Final Assignment: Linear Functions
Due Tuesday April 28, 2022

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Problem 1 *Do you understand a linear map* $L : \mathbb{R}^1 \to \mathbb{R}^1$ *with* $L(1) = 0$*?*

Solution: Main observation $Lx = L(1)x = ax$ where $L(1) = a \in \mathbb{R}$ (is considered as an element in the field $\mathbb{R}$). In this case, $Lx \equiv 0$.

Questionaire

1. $\mathcal{N}(L) = \mathbb{R}^1$, $\text{Im}(L) = \{0\}$.
2. Matrix: $A = 0$ or $A = (0)$.
3. Name(s): zero map, null map, trivial map, complete collapse (map), total annihilator.
4. Picture:

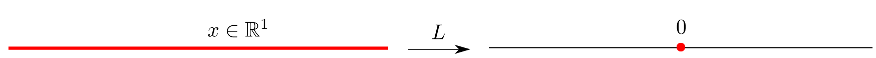

Figure 1: Illustration of the zero map $L : \mathbb{R}^1 \to \mathbb{R}^1$.

5. $0x = b$ is exactly solvable only if $b = 0$. In the case $b = 0$ any $x \in \mathbb{R}^1$ is a solution.

   If $b \neq 0$, then any $x \in \mathbb{R}^1$ may be considered an approximate solution, but no particular $x \in \mathbb{R}^1$ is better than any other.

6. The dual map $L' : \mathcal{L}(\mathbb{R}^1 \to \mathbb{R}) \to \mathcal{L}(\mathbb{R}^1 \to \mathbb{R})$ is given by

$$L'\phi = \psi_0 \qquad \text{where} \qquad \psi_0 \equiv 0$$

   is the zero map. In fact, $\psi_0 : \mathbb{R}^1 \to \mathbb{R}$ with codomain the field $\mathbb{R}$ is essentially identical to $L : \mathbb{R}^1 \to \mathbb{R}^1$ with codomain the vector space $\mathbb{R}^1$ over the field $\mathbb{R}$. This means L' is also a/the zero map.

   There is an induced/reverse map $T : \mathbb{R}^1 \to \mathbb{R}^1$ given by $T = \Phi \circ L' \circ \Psi^{-1}$ where $\Phi : \mathbb{R}^1 \to \mathcal{L}(\mathbb{R}^1 \to \mathbb{R})$ and $\Psi : \mathbb{R}^1 \to \mathcal{L}(\mathbb{R}^1 \to \mathbb{R})$ are the canonical isomorphism given by

$$\Phi(a)x = \Psi(a)x = ax,$$

   and T is also the zero map, so the alternative/composition equation

$$TLx = Tb \qquad \text{is} \qquad 0x = 0$$

   which (clearly) has each $x \in \mathbb{R}^1$ as a solution.

7. I feel like I understand this zero map pretty well. I can't think of any more interesting questions to ask about it. Can you?

Problem 2 *Do you understand a linear map $L : \mathbb{R}^1 \to \mathbb{R}^1$ with $L(1) = 1$?*

Solution: Main observation $Lx = L(1)x = ax$ where $L(1) = a \in \mathbb{R}$ (is considered as an element in the field $\mathbb{R}$). In this case, $Lx \equiv x$.

Questionaire

1. $\mathcal{N}(L) = \{0\}$, $\mathrm{Im}(L) = \mathbb{R}^1$.
2. Matrix: $A = 1$ or $A = (1)$.
3. Name(s): identity map, trivial map; this map is an isomorphism.
4. Picture:

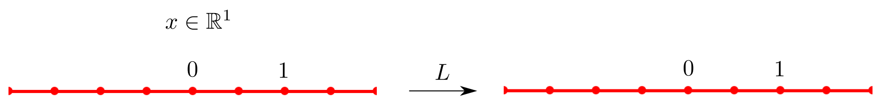

Figure 2: Illustration of the identity map $L : \mathbb{R}^1 \to \mathbb{R}^1$.

5. $1x = b$ has the unique solution $x = b$.
6. The dual map $L' : \mathcal{L}(\mathbb{R}^1 \to \mathbb{R}) \to \mathcal{L}(\mathbb{R}^1 \to \mathbb{R})$ is given by

$$L'\phi = \phi \qquad \text{or} \qquad L' = \text{id}$$

   is the identity map on the dual space $\mathcal{L}(\mathbb{R}^1 \to \mathbb{R})$.

   The induced/reverse map $T : \mathbb{R}^1 \to \mathbb{R}^1$ T is also the identity map on $\mathbb{R}^1$. The alternative/composition equation

$$TLx = Tb \qquad \text{is} \qquad x = b$$

   which is the same equation with the same unique solution $x = b$.

7. I feel like I understand this map.

Problem 3 *Do you understand a linear map* $L : \mathbb{R}^1 \to \mathbb{R}^1$ *with* $0 < L(1) < 1$*?*

Problem 4 *Do you understand a linear map* $L : \mathbb{R}^1 \to \mathbb{R}^1$ *with* $L(1) > 1$*?*

Problem 5 *Do you understand a linear map* $L : \mathbb{R}^1 \to \mathbb{R}^1$ *with* $L(1) = -1$*?*

Problem 6 *Do you understand a linear map* $L : \mathbb{R}^1 \to \mathbb{R}^1$ *with* $-1 < L(1) < 0$*?*

Problem 7 *Do you understand a linear map* $L : \mathbb{R}^1 \to \mathbb{R}^1$ *with* $L(1) < -1$*?*

Problem 8 *Which parts of the questionaiare for Problems 2-4 have essentially the same answer and could this be combined?*

Problem 9 *Which parts of the questionaiare for Problems 2-4 essentially require the consideration of cases to give a complete answer?*

Problem 10 *Which parts of the questionaiare for Problems 5-7 have essentially the same answer and could this be combined?*

Problem 11 *Which parts of the questionaiare for Problems 5-7 essentially require the consideration of cases to give a complete answer?*

Problem 12 *What is the overall moral of the big classification of linear maps* $L : \mathbb{R}^1 \to \mathbb{R}^1$*?*

Problem 13 *Which linear function* $L : \mathbb{R}^1 \to \mathbb{R}^1$ *deserves to be called the "trivial map?"*

Problem 14 *Give a big classification for linear functions* $L : \mathbb{R}^1 \to \mathbb{R}^2$*. (Make some of your own problems for the questionaire.)*

Problem 15 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $L(1) = 0$*?*

Problem 16 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $L(1) = 1$*?*

Problem 17 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $0 < L(1) < 1$*?*

Problem 18 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $L(1) > 1$*?*

Problem 19 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $L(1) = -1$*?*

Problem 20 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $-1 < L(1) < 0$*?*

Problem 21 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $L(1) < -1$*?*

Problem 22 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $L(1) = a + bi$ *with* $a^2 + b^2 = 1$*?*

Problem 23 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $L(1) = a + bi$ *with* $a^2 + b^2 < 1$*?*

Problem 24 *Do you understand a linear map* $L : \mathbb{C}^1 \to \mathbb{C}^1$ *with* $L(1) = a + bi$ *with* $a^2 + b^2 > 1$*?*

Problem 25 *Give a big classification for linear functions* $L : \mathbb{C}^1 \to \mathbb{C}^2$*. (Make some of your own problems for the questionaire.)*

Problem 26 *Show the conditions* **(i)-(iv)** *in classification category* **B** *of The Big Classification for Linear Functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *are equivalent.*

Problem 27 *Verify the equivalence of classification category* **B2** *of The Big Classification for Linear Functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$

Problem 28 *Verify the equivalence of classification category* **C1b** *of The Big Classification for Linear Functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$

Problem 29 *Verify the equivalence of classification category* **C2b** *of The Big Classification for Linear Functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$

Problem 30 *Verify the equivalence of classification category* **C3b** *of The Big Classification for Linear Functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$

Problem 31 *Verify the equivalence of classification category* **C4b** *of The Big Classification for Linear Functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$

Problem 32 *Verify the equivalence of classification category* **C5b** *of The Big Classification for Linear Functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$

Problem 33 *Verify the equivalence of classification category* **C6b** *of The Big Classification for Linear Functions* $L : \mathbb{R}^2 \to \mathbb{R}^2$

Problem 34 *Given any linear function* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *and any* $\mathbf{x} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$*, show there are real constants* a_2*,* a_1*, and* a_0 *(not all zero) for which*

$$a_2 L^2 \mathbf{x} + a_1 L\mathbf{x} + a_0 \mathbf{x} = \mathbf{0}.$$

Several of the problems below are based on this problem, and when the vector $\mathbf{x}$ *and the real numbers* a_2*,* a_1 *and* a_0 *appear below, you can assume they come from this problem and satisfy the condition/equation above.*

Problem 35 *(See Problem 34) Do you understand a linear map* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *for which* $a_2 = a_0 = 0$*?*

Problem 36 *(See Problem 34) Do you understand a linear map* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *for which* $a_1 = a_0 = 0$ *but* $L\mathbf{x} \neq \mathbf{0}$*?*

Problem 37 *(See Problem 34) Do you understand a linear map* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *for which* $a_0 = 0$*,* $a_1 \neq 0$*,* $a_2 \neq 0$*, and* $L\mathbf{x} \neq \mathbf{0}$*.*

Problem 38 *(See Problem 34) Are there any other linear maps* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *for which* $a_0 = 0$ *but none of the conditions considered in Problems 35-37 apply? Do you understand such a linear map (if there is one)?*

Problem 39 *(See Problem 34) Assume* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *is a linear map and there is no nonzero vector* $\mathbf{w} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ *for which* $L\mathbf{w} = \mathbf{0}$*. What if* $a_2 = 0$*?*

Problem 40 *(See Problem 34) Assume* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *is a linear map and there is no nonzero vector* $\mathbf{w} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ *for which* $L\mathbf{w} = \mathbf{0}$*. What if* $a_1 = 0$*?*

Problem 41 *(See Problem 34) Assume* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *is a linear map and there is no nonzero vector* $\mathbf{w} \in \mathbb{R}^2 \backslash \{\mathbf{0}\}$ *for which* $L\mathbf{w} = \mathbf{0}$*. If the polynomial* $q(z) = a_2 z^2 + a_1 z + a_0$ *has a real root* x_0*, then do you understand the linear function* L*?*

Problem 42 *(See Problem 34) Do you understand a linear map* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *for which* $a_1 = 0$ *and* $a_0 < 0$*?*

Problem 43 *(See Problem 34) Do you understand a linear map* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *for which* $a_1 = 0$ *and* $a_0 > 0$*?*

Problem 44 *(See Problem 34) Do you understand a linear map* $L : \mathbb{R}^2 \to \mathbb{R}^2$ *if the polynomial* $q(z) = a_2 z^2 + a_1 z + a_0$ *has no real roots and* $a_1 \neq 0$*?*