TEST: Quotient Spaces
NAME: ____________

MATH 3406

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This is a worksheet about a funny (ha ha) vector space called the **quotient space**.

Our main background space will be $V = \mathbb{R}^2$ or the Euclidean plane. All spaces considered are real spaces, i.e., the field is $\mathbb{R}$.

Problem 1 Draw a picture of

$$W = \{(x, 2x) \in \mathbb{R}^2 : x \in \mathbb{R}\},$$

and show W is a subspace of $\mathbb{R}^2$.

Problem 2 The elements in the quotient space $\mathbb{R}^2/W$ look like

$$\mathbf{x} + W$$

where $\mathbf{x}$ is a vector in $\mathbb{R}^2$. We want to define vector space operations on

$$\mathbb{R}^2/W = \{\mathbf{x} + W : \mathbf{x} \in \mathbb{R}^2\}$$

which is called **the quotient space of** $\mathbb{R}^2$ **by** W or sometimes just $\mathbb{R}^2$ "mod" W. This is a little tricky because the "$\mathbf{x}$" in $\mathbf{x} + W$ is not uniquely determined:

(a) We say $\mathbf{x}+W = \tilde{\mathbf{x}}+W$ if $\mathbf{x}-\tilde{\mathbf{x}} \in W$. Show that if $\mathbf{x}+W = \tilde{\mathbf{x}}+W$ and $\mathbf{y} + W = \tilde{\mathbf{y}} + W$, then

$$(\mathbf{x} + \mathbf{y}) + W = (\mathbf{x} + \tilde{\mathbf{y}}) + W = (\tilde{\mathbf{x}} + \mathbf{y}) + W = (\tilde{\mathbf{x}} + \tilde{\mathbf{y}}) + W.$$

(b) Show that for each $\mathbf{x} + W \in \mathbb{R}^2/W$ there exists a unique element $(x, 0)$ such that $\mathbf{x} + W = (x, 0) + W$. Hint: If $\mathbf{x} = (x_1, x_2)$, then what is x in terms of x_1 and x_2?

(c) Show that for each $\mathbf{x} + W \in \mathbb{R}^2/W$ there exists a unique element $(0, y)$ such that $\mathbf{x} + W = (0, y) + W$.

(d) Use part **(a)** to show

$$(\mathbf{x} + W) + (\tilde{\mathbf{x}} + W) = (\mathbf{x} + \tilde{\mathbf{x}}) + W$$

gives a well-defined associative and commutative addition on $\mathbb{R}^2/W$.

Problem 3 Find the zero element in $\mathbb{R}^2/W$ and find every element $\mathbf{x} \in \mathbb{R}^2$ so that $\mathbf{x} + W$ is the zero element in $\mathbb{R}^2/W$.

Problem 4 Show that scaling $c(\mathbf{x} + W)$ defined by $(c\mathbf{x}) + W$ is well-defined and associative on $\mathbb{R}^2/W$.

Problem 5 Verify the remaining properties required to make $\mathbb{R}^2/W$ a vector space (existence of additive inverses and the distributive properties).

Problem 6 Find a vector space U isomorphic to $\mathbb{R}^2/W$ and write down an isomorphism

$$\phi : \mathbb{R}^2/W \to U.$$

Problem 7 Show that the function $\pi : \mathbb{R}^2 \to \mathbb{R}^2/W$ by

$$\pi(\mathbf{x}) = \mathbf{x} + W$$

is linear and surjective. Find $\mathcal{N}(\pi)$.

This function π is called the **quotient map**.

Problem 8 Find the dimensions of $\mathbb{R}^2/W$ and $\mathcal{N}(\pi)$.

Problem 9 An alternative definition of the elements in $\mathbb{R}^2/W$ is the following: Each element $\mathbf{x} + W \in \mathbb{R}^2/W$ is the set

$$\mathbf{x} + W = \{\mathbf{x} + \mathbf{w} : \mathbf{w} \in W\}.$$

In terms of this definition $\mathbf{x} + W = \tilde{\mathbf{x}} + W$ means equality as sets.

(a) Show this alternative definition of equality is equivalent to the equality defined in **Problem 2** part **(a)** above.

(b) In terms of this alternative definition, the elements of the quotient sapce $\mathbb{R}^2/W$ are sets. Is it true that

$$A + B = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A, \mathbf{b} \in B\}$$

for $A, B \in \mathbb{R}^2/W$?

(c) Draw the elements of $\mathbb{R}^2/W$ as subsets of $\mathbb{R}^2$.

(d) Do you see a nice (isomorphic) copy of $\mathbb{R}^2/W$ in your picture from part **(c)**?

Problem 10 Let $L \in \mathcal{L}(\mathbb{R}^2 \to Z)$ where Z is any vector space.

The **induced linear map on the quotient** is defined to be

$$\phi : \mathbb{R}^2/\mathcal{N}(L) \to Z \qquad \text{by} \qquad \phi(\mathbf{x} + \mathcal{N}(L)) = L(\mathbf{x}).$$

(a) Show ϕ is well-defined and linear.

(b) Show ϕ is injective.

(c) Show $\mathrm{Im}(\phi) = \mathrm{Im}(L)$.

(d) Show $\mathbb{R}^2/\mathcal{N}(L)$ is isomorphic to $\mathrm{Im}(L)$.