Assignment 3 = Exam 1: Finite-Dimensional Vector Spaces Solutions

John McCuan

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Problem 1 *(Axler 2A1,6) Let v_1, v_2, v_3, and v_4 be vectors in a vector space V.*

(a) *Show that if $A = \{v_1, v_2, v_3, v_4\}$ spans V, then*

$$B = \{v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4\}$$

spans V

(b) *Show that if A is linearly independent, then B is linearly independent.*

Problem 2 *(Axler 2A2) Verify the following:*

(a) *A singleton $\{v\}$ containing one vector in a vector space is linearly independent if and only if $v \neq \mathbf{0}$.*

(b) *A doubleton $\{v_1, v_2\}$ containing two vectors in a vector space is linearly independent if and only if neither vector is a scalar multiple of the other.*

(c) $\{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)\}$ *is linearly independent in $\mathbb{R}^4$.*

(d) $\{1, z, z^2, \ldots, z^m\}$ *is linearly independent in the vector space of polynomials with complex coefficients $\mathcal{P}(\mathbb{C})$ for every $m \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$.*

Problem 3 *(Axler 2A4,5,12,13)*

(a) *Find all values of* c *for which* $\{(2,3,1),(1,-1,2),(7,3,c)\}$ *is linearly dependent in* F^3.

(b) *Show that* $\{1+i, 1-i\}$ *is linearly independent in the real vector space* $\mathbb{C}$.

(c) *Show the following: If* $A = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ *is a collection of polynomials in* $\mathcal{P}_4(F)$*, the vector space of polynomials with coefficients in* F *having degree four or less, then* A *is linearly dependent.*

(d) *Show the following: If* $B = \{p_1, p_2, p_3, p_4\}$ *is a collection of polynomials in* $\mathcal{P}_4(F)$*, then*

$$\operatorname{span} B \neq \mathcal{P}_4(F).$$

Problem 4 *(Axler 2B2,5) Verify the following:*

(a) $\{1, z, z^2, \ldots, z^m\}$ *is a basis for* $\mathcal{P}_m(\mathbb{C})$ *the vector space of polynomials with complex coefficients and order less than or equal to* m.

(b) *There exists a basis* $\{p_1, p_2, p_3, p_4\}$ *of* $\mathcal{P}_3(\mathbb{C})$ *such that none of the polynomials* p_1, p_2, p_3, p_4 *is of degree 2.*

Problem 5 *(Axler 2B7) Prove or disprove: If* $\{v_1, v_2, v_3, v_4\}$ *is a basis of* V *and* W *is a subspace of* V *such that* $v_1, v_2 \in W$ *and* $v_3 \notin W$ *and* $v_4 \notin W$*, then* $\{v_1, v_2\}$ *is a basis of* W.

Problem 6 *(Axler 2C8) Let*

$$W = \left\{ p \in \mathcal{P}_4(\mathbb{R}) : \int_{-1}^{1} p(x)\,dx = 0 \right\}.$$

(a) *Show that* W *is a subspace of* $\mathcal{P}_4(\mathbb{R})$.

(b) *Find a basis* B *for* W.

(c) *Extend the basis* B *to a basis* A *for* $\mathcal{P}_4(\mathbb{R})$.

(d) *Find a subspace* V *of* $\mathcal{P}_4(\mathbb{R})$ *such that* $\mathcal{P}_4(\mathbb{R}) = W \oplus V$.

Problem 7 *(Axler 2C10) Show that if* $A = \{p_0, p_1, p_2, \ldots, p_n\} \subset \mathcal{P}(F)$ *with* $\deg(p_j) = j$ *for* $j = 0, 1, 2, \ldots, n$*, then* A *is a basis for* $\mathcal{P}_n(F)$.

Solution: Remember that $\mathcal{P}_n = \mathcal{P}_n(F)$ is the vector space of polynomials of degree at most n, and to be a basis means to be a **linearly indepdendent spanning set**. Thus, we wish to show A is linearly independent and $\text{span}(A) = \mathcal{P}_n$.

As the notation gets a little cumbersome here, let me illustrate the situation in a case when n is relatively small. Say $n = 2$ and we have polynomials

$$\begin{aligned} p_0 &= a_{00} \\ p_1 &= a_{10} + a_{11}x \\ p_2 &= a_{20} + a_{21}x + a_{22}x^2. \end{aligned}$$

The assumption that $\deg(p_j) = j$ here may be interpreted to mean $a_{j0} \neq 0$ for $j = 0, 1, 2$. That is, the top order coefficient of each polynomial is nonzero. In particular, $p_0 = a_{00} \neq 0$ giving the first condition of Axler's lemma for $\{p_0, p_1, p_2\}$ to be a linearly independent set. It is pretty clear that

$$\text{span}\{p_0, p_1, \dots, p_{\ell-1}\} \subset \mathcal{P}_{\ell-1}.$$

That is, every linear combination of the polynomials $p_0, p_1, \dots, p_{\ell-1}$ is a polynomial of degree less than or equal to $\ell-1$. This holds for us when $\ell = 1$ or $\ell = 2$, but if we had more polynomials it is also clear that this argument holds in general. Consequently, it is clear that

$$p_\ell \notin \{p_0, p_1, \dots, p_{\ell-1}\} \qquad \text{for} \qquad \ell = 1, 2.$$

And for $\{p_0, p_1, \dots, p_n\}$ in general

$$p_\ell \notin \{p_0, p_1, \dots, p_{\ell-1}\} \qquad \text{for} \qquad \ell = 1, 2, \dots n.$$

Axler's lemma implies $\{p_0, p_1, \dots, p_n\}$ is linearly independent (particularly in the case $n = 2$).

It remains to show $\{p_0, p_1, p_2\}$ spans $\mathcal{P}_2$. Let q be any polynomial of degree less than or equal to 2. This means we can write

$$q = b_0 + b_1x + \cdots + b_mx^m = \sum_{j=0}^{m} b_jx^j$$

where $m \leq 2$ and $b_m \neq 0$. Actually, there is another possibility, namely that $q = 0$, but we can either ignore this case or simply note that

$$0 = \sum_{j=0}^{n} 0p_j$$

so the zero polynomial is certainly in the span of $A = \{p_0, p_1, p_2\}$. Returning to the "real" case in which $b_m \neq 0$, we consider cases:

$m = 0$**:** In this case, $q = b_0$ is (a) constant and

$$q = \frac{b_0}{a_{00}} p_0 + 0p_1 + 0p_2.$$

We conclude $q \in \text{span}\{p_0, p_1, p_2\}$.

$m = 1$**:** In this case, $q = b_0 + b_1 x$ with $b_1 \neq 0$. We can take

$$q = \frac{b_1}{a_{11}} p_1 + \left(\frac{b_0}{a_{00}} - \frac{b_1 a_{10}}{a_{00} a_{11}} \right) p_0 + 0p_2. \tag{1}$$

This looks a little complicated, so let's think about it a bit more.

In the end, we want to write q as a linear combination of p_0, p_1, p_2:

$$q = c_0 p_0 + c_1 p_1 + c_2 p_2 = \sum_{j=0}^{2} c_j p_j.$$

Notice that we've put $c_2 = 0$ as the coefficient of p_2 in (1) because p_2 has degree two and q in this case has degree one. (So if we had a nonzero coefficient c_2 for p_2 we would definitely get a degree two polynomial for every linear combination

$$c_0 p_0 + c_1 p_1 + c_2 p_2 = \sum_{j=0}^{2} c_j p_j,$$

and that couldn't be q. Next, we want the coefficient of x in

$$c_0 p_0 + c_1 p_1 + 0p_2 = \sum_{j=0}^{1} c_j p_j$$

to be b_1. This coefficent, however, is

$$c_1 a_{11} \qquad \text{since} \qquad c_1 p_1 = c_1 a_{11} x + c_1 a_{10}$$

and $c_0 p_0 = c_0 a_{00}$ is constant. Therefore, **we need**

$$c_1 a_{11} = b_1.$$

That is,

$$c_1 = \frac{b_1}{a_{11}}.$$

And you see this is the choice we have made for c_1 in (1). It remains to determine c_0. So far we have

$$c_2 p_2 + c_1 p_1 + c_0 p_0 = \frac{b_1}{a_{11}}(a_{11}x + a_{10}) + c_0 a_{00}.$$

Gathering together the constant terms we must have

$$\frac{b_1}{a_{11}} a_{10} + c_0 a_{00} = b_0.$$

That is,

$$c_0 = \frac{1}{a_{00}} \left(b_0 - \frac{b_1}{a_{11}} a_{10} \right)$$

which is the value we used in (1).

$m = 2$: In this case, $q = b_0 + b_1 x + b_2 x^2$ with $b_2 \neq 0$. Hopefully, we can see from the previous case how to choose the coefficients c_0, c_1, c_2. Writing

$$c_2 p_2 + c_1 p_1 + c_0 p_0 = b_2 x^2 + b_1 x + b_0,$$

we can start with

$$c_2 = \frac{b_2}{a_{22}}.$$

This choice means the coefficient of x in $c_2 p_2 + c_1 p_1 + c_0 p_0$ is

$$\frac{b_2}{a_{22}} a_{21} + c_1 a_{11}.$$

This means we need

$$c_1 = \frac{1}{a_{11}} \left(b_1 - \frac{b_2}{a_{22}} a_{21} \right).$$

With this choice we can see the constant term in $c_2 p_2 + c_1 p_1 + c_0 p_0$ is

$$\frac{b_2}{a_{22}} a_{20} + \frac{1}{a_{11}} \left(b_1 - \frac{b_2}{a_{22}} a_{21} \right) a_{10} + c_0 a_{00}.$$

Hence we take

$$c_0 = \frac{1}{a_{00}} \left[b_0 - \frac{b_2}{a_{22}} a_{20} - \frac{1}{a_{11}} \left(b_1 - \frac{b_2}{a_{22}} a_{21} \right) a_{10} \right].$$

Indeed with this choice we see

$$\begin{aligned} q &= b_2 x^2 + b_1 x + b_0 \\ &= \frac{b_2}{a_{22}} p_2 + \frac{1}{a_{11}} \left(b_1 - \frac{b_2}{a_{22}} a_{21} \right) p_1 \\ &\qquad + \frac{1}{a_{00}} \left[b_0 - \frac{b_2}{a_{22}} a_{20} - \frac{1}{a_{11}} \left(b_1 - \frac{b_2}{a_{22}} a_{21} \right) a_{10} \right] p_0. \end{aligned}$$

The General Case The solution we have given for $n = 2$ looks quite complicated, and in the general case, it is probably convenient to use some sort of formal iterative procedure (or perhaps a kind of induction). Let's see:

We know as above that if we want to write

$$q = \sum_{j=0}^{m} b_j x^j = \sum_{j=0}^{n} c_j p_j$$

where $m \le n$ and $b_m \ne 0$, then we need $c_n = c_{n-1} = \cdots = c_{m+1} = 0$ and

$$c_m = \frac{b_m}{a_{mm}}.$$

Here we are writing our given polynomials in $A = \{p_0, p_1, \ldots, p_n\}$ as

$$p_k = \sum_{j=0}^{k} a_{kj} x^j$$

with $a_{kk} \ne 0$ for $k - 0, 1, \ldots, n$. Working backwards, say we have determined the coefficients $c_n, c_{n-1}, \ldots, c_\ell$ for some $\ell \le m$. By this we mean

$$\sum_{j=\ell}^{m} c_j p_j$$

is a polynomial of degree m which we can write as

$$\sum_{j=\ell}^{m} c_j p_j = \sum_{j=0}^{m} \beta_{\ell j} x^j$$

with coefficients $\beta_{\ell 0}, \beta_{\ell 1}, \ldots, \beta_{\ell m}$ satisfying

$$\beta_{\ell j} = b_j \qquad \text{for} \qquad j = \ell, \ell+1, \ldots, m. \tag{2}$$

We then consider a linear combination

$$c_{\ell-1} p_{\ell-1} + \sum_{j=\ell}^{m} c_j p_j = \sum_{j=0}^{m} \beta_{\ell-1,j} x^j. \tag{3}$$

Clearly since $p_{\ell-1}$ has degree $\ell - 1$, the condition (2) implies

$$\beta_{\ell-1,j} = \beta_{\ell j} = b_j \qquad \text{for} \qquad j = \ell, \ell+1, \ldots, m. \tag{4}$$

Furthermore, we can see the coefficent of $x^{\ell-1}$ in (3) is

$$c_{\ell-1} a_{\ell-1,\ell-1} + \sum_{j=\ell}^{m} c_j a_{j,\ell-1}.$$

Therefore, by choosing

$$c_{\ell-1} = \frac{1}{a_{\ell-1,\ell-1}} \left(b_{\ell-1} - \sum_{j=\ell}^{m} c_j a_{j,\ell-1} \right)$$

we ensure the last/next relation to go along with (4), namely

$$\beta_{\ell-1,\ell-1} = b_{\ell-1}.$$

Repeating this procedure finitely many times, we obtain the condition of the recursion with $\ell = 0$ according to which

$$\sum_{j=0}^{m} c_j p_j = \sum_{j=0}^{m} \beta_{0j} x^j$$

is a polynomial with coefficients satisfying

$$\beta_{\ell j} = b_j \qquad \text{for} \qquad j = 0, 1, \ldots, m.$$

That is,

$$q = \sum_{j=0}^{m} c_j p_j + \sum_{j=m+1}^{n} 0 p_j$$

as was to be shown.

I guess this argument is pretty convincing and pretty good. Perhaps a more formal induction on the index n is possible. Let's see. A base case, of course, is when $n = 0$. For this we have

$$A = A_0 = \{p_0 = a_{00}\}.$$

Every constant $q = b_0$ is a linear combination of p_0 with

$$q = \frac{b_0}{a_{00}} p_0.$$

Thus, we have established the base case in the assertion

> $A = A_n = \{p_0, p_1, \ldots, p_n\}$ *spans* $\mathcal{P}_n$ *where* A *is any collection of polynomials satisfying* $\deg(p_j) = j$ *for* $j = 0, 1, \ldots, n$.

As an inductive hypothesis we may take

> *Every collection* $B = B_\nu = \{Q_0, Q_1, \ldots, Q_\nu\}$ *spans* $\mathcal{P}_\nu$ *where* B *is any collection of polynomials satisfying* $\deg(Q_j) = j$ *for* $j = 0, 1, \ldots, \nu$ *and* $\nu \leq k$.

We then consider a collection $C = \{p_0, p_2, \ldots, p_{k+1}\}$ of polynomials with $\deg(p_j) = j$ for $j = 0, 1, \ldots, k+1$. Letting q be any polynomial in $\mathcal{P}_{k+1}$, we can write

$$q = \sum_{j=0}^{m} b_j x^j$$

where $\deg(q) = m \leq k+1$. If $m < k+1$, then $q \in \mathcal{P}_k$ and $q \in \text{span}\{p_0, p_1, \ldots, p_k\}$ by the inductive hypothesis. If $m = k+1$, then we consider

$$q - \frac{b_k}{a_{k+1,k+1}} p_{k+1}.$$

This is a polynomial of degree $\mu < k+1$. By the inductive hypothesis, we can write

$$q - \frac{b_k}{a_{k+1,k+1}} p_{k+1} = \sum_{j=0}^{k} c_j p_j$$

for some $c_0, c_1, \ldots, c_k \in F$. Consequently,

$$q = \sum_{j=0}^{k} c_j p_j + \frac{b_k}{a_{k+1,k+1}} p_{k+1} \in \text{span}\{p_0, p_1, \ldots, p_{k+1}\}.$$

This completes the induction. I guess this is also a good proof, and maybe even better than the first one.

Problem 8 *(sums of subspaces and direct sums of subspaces) Let*

$$\begin{aligned} V &= \{(x, y, 0) : x, y \in \mathbb{R}\}, \\ W &= \{(x, 0, x) : x \in \mathbb{R}\}, \text{ and} \\ Z &= \{(0, y, y) : y \in \mathbb{R}\} \end{aligned}$$

be subspaces in $\mathbb{R}^3$.

(a) *Find* $V + W$.

(b) *Find* $V + Z$.

(c) *Find* $V + W + Z$.

(d) *Show that* $V \cap W = V \cap Z = W \cap Z = \{\mathbf{0}\}$.

(e) *Which of the sums in (a-c) are direct sums?*

Problem 9 *(Axler 2C13) If* W_1 *and* W_2 *are both four-dimensional subspaces of* $\mathbb{R}^6$, *find the smallest integer* n *and the largest integer* m *for which*

$$n \leq \dim(W_1 \cap W_2) \leq m,$$

and justify your answer.

Problem 10 *(Axler 2C15) If* V *is a finite dimensional vector space with dimension* $\dim(V) = n$, *then there are one-dimensional subspaces* $W_1, W_2, \ldots, W_n$ *of* V *such that*

$$V = \bigoplus_{j=1}^{n} W_j.$$