TEST: Homogeneity, Additivity, and Linearity corrected

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Let V and W be vector spaces *over the same field*[1] and consider a function $f : V \to W$.

Problem 1 *Give a precise definition of what it means for f to be* **additive**.

Problem 2 *Define a function $\phi : \mathbb{C}^2 \to \mathbb{C}^2$ by*

$$\phi(a + bi, c + di) = (a + 2bi, 3c + 4di).$$

(a) *If we consider $\mathbb{C}^2$ as a complex vector space (as usual) what is the dimension of $\mathbb{C}^2$?*

(b) *Show that ϕ is additive.*

Problem 3 *Give a precise definition of what it means for f to be* **homogeneous**.

Problem 4 *Define a function $g : \mathbb{R}^2 \to \mathbb{R}$ by*

$$g(x, y) = \sqrt[3]{x^3 + y^3}.$$

(a) *Show that g is homogeneous.*

[1] correction

(b) *Show that g is additive on every one-dimensional subspace of $\mathbb{R}^2$, but $g(p+q) \neq g(p) + g(q)$ for any nonzero vectors $p, q \in \mathbb{R}^2$* **not** *in the same one-dimensional subspace.*

Correction: *Note that $(1,0)$ and $(0,1)$ are nonzero vectors that are not in the same one-dimensional subspace, but $(1,0) + (0,-1) = (1,-1)$, and*

$$g(1,0) + g(0,-1) = 1 + (-1) = 0 = g(1,-1).$$

Thus, the claimed assertion is not true.[2] *Can you characterize the pairs of points $((x,y),(z,w)) \in \mathbb{R}^4$ (!) for which additivity holds/fails? Something that is "probably" true: If you pick any nonzero point $p = (x,y) \in \mathbb{R}^2$, then given any one-dimensional subspace Z distinct from $\operatorname{span}\{(x,y)\}$, there exists a point $q = (z,w) \in W$ for which additivity fails: $g(p+q) \neq g(p) + g(q)$.*

(c) *Show the function ϕ from Problem 2 is not homogeneous.*

Problem 5 *Show that if $\dim V = 1$ and $f : V \to W$ is homogeneous, then f is linear.*

[2] Thanks go to Leo Wang for the counterexample.