Assignment 3 = Exam 1:
Finite-Dimensional Vector Spaces
Due Tuesday February 15, 2022

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Problem 1 *(Axler 2A1,6) Let* v_1, v_2, v_3, *and* v_4 *be vectors in a vector space* V.

(a) *Show that if* $A = \{v_1, v_2, v_3, v_4\}$ *spans* V, *then*

$$B = \{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$$

spans V

(b) *Show that if* A *is linearly independent, then* B *is linearly independent.*

Problem 2 *(Axler 2A2) Verify the following:*

(a) *A singleton* $\{v\}$ *containing one vector in a vector space is linearly independent if and only if* $v \neq \mathbf{0}$.

(b) *A doubleton* $\{v_1, v_2\}$ *containing two vectors in a vector space is linearly independent if and only if neither vector is a scalar multiple of the other.*

(c) $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$ *is linearly independent in* $\mathbb{R}^4$.

(d) $\{1, z, z^2, \ldots, z^m\}$ *is linearly independent in the vector space of polynomials with complex coefficients* $\mathcal{P}(\mathbb{C})$ *for every* $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

Problem 3 *(Axler 2A4,5,12,13)*

(a) *Find all values of* c *for which* $\{(2,3,1),(1,-1,2),(7,3,c)\}$ *is linearly dependent in* F^3.

(b) *Show that* $\{1+i, 1-i\}$ *is linearly independent in the real vector space* $\mathbb{C}$.

(c) *Show the following: If* $A = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ *is a collection of polynomials in* $\mathcal{P}_4(F)$*, the vector space of polynomials with coefficients in* F *having degree four or less, then* A *is linearly dependent.*

(d) *Show the following: If* $B = \{p_1, p_2, p_3, p_4\}$ *is a collection of polynomials in* $\mathcal{P}_4(F)$*, then*

$$\operatorname{span} B \neq \mathcal{P}_4(F).$$

Problem 4 *(Axler 2B2,5) Verify the following:*

(a) $\{1, z, z^2, \ldots, z^m\}$ *is a basis for* $\mathcal{P}_m(\mathbb{C})$ *the vector space of polynomials with complex coefficients and order less than or equal to* m.

(b) *There exists a basis* $\{p_1, p_2, p_3, p_4\}$ *of* $\mathcal{P}_3(\mathbb{C})$ *such that none of the polynomials* p_1, p_2, p_3, p_4 *is of degree 2.*

Problem 5 *(Axler 2B7) Prove or disprove: If* $\{v_1, v_2, v_3, v_4\}$ *is a basis of* V *and* W *is a subspace of* V *such that* $v_1, v_2 \in W$ *and* $v_3 \notin W$ *and* $v_4 \notin W$*, then* $\{v_1, v_2\}$ *is a basis of* W.

Problem 6 *(Axler 2C8) Let*

$$W = \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p(x)\, dx = 0 \right\}.$$

(a) *Show that* W *is a subspace of* $\mathcal{P}_4(\mathbb{R})$.

(b) *Find a basis* B *for* W.

(c) *Extend the basis* B *to a basis* A *for* $\mathcal{P}_4(\mathbb{R})$.

(d) *Find a subspace* V *of* $\mathcal{P}_4(\mathbb{R})$ *such that* $\mathcal{P}_4(\mathbb{R}) = W \oplus V$.

Problem 7 *(Axler 2C10) Show that if* $A = \{p_0, p_1, p_2, \ldots, p_n\} \subset \mathcal{P}(F)$ *with* $\deg(p_j) = j$ *for* $j = 0, 1, 2, \ldots, n$*, then* A *is a basis for* $\mathcal{P}_n(F)$.

Problem 8 *(sums of subspaces and direct sums of subspaces) Let*

$$V = \{(x, y, 0) : x, y \in \mathbb{R}\},$$
$$W = \{(x, 0, x) : x \in \mathbb{R}\}, \text{ and}$$
$$Z = \{(0, y, y) : y \in \mathbb{R}\}$$

be subspaces in $\mathbb{R}^3$.

(a) *Find* $V + W$.

(b) *Find* $V + Z$.

(c) *Find* $V + W + Z$.

(d) *Show that* $V \cap W = V \cap Z = W \cap Z = \{\mathbf{0}\}$.

(e) *Which of the sums in (a-c) are direct sums?*

Problem 9 *(Axler 2C13) If* W_1 *and* W_2 *are both four-dimensional subspaces of* $\mathbb{R}^6$, *find the smallest integer* n *and the largest integer* m *for which*

$$n \leq \dim(W_1 \cap W_2) \leq m,$$

and justify your answer.

Problem 10 *(Axler 2C15) If* V *is a finite dimensional vector space with dimension* $\dim(V) = n$, *then there are one-dimensional subspaces* $W_1, W_2, \ldots, W_n$ *of* V *such that*

$$V = \bigoplus_{j=1}^{n} W_j.$$