TEST 4: Duality
NAME: ____________

MATH 3406

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Consider $L : \mathbb{R}^3 \to \mathbb{R}^4$ by

$$L \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Remember Problem 3 from PRETEST 1: Classify all subspaces U of $\mathbb{R}^3$ such that

$$\mathbb{R}^3 = \mathcal{N}(L) \oplus U.$$

Fix standard bases $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ for $\mathbb{R}^3$, $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ for $\mathbb{R}^4$, $\{\phi_1, \phi_2, \phi_3\}$ for $(\mathbb{R}^3)'$, and $\{\psi_1, \psi_2, \psi_3, \psi_4\}$ for $(\mathbb{R}^4)'$.

Fix standard isomorphisms $\Phi : \mathbb{R}^3 \to (\mathbb{R}^3)'$ and $\Psi : \mathbb{R}^4 \to (\mathbb{R}^4)'$.

Problem 1 Given a subspace U of $\mathbb{R}^3$ from Problem 3 of PRETEST 1 and a vector $\mathbf{b} \in \mathbb{R}^4$, find a formula for

$$\left(T \circ L\big|_U \right)^{-1} L\mathbf{b}.$$

Problem 2 Let

$$\mathbf{x} = \left(T \circ L\big|_U \right)^{-1} L\mathbf{b}$$

from Problem 1 above. What is the signficance of $L\mathbf{x}$?

Problem 3 Is the subspace $\mathrm{Im}(T)$ an example of one of the subspaces U of $\mathbb{R}^3$ from Problem 3 of PRETEST 1? If so, what is special about this choice of U?