PRETEST 3: Duality
NAME: ____________

MATH 3406

March 29, 2022

Consider $L : \mathbb{R}^3 \to \mathbb{R}^4$ by

$$L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Remember Problem 3 from PRETEST 1: Classify all subspaces U of $\mathbb{R}^3$ such that

$$\mathbb{R}^3 = \mathcal{N}(L) \oplus U.$$

Fix standard bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ for $\mathbb{R}^4$, $\{\phi_1, \phi_2, \phi_3\}$ for $(\mathbb{R}^3)'$, and $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ for $(\mathbb{R}^4)'$.

Fix standard isomorphisms $\Phi : \mathbb{R}^3 \to (\mathbb{R}^3)'$ and $\Psi : \mathbb{R}^4 \to (\mathbb{R}^4)'$.

Problem 1 Given $\mathbf{u} \in \mathbb{R}^4$, show

$$\Psi(\mathbf{u})\mathbf{w} = \sum_{j=1}^{4} u_j w_j \qquad \text{for} \qquad \mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \in \mathbb{R}^4.$$

Problem 2 Given $\phi \in (\mathbb{R}^3)'$, show

$$\Phi^{-1}(\phi) = \sum_{j=1}^{3} \phi(\mathbf{e}_j)\mathbf{e}_j.$$

Problem 3 Show $\psi_j(\mathbf{w}) = \langle \mathbf{e}_j, \mathbf{w} \rangle_{\mathbb{R}^4}$ where

$$\langle \mathbf{x}, \mathbf{y} \rangle_{\mathbb{R}^4} = \sum_{j=1}^{4} x_j y_j$$

is the "dot product" or standard inner product on $\mathbb{R}^4$.