# Assignment 4:
# Linear Functions
# Due Tuesday February 22, 2022

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**Problem 1** *(Axler 3A4) Let $V$ and $W$ be vector spaces over a field $F$.*

**(a)** *What does it mean for a function $L : V \to W$ to be* **linear***?*

**(b)** *If $L : V \to W$ is linear and $\{v_1, v_2, \ldots, v_k\} \subset V$ with*

$$\{Lv_1, Lv_2, \ldots, Lv_k\} \quad \textit{linearly independent},$$

*then show $\{v_1, v_2, \ldots, v_k\}$ is linearly independent.*

**Problem 2** *(Axler 3A5) Let $V$ and $W$ be vector spaces over a field $F$.*

**(a)** *If $L : V \to W$ is linear and $T : V \to W$ is linear, what is the* **sum** *of $L$ and $T$?*

**(b)** *If $L : V \to W$ is linear and $a \in F$, what is the* **scaling** *of $L$ by $a$?*

**(c)** *Denote by $\mathcal{L}(V \to W)$ the set of all linear functions from $V$ to $W$. With the operations from parts* **(a)** *and* **(b)** *above prove $\mathcal{L}(V \to W)$ is a vector space. Hint: $\mathcal{L}(V \to W) \subset V^W$.*

**Problem 3** *(Axler 3A3) Given real constants* $a_{1j}$ *for* $j = 1, 2, \ldots, n$*, define a function* $L : \mathbb{R}^n \to \mathbb{R}$ *by*

$$L(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_{1j} x_j.$$

**(a)** *Show that* $L$ *is linear, i.e.,* $L \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R})$.

**(b)** *What are the partial derivatives of* $L$*?*

**(c)** *If* $\{v_1, v_2, \ldots, v_n\}$ *is a basis of* $\mathbb{R}^n$ *and* $T \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R})$ *satisfies*

$$T(v_j) = L(v_j) \quad \textit{for } j = 1, 2, \ldots, n,$$

*show* $T(v) = L(v)$ *for all* $v \in \mathbb{R}^n$*, i.e.,* $T = L$.

**Problem 4** *(Axler 3A3) Given complex constants* $a_{1j}$ *for* $j = 1, 2, \ldots, n$*, and*

$$a_{2j}, \quad \textit{for } j = 1, 2, \ldots, n,$$

*define a function* $L : \mathbb{R}^n \to \mathbb{R}^2$ *by*

$$L(x_1, x_2, \ldots, x_n) = \left( \sum_{j=1}^{n} a_{1j} x_j, \sum_{j=1}^{n} a_{2j} x_j \right).$$

**(a)** *Show that* $L$ *is linear, i.e.,* $L \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^2)$.

**(b)** *If* $T \in \mathcal{L}(\mathbb{R}^n \to \mathbb{R}^2)$*, then show there are constants*

$$b_{1j} \text{ for } j = 1, 2, \ldots, n \qquad \text{and} \qquad b_{2j} \text{ for } j = 1, 2, \ldots, n$$

*such that*

$$T(x_1, x_2, \ldots, x_n) = \left( \sum_{j=1}^{n} b_{1j} x_j, \sum_{j=1}^{n} b_{2j} x_j \right).$$

**Problem 5** *(Axler 3A6) If* $V$*,* $W$*,* $Y$*, and* $Z$ *are four vector spaces over the same field and*

$$L \in \mathcal{L}(V, W), \ M \in \mathcal{L}(W, Y), \ \textit{and } T \in \mathcal{L}(Y, Z),$$

*recall that the* **compositions** $M \circ L$ *and* $T \circ M$ *are defined by*

$$M \circ L(v) = M(V(v)) \qquad \text{and} \qquad T \circ M(w) = T(M(w)).$$

**(a)** *Show that* $M \circ L \in \mathcal{L}(V \to Y)$ *and* $T \circ M \in \mathcal{L}(W \to Z)$.

**(b)** *To what sets do* $(T \circ M) \circ L$ *and* $T \circ (M \circ L)$ *belong?*

**(c)** *Show*

$$(T \circ M) \circ L = T \circ (M \circ L).$$

**Problem 6** *Given any vector space* $V$*, show that the* **identity** *function* $\mathrm{id}_V : V \to V$ *by*

$$\mathrm{id}_V(v) = v$$

*satisfies* $\mathrm{id}_V \in \mathcal{L}(V \to V)$.

**Problem 7** *(Axler 3A8) Give an example of a function* $f : \mathbb{R}^2 \to \mathbb{R}$ *satisfying* $f(av) = af(v)$ *for every* $a \in \mathbb{R}$ *and* $v \in \mathbb{R}^2$*, but*

$$f \notin \mathcal{L}(\mathbb{R}^2 \to \mathbb{R}).$$

**Problem 8** *(Axler 3A10) Let* $W$ *be a* **proper** *subspace of a vector space* $V$*. (This just means* $W$ *is a subspace of* $V$*, but* $W \neq V$*.) Also, let* $Z$ *be a vector space over the same field, and let* $L \in \mathcal{L}(W \to Z)$*. Define*

$$f : V \to Z \qquad \text{by} \qquad f(v) = \begin{cases} Lv, & \text{if } v \in W \\ \mathbf{0}, & \text{if } v \notin W. \end{cases}$$

*If there is some* $v_1 \in W$ *such that* $Lv_1 \neq \mathbf{0}$*, then show* $f \notin \mathcal{L}(V \to Z)$.

**Problem 9** *(Axler 3A11) Let* $V$ *be a two-dimensional subspace of* $\mathbb{R}^3$ *and assume* $L \in \mathcal{L}(V \to W)$*. Show there exists a linear function* $T \in \mathcal{L}(\mathbb{R}^3 \to W)$ *with*

$$T(v) = L(v) \qquad \text{for every } v \in V.$$

**Problem 10** *(Axler 3A14) Find linear funtions* $L, T \in \mathcal{L}(\mathbb{R}^2 \to \mathbb{R}^2)$ *for which*

$$L \circ T \neq T \circ L.$$

*Remember that in this case* $T \circ L$ *and* $L \circ T$ *are in* $\mathcal{L}(\mathbb{R}^2 \to \mathbb{R}^2$ *and we call these compositions* **products** *and write* $TL = T \circ L$ *and* $LT = L \circ T$*. Thus, the "product" is not commutative in the ring* $\mathcal{L}(\mathbb{R}^2 \to \mathbb{R}^2)$.