Assignment 8:
Invertibility (Section 3D)
Due Tuesday March 22, 2022

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Problem 1 *(Axler 3D1) If* $L \in \mathcal{L}(V \to W)$ *and* $M \in \mathcal{L}(W \to Z)$ *with* L *and* M *both invertible, then* $ML = M \circ L$ *is invertible.*

(a) *Find the function* $T \in \mathcal{L}(Z \to V)$ *for which*

$$ML \circ T = \text{id}_Z \qquad \text{and} \qquad T \circ ML = \text{id}_V \tag{1}$$

(b) *Verify the conditions in (1).*

Problem 2 *(Axler 3D2) Let* V *be a finite dimensional vector space with* $\dim(V) > 1$.

(a) *If* $L \in \mathcal{L}(V \to V)$ *is* **not** *invertible, then show* cL *is not invertible for every* $C \in F$.

(b) *Find noninvertible operators* $L, M \in \mathcal{L}(V \to V)$ *with* $L + M$ *invertible.*

(c) *Find invertible operators* $L, M \in \mathcal{L}(V \to V)$ *with* $L + M$ *not invertible.*

(d) *In part* **(c)** *can you find an example with* $\dim \mathcal{N}(L + M) = 1$?

Problem 3 *(Axler 3D3, extension) Let V be a finite dimensional vector space and W a subspace of V. Show the following:*

(a) *If $M \in \mathcal{L}(W \to V)$ is injective, there exists some $L \in \mathcal{L}(V \to V)$ with*

  (i) *L is invertible and*

  (ii) *$Lv = Mv$ for all $v \in W$.*

(b) *If $L \in \mathcal{L}(V \to V)$ with L is invertible, then $M : W \to V$ by $Mv = Lv$ satisfies*

  (i) *$M \in \mathcal{L}(W \to V)$ and*

  (ii) *M is injective.*

Problem 4 *(Axler 3D5) Let V be a finite dimensional vector space and $L, M \in \mathcal{L}(V \to W)$. Show the following:*

(a) *If $\mathrm{Im}(L) = \mathrm{Im}(M)$, then there exists an invertible operator $T \in \mathcal{L}(V \to V)$ with $L = MT$.*

(b) *there exists an invertible operator $T \in \mathcal{L}(V \to V)$ with $L = MT$, then $\mathrm{Im}(L) = \mathrm{Im}(M)$.*

Problem 5 *(Axler 3D6) Let V and W be finite dimensional vector spaces and $L, M \in \mathcal{L}(V \to W)$. Show the following:*

(a) *If $\dim \mathcal{N}(L) = \dim \mathcal{N}(M)$, then there are invertible operators $T \in \mathcal{L}(V \to V)$ and $S \in \mathcal{L}(W \to W)$ for which*

$$L = SMT.$$

(b) *If there are invertible operators $T \in \mathcal{L}(V \to V)$ and $S \in \mathcal{L}(W \to W)$ for which*

$$L = SMT,$$

then $\dim \mathcal{N}(L) = \dim \mathcal{N}(M)$.

Problem 6 *(Axler 3C14) Given a basis $\{v_1, v_2, \ldots, v_n\}$ of a vector space V, show that $L \in \mathcal{L}(V \to F^n)$ by*

$$Lv = (a_1, a_2, \ldots, a_n) \qquad \text{where} \qquad v = \sum_{j=1}^{n} a_j v_j$$

is a linear isomorphism.

Problem 7 *(Axler 3D16) Let V be a finite dimensional vector space and $L \in \mathcal{L}(V \to V)$. Show the following: If*

$$LM = ML \qquad \text{for every } M \in \mathcal{L}(V \to V),$$

then there exists some $c \in F$ such that L has the form

$$Lv = c \underset{V}{\mathrm{id}}(v).$$

Problem 8 *(Axler 3D17) Let V be a finite dimensional vector space and $\mathcal{W}$ a subspace of $\mathcal{L}(V \to V)$. Show the following: If*

$$LM = ML \in \mathcal{W} \qquad \text{for every } L \in \mathcal{L}(V \to V) \text{ and } M \in \mathcal{W},$$

then either $\mathcal{W} = \{0\}$ contains only the zero map or $\mathcal{W} = \mathcal{L}(V \to V)$ is the entire collection of linear operators on V.

Problem 9 *(Axler 3D19) If $L \in \mathcal{L}(\mathcal{P} \to \mathcal{P})$, where $\mathcal{P} = \mathcal{P}(F)$ denotes the vector space of polynomials with coefficients in a field F, and*

(i) *L is injective and*

(ii) $\deg(Lp) \leq \deg p$ *for every $p \in \mathcal{P}$,*

then show

(a) *L is onto and*

(b) $\deg(Lp) = \deg p$ *for every $p \in \mathcal{P}$.*

Problem 10 *(Axler 3D20) Let $A = (a_{ij})$ be an $n \times n$ matrix with entries in a field F. Show that if $x_1 = x_2 = \cdots = x_n = 0$ is the* **only solution** *of the system of equations*

$$\begin{aligned}\sum_{j=1}^{n} a_{1j}x_j &= 0\\ \sum_{j=1}^{n} a_{2j}x_j &= 0\\ &\vdots\\ \sum_{j=1}^{n} a_{nj}x_j &= 0,\end{aligned}$$

then the system of equations

$$\begin{aligned}\sum_{j=1}^{n} a_{1j}x_j &= c_1\\ \sum_{j=1}^{n} a_{2j}x_j &= c_2\\ &\vdots\\ \sum_{j=1}^{n} a_{nj}x_j &= c_n\end{aligned}$$

has a (unique) solution for each $(c_1, c_2, \ldots, c_n) \in F^n$.