Assignment 6 = Exam 2: Linear Functions (Section 3B) Due Tuesday March 8, 2022

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Problem 1 *Let V and W be vector spaces over the same field. Prove the following:*

(a) *If $L : V \to W$ is linear, then $L(\mathbf{0}_V) = \mathbf{0}_W$.*

(b) *If $h : V \to W$ is homogeneous, then $h(\mathbf{0}_V) = \mathbf{0}_W$.*

(c) *If $\alpha : V \to W$ is additive, then $\alpha(\mathbf{0}_V) = \mathbf{0}_W$.*

Problem 2 *(Axler 3B10) If $\{v_1, v_2, \ldots, v_n\}$ spans a vector space V and $L : V \to W$ is linear, then $\{L(v_1), L(v_2), \ldots, L(v_n)\}$ spans the image of L.*

Problem 3 *(Axler 3B14) If $L : V \to \mathbb{C}^5$ is linear and has null space a three-dimensional subspace of a complex vector space V with $\dim(V) = 8$, then show L is surjective.*

Problem 4 *(Axler 3B17) Let V and W be finite dimensional vector spaces over the same field.*

(a) *Show that if $\dim(V) \leq \dim(W)$, then there exists an injective linear function $L : V \to W$.*

(b) *Show that if there exists an injective linear function $L : V \to W$, then $\dim(V) \leq \dim(W)$.*

Problem 5 *(Axler 3B20) Let W be a finite dimensional vector space and consider a linear function $L : V \to W$. Show the following:*

(a) *If there exists a linear function $T : W \to V$ such that the composition $TL = T \circ L = \mathrm{id}_V$, then L is injective.*

(b) *If L is injective, then there exists a linear function $T : W \to V$ such that the composition $TL = T \circ L = \mathrm{id}_V$.*

Problem 6 *(Axler 3B22) If V and W are finite dimensional vector spaces and $L : V \to W$ and $T : W \to Z$ are linear, then (show that)*

$$\dim \mathcal{N}(T \circ L) \leq \dim \mathcal{N}(L) + \dim \mathcal{N}(T) \tag{1}$$

where $\mathcal{N}$ denotes the null space of a linear function.

Problem 7 *(Axler 3B22) Find an example of linear functions $L : V \to W$ and $T : W \to Z$ satisfying the hypotheses of the previous problem and for which* **equality** *holds in* (1).

Problem 8 *(systems of linear equations) Show that a homogeneous system of 26 linear equations*

$$a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{i,27}x_{27} = 0, \qquad i = 1, 2, \ldots, 26$$

in 27 unknowns $x_1, x_2, \ldots, x_{27}$ has a solution

$$\mathbf{x} = (x_1, \ldots, x_{27}) \in F^{27} \backslash \{\mathbf{0}\}.$$

Problem 9 *(Axler 3B27) Let $\mathcal{P} = \mathcal{P}(F)$ denote the vector space of polynomials with subspaces $\mathcal{P}_n = \mathcal{P}_n(F)$ of polynomials of degree no more than n as usual.*

(i) *Using what you know about the properties of differentiation from calculus, show that $D : \mathcal{P} \to \mathcal{P}$ by*

$$Dp = \frac{d}{dx}p$$

is linear.

(ii) *If D is restricted to $\mathcal{P}_n$, what is the image of the restriction?*

(iii) *Show that $L : \mathcal{P} \to \mathcal{P}$ given by $Lq = 5DDq + 3Dq$ is a linear function.*

(iv) *Use the linear function L of the previous part to show that given any $p \in \mathcal{P}$ there exists a polynomial $q \in \mathcal{P}$ for which $Lq = p$. Hint(s): Restrict L to an appropriate subspace and find the null space of L.*

Problem 10 *(Axler 3B30) Let V be a vector space over the field F and ϕ and ψ two elements of $\mathcal{L}(V \to F)$. Sometimes the functions in $\mathcal{L}(V \to F)$ are called* **linear functionals**. *Show that if the null spaces of ϕ and ψ satisfy*

$$\mathcal{N}(\phi) = \mathcal{N}(\psi),$$

then there exists a constant $c \in F$ such that $\phi = c\psi$.