Assignment 7:
Matrices (Section 3C)
Due Tuesday March 15, 2022

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Problem 1 *Let V and W be vector spaces over the same field. Remember that the collection of linear functions $\mathcal{L}(V \to W)$ of linear functions from V to W is a vector space.*

(a) *Show $\mathcal{L}(V \to W)$ is finite dimensional.*

(b) *Find a basis for $\mathcal{L}(V \to W)$.*

(c) *Find the dimension of $\mathcal{L}(V \to W)$.*

Problem 2 *(Axler 3C3) Given $L \in \mathcal{L}(V \to W)$ where $\dim(V) = n$ and $\dim(W) = m$, show there is a basis $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ of V and a basis $\mathcal{C} = \{w_1, w_2, \ldots, w_m\}$ of W such that the matrix A of L with respect to the bases $\mathcal{B}$ and $\mathcal{C}$ has block form*

$$\begin{pmatrix} I_{q\times q} & Z_{q\times(n-q)} \\ Z_{(m-q)\times q} & Z_{(m-q)\times(n-q)} \end{pmatrix}$$

where $I_{q\times q}$ is the $q \times q$ identity matrix with $q = \dim \mathrm{Im}(L)$ and $Z_{p\times r}$ is the $p \times r$ matrix with all zero entries.

Problem 3 *(Axler 3C6—hard) If V and W are finite dimensional and $L : V \to W$ is linear with $\dim \mathrm{Im}(L) = 1$, then there are bases $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ of V and $\mathcal{C} = \{w_1, w_2, \ldots, w_m\}$ of W for which the matrix of L with respect to $\mathcal{B}$ and $\mathcal{C}$ is the $m \times n$ matrix with all entries $a_{ij} = 1$.*

Problem 4 *(Axler 3C9) If A is an $m \times n$ matrix, show that the matrix multiplcation*

$$A\mathbf{x}$$

where $\mathbf{x} = (x_1, x_2, \ldots, x_n)^T$ is a column vector in F^n is the linear combination

$$\sum_{j=1}^{n} x_j A_j$$

of the columns of $A_1, A_2, \ldots, A_n$ of the matrix A with coefficients $x_1, x_2, \ldots, x_n$ the components of $\mathbf{x}$.

Problem 5 *(Axler 3B21) Let V be a finite dimensional vector space and consider a linear function $L : V \to W$. Show the following:*

(a) *If there exists a linear function $T : W \to V$ such that the composition $LT = L \circ T = \mathrm{id}_W$, then L is surjective.*

(b) *If L is surjective, then there exists a linear function $T : W \to V$ such that the composition $LT = L \circ T = \mathrm{id}_W$.*

Problem 6 *(Axler 3C10) If A is an $m \times n$ matrix and B is an $n \times q$ matrix, show that each row of the product AB is a linear combination of the rows of B.*

Problem 7 *(Axler 3C11) If $\mathbf{y} = (y_1, y_2, \ldots, y_m)$ is a row vector and B is an $m \times n$ matrix, then show the product $\mathbf{y}B$ is*

$$\sum_{k=1}^{m} y_k B_k$$

where $B_1, B_2, \ldots, B_m$ are the rows of B.

Problem 8 *(rotate a book, Axler 3C12) Show that a rotation $\rho : \mathbb{R}^3 \to \mathbb{R}^3$ of Euclidean three-dimensional space about the origin is linear. Hint: A rotation of*

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

is some vector $(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ for some ϕ with $0 \le \phi \le \pi$ and some θ with $0 \le \theta \le 2\pi$.

Take your textbook with the front cover facing up in front of you (binding on your left as if you were about to open it to read it). Rotate the book clockwise by ninety degrees in a horizontal plane so the binding is facing away from you. Now rotate it ninety degrees toward you so that the binding is facing up.

Start back in the original position with the cover up and binding left. Reverse these rotations: Toward you nintely degrees and then ninety degrees clockwise about the z-axis.

What do you observe about rotations of $\mathbb{R}^3$*?*

Problem 9 *(Linear Functions)*

(i) *If* $L : \mathbb{R}^1 \to \mathbb{R}^1$ *is linear, then (show) there is a constant* $m \in \mathbb{R}$ *such that*

$$Lx = mx \qquad \text{for every } x \in \mathbb{R}^1.$$

(ii) *If* $L : V \to W$ *is linear and* V *is a one-dimensional real vector space, is it always true that there is some* $m \in \mathbb{R}$ *such that*

$$Lv = mv \qquad \text{for every } v \in V?$$

(iii) *If* $L : \mathbb{R}^1 \to \mathbb{R}^2$ *is linear, then (show) there is some* $\mathbf{v} \in \mathbb{R}^2$ *for which*

$$Lx = x\mathbf{v} \qquad \text{for every } x \in \mathbb{R}^1.$$

Problem 10 *(Axler 3C12) Find matrices* A *and* B *in* $\mathcal{M}_{2\times 2}(F)$*, the collection of all* 2×2 *matrices with entries in the field* F*, such that*

$$AB \neq BA.$$