S_k \) is empty if \( k \leq j \), as no vertex in \( T \) has a neighbor in \( S_{k-1} \) by the choice of \( j \). Thus \( S'_k \subseteq S_k \). Moreover,
if \( k < j \) then \( D'(v) = D(v) \) for every \( v \in S_k \) by (ii)', and therefore \( g'_k(v) = g_k(v) \) by (3.6). It follows in this case that \( S_k \subseteq S'_k \), and thus that \( S_k = S'_k \), which proves (3.5).

It remains only to prove (3.4). By (ii)' and (3.5), for all \( i \leq j \) we have \( g'_i(v) \geq g_i(v) \) for every \( v \in S_i \), and the inequality is strict for every vertex in \( S_i \) that has a neighbor in \( T \). Therefore, for all \( i \leq j \) we have

\[
(3.8) \quad \sum_{v \in S_i} g_i(v) \leq \sum_{v \in S'_i} g'_i(v) + \sum_{v \in S_i - S'_i} g_i(v) \leq \sum_{v \in S'_i} g'_i(v).
\]

When \( i = j \), the first inequality in (3.8) is strict if \( S_j = S'_j \), and otherwise the second one is. Therefore (3.4) holds.

As a consequence of Lemma 3.1 and Theorem 3.2, we obtain:

**Corollary 3.9.** For \( D \in \text{Div}(G) \) with \( \text{deg}(D) = g - 1 \) we have \( D \in \mathcal{N} \) if and only if there exists a linear order \( <_P \) on \( V(G) \) such that \( D \sim \nu_P \).

**Proof.** It suffices to note that if \( \nu_P - D \sim E \) with \( E \geq 0 \), then \( \text{deg}(E) = 0 \) and thus \( E = 0 \), so that \( D \sim \nu_P \). □

We can now prove our graph-theoretic version of the Riemann-Roch theorem.

**Proof of Theorem 1.11.** By Theorem 2.2, it suffices to prove that conditions (RR1) and (RR2) are satisfied.

Let \( D \in \text{Div}(G) \), and suppose first that \( r(D) \geq 0 \). By Lemma 2.1, it follows that \( r(\nu - D) \leq r(\nu) - r(D) < 0 \) for every \( \nu \in \mathcal{N} \). Therefore \( \epsilon(D) + \epsilon(\nu - D) = 0 + 1 = 1 \) for every \( \nu \in \mathcal{N} \) and (RR1) holds. Suppose, on the other hand, that \( r(D) < 0 \). Then by Theorem 3.2, we must have \( r(\nu_P - D) \geq 0 \) for some order \( <_P \) on \( V(G) \), and then \( \epsilon(D) + \epsilon(\nu_P - D) = 1 + 0 = 1 \). As \( \nu_P \in \mathcal{N} \) by Lemma 3.1, it follows once again that (RR1) holds.

To prove (RR2), it suffices to show that for every \( D \in \mathcal{N} \) we have \( K - D \in \mathcal{N} \). By Corollary 3.9, we have \( D \sim \nu_P \) for some linear order \( <_P \) on \( V(G) \). Let \( \bar{P} \) be the reverse of \( P \) (i.e., \( v <_P w \Leftrightarrow w <_{\bar{P}} v \)). Then for every \( v \in V(G) \), we have

\[
\nu_P(v) + \nu_P'(v) = (\{|e = vw \in E(G) : w <_P v| - 1\})
+ (\{|e = vw \in E(G) : w <_{\bar{P}} v| - 1\})
= \text{deg}(v) - 2 = K(v).
\]

Therefore \( K - D \sim K - \nu_P = \nu_P \in \mathcal{N} \). □
3.2. **Consequences of the Riemann-Roch theorem.** As in the Riemann surface case, one can derive a number of interesting consequences from the Riemann-Roch formula. As just one example, we prove a graph-theoretic analogue of Clifford’s theorem (see Theorem VII.1.13 of [23]). For the statement, we call a divisor \( D \) **special** if \( |K - D| \neq \emptyset \), and **non-special** otherwise.

**Corollary 3.10** (Clifford’s Theorem for Graphs). Let \( D \) be an effective special divisor on a graph \( G \). Then

\[
r(D) \leq \frac{1}{2} \deg(D) .
\]

**Proof.** If \( D \) is effective and special, then \( K - D \) is also effective, and by Lemma 2.1 we have

\[
r(D) + r(K - D) \leq r(K) = g - 1 .
\]

On the other hand, by Riemann-Roch we have

\[
r(D) - r(K - D) \leq r(K) = \deg(D) + 1 - g .
\]

Adding these two expressions gives \( 2r(D) \leq \deg(D) \) as desired. \( \Box \)

As pointed out in §IV.5 of [14], the interesting thing about Clifford’s theorem is that for a non-special divisor \( D \), we can compute \( r(D) \) exactly as a function of \( \deg(D) \) using Riemann-Roch. However, for a special divisor, \( r(D) \) does not depend only on the degree. Therefore it is useful to have a non-trivial upper bound on \( r(D) \), and this is what Corollary 3.10 provides.

4. **The Abel-Jacobi map from a graph to its Jacobian**

Let \( G \) be a graph, let \( v_0 \in V(G) \) be a base point, and let \( k \) be a positive integer. In this section, we discuss the injectivity and surjectivity of the map \( S^{(k)}_{v_0} \).

We leave it to the reader to verify the following elementary observations:

**Lemma 4.1.**

1. \( S^{(k)}_{v_0} \) is injective if and only if whenever \( D, D' \) are effective divisors of degree \( k \) with \( D \sim D' \), we have \( D = D' \). If \( S^{(k)}_{v_0} \) is injective, then \( S^{(k')}_{v_0} \) is injective for all positive integers \( k' \leq k \).

2. \( S^{(k)}_{v_0} \) is surjective if and only if every divisor of degree \( k \) is linearly equivalent to an effective divisor. If \( S^{(k)}_{v_0} \) is surjective, then \( S^{(k')}_{v_0} \) is surjective for all integers \( k' \geq k \).
In particular, whether or not $S_{v_0}$ is injective (resp. surjective) is independent of the base point $v_0$. We therefore write $S$ instead of $S_{v_0}$ in what follows.

4.1. **Surjectivity of the maps $S^{(k)}$.** We recall the statement of Theorem 1.7:

**Theorem.** The map $S^{(k)}$ is surjective if and only if $k \geq g$.

**Proof of Theorem 1.7.** This is an easy consequence of the Riemann-Roch theorem for graphs. If $D$ is a divisor of degree $d \geq g$, then since $r(K - D) \geq -1$, Riemann-Roch implies that $r(D) \geq 0$, so that $D$ is linearly equivalent to an effective divisor. Thus $S^{(d)}$ is surjective. (Alternatively, we can apply (RR1) directly: if $\text{deg}(D) \geq g$, then for all $\nu \in \mathcal{N}$ we have $\text{deg}(\nu - D) < 0$ and thus $r(\nu - D) = -1$. By (RR1) we thus have $r(D) \geq 0$.)

Conversely, (RR1) implies that $\mathcal{N} \neq \emptyset$, and therefore $S^{(g-1)}$ is not surjective. □

**Remark 4.2.** This result was left as an open problem on p. 179 of [1].

4.2. **The chip-firing game revisited.** As mentioned earlier, Theorems 1.9 and 1.7 are equivalent. To see this, we note the following easy lemma:

**Lemma 4.3.** Two divisors $D$ and $D'$ on $G$ are linearly equivalent if and only if there is a sequence of moves in the chip firing game which transforms the configuration corresponding to $D$ into the configuration corresponding to $D'$.

**Proof.** A sequence of moves in the chip-firing game can be encoded as the $n \times 1$ column vector $\vec{v}$ whose $i$th entry is the number of times vertex $i$ “borrows” a dollar minus the number of time it “lends” a dollar. (Note that the game is “commutative”, in the sense that the order of the moves does not matter.) The end configuration, starting from an initial configuration of 0 dollars at each vertex and playing the moves corresponding to $\vec{v}$, is given by the vector $Q\vec{v}$. So the dollar distributions achievable from the initial configuration $\vec{0}$ are precisely the vectors of the form $Q(\vec{v})$ for $\vec{v} \in \mathbb{Z}^n$. These are the same as the divisors linearly equivalent to zero (after identifying divisors on $G$ with vectors in $\mathbb{Z}^n$).

The equivalence between Theorem 1.9 and Theorem 1.7 is now an immediate consequence of Lemma 4.1(1), since as we have already noted, there is a winning strategy in the chip-firing game whose initial configuration corresponds to $D$ if and only if $D$ is linearly equivalent to an effective divisor. In particular, we have now proved Theorem 1.9.
4.3. Injectivity of the maps $S^{(k)}$. We recall the statement of Theorem 1.8.

**Theorem.** The map $S^{(k)}$ is injective if and only if $G$ is $(k+1)$-edge-connected.

**Proof of Theorem 1.8.** Suppose $G$ is not $(k+1)$-edge-connected. Let $C \subseteq E(G)$ be an edge cut of size $j \leq k$, and let $X \subseteq V(G)$ be one of the components of $G - C$. Let $D = \sum_{v \in X} |E_v \cap C|(v)$, and define $f \in \mathcal{M}(G)$ by the formula

$$f(v) = \begin{cases} 1 & v \in X \\ 0 & v \not\in X. \end{cases}$$

Let $D' = D - \Delta(f)$. Then for each $v \in V(G)$, we have

$$D'(v) = |E_v \cap C| \cdot f(v) - \sum_{e = vw \in E_v} (f(v) - f(w))$$

$$= \begin{cases} 0 & v \in X \\ |\{e = vw \in E_v : w \in X\}| & v \not\in X. \end{cases}$$

Thus $D, D' \geq 0$, $D \sim D'$, and $D \neq D'$. It follows that the map $S^{(j)}$ is not injective, and consequently neither is $S^{(k)}$.

Conversely, suppose we are given $D, D' \in \text{Div}^+_k(G)$ such that $D \sim D'$ and $D \neq D'$. Let $f \in \mathcal{M}(G)$ be the unique function for which $D' - D = \Delta(f)$, $f(v) \geq 0$ for all $v \in G$, and $X := \{v \in V(G) : f(v) = 0\}$ is neither empty nor equal to $V(G)$. Such a function $f$ exists and is unique because the kernel of $\Delta$ is spanned by the constant function 1. If $C$ is the cut separating $X$ from $V(G) - X$, then

$$0 \leq \sum_{v \in X} D'(v) = \sum_{v \in X} \left( D(v) + \sum_{e = vw \in E_v} (f(v) - f(w)) \right)$$

$$\leq \deg(D) - \sum_{v \in X} |\{e = vw \in E_v : w \not\in X\}|$$

$$= k - |C| .$$

It follows that $|C| \leq k$, and therefore $G$ is not $(k+1)$-edge-connected. \qed

In particular, $S$ is injective if and only if every edge of $G$ is contained in a cycle.

**Remark 4.4.** In part (iv) of Proposition 7 in [1], the authors state that $S$ is injective if $G$ is has vertex connectivity at least 2, and is not the graph consisting of one edge connecting two vertices. However, their proof contains an error (the map $h : V \to \mathbb{Z}/n\mathbb{Z}$ which they define
need not be harmonic). In any case, Theorem 1.8 for \( k = 1 \) is a more
precise result.

4.4. Injectivity of the Abel-Jacobi map via circuit theory. There
is an alternate way to see that \( S \) is injective if and only if \( G \) is 2-edge-
connected using the theory of electrical networks (which we refer to as
circuit theory). We sketch the argument here; see §15 of [3] for some
background on electrical networks.

Consider \( G \) as an electric circuit where the edges are resistors of
resistance 1, and let \( \iota_{v_0}^v(e) \) be the current flowing through the oriented
edge \( e \) when one unit of current enters the circuit at \( v \) and exits at \( v_0 \).
Let \( d : C^0(G, \mathbb{R}) \to C^1(G, \mathbb{R}) \) and \( d^* : C^1(G, \mathbb{R}) \to C^0(G, \mathbb{R}) \) be the
usual operators on cochains (see §1 of [1]). By Kirchhoff’s laws, \( \iota_{v_0}^v(e) \) is
the unique element \( \iota_{v_0}^v \in C^1(G, \mathbb{R}) \cap \text{Im}(d) \) for which \( d^*(\iota_{v_0}^v) = (v) - (v_0) \). It
follows from the fact that \( d(C^0(G, \mathbb{Z})) = C^1(G, \mathbb{Z}) \) that \( \iota_{v_0}^v \in C^1(G, \mathbb{Z}) \) if
and only if \( (v) - (v_0) \in d^*(C^1(G, \mathbb{Z})) = (d^*d)(C^0(G, \mathbb{Z})) \), which happens
if and only if \( S_{v_0}(v) = 0 \).

Circuit theory implies that \( 0 < |\iota_{v_0}^v(e)| \leq 1 \) for every edge \( e \) which
belongs to a path connecting \( v \) and \( v_0 \). In other words, the magnitude
of the current flow is at most 1 everywhere in the circuit, and a nonzero
amount of current must flow along every path from \( v \) to \( v_0 \).

Recall that a graph \( G \) is 2-edge-connected if and only if every edge
of \( G \) is contained in a cycle. So if \( G \) is 2-edge-connected, then circuit
theory implies that \( |\iota_{v_0}^v(e)| < 1 \) for every edge \( e \) belonging to a path
connecting \( v \) and \( v_0 \). (Some current flows along each path from \( v \) to \( v_0 \),
and there are at least two such edge-disjoint paths.) Therefore
\( \iota_{v_0}^v \notin C^1(G, \mathbb{Z}) \), so \( S_{v_0}(v) \neq 0 \). Since \( S_{v_0}(v) - S_{v_0}(v') = S_{v'}(v) \), this
implies that \( S_{v_0} \) is injective.

Conversely, if an edge \( e' \) of \( G \) is not contained in any cycle, then
letting \( v, v' \) denote the endpoints of \( e' \), it follows from circuit theory
that
\[
\iota_{v_0}^v(e) = \begin{cases} 1 & \text{if } e = e' \\ 0 & \text{otherwise.} \end{cases}
\]
Therefore \( S_{v_0}(v) = S_{v_0}(v') \) and \( S_{v_0} \) is not injective.

Remark 4.5. A similar argument is given in §9 of [8], although the con-
nection with the Jacobian of a finite graph is not explicitly mentioned.

The circuit theory argument actually tells us something more precise
about the failure of \( S \) to be injective on a general graph \( G \). Let \( \overline{G} \) be
the graph obtained by contracting every edge of \( G \) which is not part of
a cycle, and let \( \rho : G \to \overline{G} \) be the natural map.

Lemma 4.6. \( \rho(v_1) = \rho(v_2) \) if and only if \( (v_1) \sim (v_2) \).
Proof. \( \rho(v_1) = \rho(v_2) \) if and only if there is a path from \( v_1 \) to \( v_2 \) in \( G \), none of whose edges belong to a cycle. By circuit theory, this occurs if and only if there is a unit current flow from \( v_1 \) to \( v_2 \) which is integral along each edge. By the above discussion, this happens if and only if \( (v_1) \sim (v_2) \). \qed

As a consequence of Lemma 4.6 and Theorem 1.8, we obtain:

**Corollary 4.7.** For every graph \( G \) and every base point \( v_0 \in G \), there is a commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\rho} & \overline{G} \\
\downarrow S & & \downarrow \overline{S} \\
\text{Jac}(G) & \xrightarrow{\rho_* \cong} & \text{Jac}(\overline{G})
\end{array}
\]

in which \( \rho_* \) is an isomorphism, \( \rho \) is surjective, and \( \overline{S} = S_{\rho(v_0)} \) is injective.

**Remark 4.8.** (i) It is not hard to give a rigorous proof of Corollary 4.7 which does not rely on circuit theory by showing that the natural map \( \rho_* : \text{Div}(G) \to \text{Div}(G') \) given by \( \rho_*(\sum a_v(v)) = \sum a_v(\rho(v)) \) sends principal divisors to principal divisors and induces a bijection \( \text{Jac}(G) \to \text{Jac}(G') \). We leave this as an exercise for the interested reader.

(ii) Theorem 1.8 and Corollary 4.7 suggest that from the point of view of Abel-Jacobi theory, the “correct” analogue of a Riemann surfaces is a 2-edge-connected graph. This point of view resonates with the classification of Riemann surfaces by genus. For example, there is a unique Riemann surface of genus 0 (the Riemann sphere), and there is a unique 2-edge-connected graph of genus 0 (the graph with one vertex and no edges). Similarly, Riemann surfaces of genus 1 are classified up to isomorphism by a single complex number known as the “\( j \)-invariant”, and a 2-edge-connected graph of genus 1 is isomorphic to a cycle of length \( n \geq 2 \), so is determined up to isomorphism by the integer \( n \).

5. **Complements**

5.1. **Morphisms between graphs.** In algebraic geometry, one is usually interested not just in Riemann surfaces themselves but also in the holomorphic maps between them. The most general graph-theoretic analogue of a holomorphic map between Riemann surfaces in the context of the present paper appears to be the notion of a **harmonic morphism**, as defined in [31]. For a non-constant harmonic morphism
f : X_1 → X_2, there is a graph-theoretic analogue of the classical
Riemann-Hurwitz formula relating the canonical divisor on X_1 to the
pullback of the canonical divisor on X_2. Moreover, a non-constant har-
monic morphism f : X_1 → X_2 induces maps f_* : Jac(X_1) → Jac(X_2)
and f^* : Jac(X_2) → Jac(X_1) between the Jacobians of X_1 and X_2
in a functorial way. We will discuss these and other matters, includ-
ing several characterizations of “hyperelliptic” graphs, in a subsequent
paper.

5.2. Generalizations. There are some obvious ways in which one
might attempt to generalize the results of this paper. For example:
1. We have dealt in this paper only with finite unweighted graphs,
but it would be interesting to generalize our results to certain infinite
graphs, as well as to weighted and/or metric graphs.
2. Can the quantity \( r(D) - r(K - D) \) appearing in Theorem 1.11
be interpreted in a natural way as an Euler characteristic? In other
words, is there a Serre duality theorem for graphs?
3. One could try to generalize some of the results in this paper
to higher-dimensional simplicial complexes. For example, is there a
higher-dimensional generalization of Theorem 1.11 analogous to the
Hirzebruch-Riemann-Roch theorem in algebraic geometry?

5.3. Other Riemann-Roch theorems.
1. Metric graphs are closely related to “tropical curves”, and in this
context Mikhalkin and Zharkov have recently announced a tropical
Abel-Jacobi theorem and a tropical Riemann-Roch inequality (see §5.2
of [22]). It appears, however, that their definition of \( r(D) \) is different
from ours (this is related to the discussion in Remark 1.12).
2. There is a Riemann-Roch formula in toric geometry having to do
with lattice points and volumes of polytopes (see, e.g., §5.3 of [10]).
Our Theorem 1.11 appears to be of a rather different nature.

5.4. Connections with number theory. The first author’s origi-
nal motivation for looking at the questions in this paper came from
connections with number theory. We briefly discuss a few of these
connections.
1. The Jacobian of a finite graph arises naturally in the branch of
number theory known as arithmetic geometry. One example is the
theorem of Raynaud [25] relating a proper regular model \( \mathcal{X} \) for a curve
\( X \) over a discrete valuation ring to the group of connected components
\( \Phi \) of the Néron model of the Jacobian of \( X \). Although not usually stated
in this way, Raynaud’s result essentially says that \( \Phi \) is the Jacobian
of the dual graph of the special fiber of \( \mathcal{X} \). See [8, 16, 17, 18] for
Raynaud's theorem plays an important supporting role in a number of seminal papers in number theory (see, for example, [19] and [26]).

2. The canonical divisor $K$ on a graph, as defined in (1.10), plays a prominent role in Zhang’s refinement of Arakelov’s intersection pairing on an arithmetic surface (see [33]).

3. By its definition as a “Picard group”, the Jacobian of a finite graph $G$ can be thought of as analogous to the ideal class group of a number field. In particular, the number $\kappa(G)$ of spanning trees in a graph $G$, which is the order of $\text{Jac}(G)$, is analogous to the class number of a number field. This analogy appears in the functional equation for the Ihara zeta function of $G$ (see e.g. [15, 27, 28, 29]), where $\kappa(G)$ plays the same role as the class number does in the functional equation for the Dedekind zeta function of a number field.

References


