Assignment 9:
Products, Quotients, and Duality (Sections 3E-F)
Due Tuesday April 5, 2022

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Problem 1 *(Axler 3E1) If $f : X \to Y$ is any function from X to Y, then the* **graph** *of f is defined to be the set*

$$\{(x, f(x)) : x \in X\}.$$

Note that this is a set of **ordered pairs**, *and the set of all ordered pairs is called the* **cross product***:*

$$X \times Y = \{(x, y) : x \in X,\ y \in Y\}.$$

Let V and W be vector spaces, and consider a function $\phi : V \to W$. Show that ϕ is linear if and only if the graph of ϕ is a subspace of $V \times W$.

Problem 2 *(Axler 3E2) Show that if $V_1, V_2, \ldots, V_n$ are vector spaces and $V_1 \times V_2 \times \cdots \times V_n$ is finite dimensional, then each vector space V_j for $j = 1, 2, \ldots, n$ is finite dimensional.*

Problem 3 *(Axler 3E3, extension) Give an example of a vector space V having subspaces W_1 and W_2 such that $W_1 \times W_2$ is isomorphic to $W_1 + W_2$, but $W_1 + W_2$ is not a direct sum.*

Problem 4 *(Axler 3E7) Let V be a vector space with subspaces W_1 and W_2. Show the following: If v and $\tilde{v}$ are vectors in V for which*

$$v + W_1 = \tilde{v} + W_2,$$

when $W_1 = W_2$.

Problem 5 *(Axler 3E8) Let V be a vector space and $A \subset V$. Show that A is an affine subspace of V if and only if the following hold*

(i) *V is nonempty, and*

(ii) $(1-t)v + t\tilde{v} \in A$ *whenever* $v, \tilde{v} \in V$ *and* $t \in F$.

Problem 6 *(Axler 3E8) Given an affine subspaces A_1 and A_2 of a vector space V, so that either $A_1 \cap A_2$ is an affine subspace or $A_1 \cap A_2$ is empty.*

Problem 7 *(Axler 3E12) Let V be a vector space with a subspace W such that V/W is finite dimensional. Show V is isomorphic to*

$$W \times (V/W).$$

Problem 8 *(Axler 3E13) Let V be a vector space and $\mathcal{W}$ a subspace of V. Show the following: If*

(i) $\{w_1, w_2, \ldots, w_k\}$ *is a basis for W, and*

(ii) $\{v_1 + W, v_2 + W, \ldots, v_\ell + W\}$ *is a basis for V/W,*

then $\{v_1, v_2, \ldots, v_\ell, w_1, w_2, \ldots, w_k\}$ *is a basis for V.*

Problem 9 *(Axler 3E17) Let V be a vector space with a subspace W for which V/W is finite dimensional. Show there exists a subspace M of V for which*

(i) $\dim M = \dim V/W$, *and*

(ii) $V = M \oplus W$.

Problem 10 *(Axler 3F1-2) Consider the real vector space $V = C^0[0,1]$ of real valued continuous functions on the unit interval, and let $V' = \mathcal{L}(V \to \mathbb{R})$ denote the* **dual space** *of linear functionals $\phi : V \to \mathbb{R}$ on V. Show the following:*

(a) *Each element $\phi \in V'$ is either surjective or the zero map.*

(b) $\psi_1 : C^0[0,1] \to \mathbb{R}$ *by* $\psi_1[f] = f(0)$ *is in* V'.

(c) $\psi_2 : C^0[0,1] \to \mathbb{R}$ *by*

$$\psi_2[f] = \int_0^1 f(x)\,dx$$

is in V'.

(d) *Give three more (different) examples of linear functionals on $C^0[0,1]$.*