Geometric Linear Algebra: Ellipses

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I would like to establish some basic assertions about bijective linear functions $L : \mathbb{R}^2 \to \mathbb{R}^2$. The first is that the image

$$\mathcal{E} = \left\{ L\mathbf{x} : |\mathbf{x}| = \sqrt{x^2 + y^2} = r \right\}$$

of every circle

$$\left\{ \mathbf{x} = (x, y) \in \mathbb{R}^2 : |\mathbf{x}| = \sqrt{x^2 + y^2} = r \right\}$$

centered at the origin and having radius $r > 0$ is an ellipse. This ellipse will also be centered at the origin $(0, 0)$ in the codomain. In addition, I would like to obtain precise geometric information about the ellipse $\mathcal{E}$. Namely I would like to identify the two orthogonal directions within the codomain along which the axes of the ellipse lie and the length of the axes within the ellipse. I would like to express this geometric information in terms of the linear function L. To be specific, I should like to consider at least three specific cases:

1. If L is diagonalizable with eigenvalue/eigenvector pairs $(\lambda, \mathbf{v})$ and $(\tilde{\lambda}, \tilde{\mathbf{v}})$ with $\{\mathbf{v}, \tilde{\mathbf{v}}\}$ a basis for $\mathbb{R}^2$, I would like to express the information explicitly in terms of $\lambda, \tilde{\lambda}, \mathbf{v}, \tilde{\mathbf{v}}$ and, of course, the radius of the circle r.

2. If L is a Jordan shear transformation having a single eigenvalue $\lambda \neq 0$ and a unique one-dimensional eigenspace spanned by a vector $\mathbf{v}$, I would like to express the characteristics of the image ellipse in terms of λ and $\mathbf{v}$.

3. Finally, given any bijective linear function $L : \mathbb{R}^2 \to \mathbb{R}^2$, one can find the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

for which $L\mathbf{x} = A\mathbf{x}$ is given by matrix multiplication with respect to the standard basis

$$\left\{ \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

In this instance, we recall that it is the usual practice to consider $\mathbf{x}$ and $L\mathbf{x}$ as column vectors so that

$$L\mathbf{x} = L\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11}x + a_{12}y \\ a_{21}x + a_{22}y \end{pmatrix}.$$

We would like to obtain the geometric information about the image ellipse in terms of the four entries of the matrix

$$L\mathbf{e}_1 = \begin{pmatrix} a_{11} \\ a_{21} \end{pmatrix} \qquad \text{and} \qquad L\mathbf{e}_2 = \begin{pmatrix} a_{12} \\ a_{22} \end{pmatrix}.$$

We first review the analytic geometry of ellipses. This is done, among other things, in order to make clear what we mean by the "geometric information" associated with an ellipse. We also consider some special cases of the general problem suggested above.

1 Ellipses

An ellipse is given by

$$\mathcal{E}_0 = \left\{ (x, y) \in \mathbb{R}^2 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}$$

where a and b are positive real numbers. Of course, this is not the most general ellipse. One can translate the center to a different location by considering the relation

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1,$$

and one can also consider a rotation by some specified angle, or one can consider both a rotation and a translation. We will restrict attention, at least for the moment, to ellipses with center at the origin $(0, 0) \in \mathbb{R}^2$, but rotations will be important for us. The numbers a and b are called the **lengths of the semi-axes**, and if $a = b = r > 0$, then the set $\mathcal{E}_0$ is a circle of radius r. In general, the ellipse $\mathcal{E}_0$ intersects the x-axis at $x = \pm a$ and the y-axis at $y = \pm b$. It will be convenient for us to consider a **parametric representation** of the ellipse $\mathcal{E}_0$ (and other ellipses as well), namely, consider

$$\gamma(t) = (a\cos t, b\sin t) \qquad \text{for} \qquad 0 \le t < 2\pi. \tag{1}$$

Exercise 1 *Show the parameter t appearing in (1) is, in general,* **not** *the angle associated with* **polar coordinates** *in the ellipse. Under what circumstances is t the polar angle? What is the equation*

$$r = r(\theta)$$

of the ellipse in polar coordinates?

Exercise 2 *Can you give a geometric interpretation of the angle t appearing in (1)?*

My main theorem on ellipses involves being able to recognize an ellipse in terms of the parametric representation given in (1). To motivate this result I will make a computation:

$$\begin{aligned}
|\gamma(t)|^2 &= a^2\cos^2 t + b^2\sin^2 t \\
&= (a^2 - b^2)\cos^2 t + b^2 \\
&= \frac{a^2 - b^2}{2}\cos(2t) + \frac{a^2 - b^2}{2} + b^2 \\
&= \frac{a^2 - b^2}{2}\cos(2t) + \frac{a^2 + b^2}{2}.
\end{aligned}$$

I have used the identity

$$\cos^2 t = \frac{\cos(2t) + 1}{2}. \tag{2}$$

Exercise 3 *Derive or verify the identity (2).*