Russell Newton's Homogeneous Function

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In MATH 3406 (A Second Course in Linear Algebra) Spring semester 2022 we were considering briefly homogeneous functions $f : V \to W$ where V and W are vector spaces over the same field and particularly, we were looking for functions that were homogeneous but not linear, i.e., not additive. One particularly nice example was given by Russell Newton.

Here we consider Russell's function $g : \mathbb{R}^2 \to \mathbb{R}$ given by

$$g(x, y) = \sqrt[3]{x^3 + y^3}.$$

This is a homogeneous function and is therefore linear when restricted to each one-dimensional subspace of $\mathbb{R}^2$, but is not additive on all of $\mathbb{R}^2$.

Leo Wang noticed that there are nonzero points p and q in different one-dimensional subspaces satisfying

$$g(p + q) = g(p) + g(q).$$

For example, if we take $p = (1, 0)$ and $q = (0, -1)$, then

$$g(p) + g(q) = 1 + (-1) = 0 = g(p + q).$$

On the other hand, for $p = (1, 0)$ and $q = (0, 1)$ we have

$$g(p) + g(q) = 2 \neq \sqrt[3]{2} = g(p + q).$$

Ideally we can characterize (and understand exactly) the extent to which g is additive and/or fails to be additive. This seems to be not so easy. From what I can tell is does seem to be the case that g fails to be additive on "most" points. Here is a conjecture I don't know how to prove (and may not be true):

Conjecture 1 *If $p = (x, y)$ is fixed and nonzero and Z is a one-dimensional subspace of $\mathbb{R}^2$ different from* $\operatorname{span}\{p\}$*, then there are at most finitely many points $q = (z, w) \in Z$ for which*

$$g(p + q) = g(p) + g(q).$$

The question of how many additive points there are in a subspace suggested by this conjecture can at least be examined computationally. From what I have tried, the “finite number” of the conjecture above seems to be “one” in many cases. On the other hand, there is this:

Conjecture 2 *If $p = (x, y)$ is fixed and nonzero, then there exists at least one one-dimensional subspace Z of $\mathbb{R}^2$ such that no nonzero point $q = (z, w) \in Z$ satisfies*

$$g(p + q) = g(p) + g(q).$$

I will give an example below where there is one such subspace and and example where there are two distinct such subspaces. I have no idea how many are possible.

1 Computations

Let us ”fix” a point/vector $p = (x, y) = r(\cos\theta, \sin\theta) \in \mathbb{R}^2\backslash\{(0, 0)\}$ and consider a second point

$$q = (z, w) = \alpha(\cos t, \sin t)$$

also in $\mathbb{R}^2$. Given a point in polar coordinates, we have

$$g(p) = r\sqrt[3]{\cos^3\theta + \sin^3\theta}. \tag{1}$$

The condition that $g(p + q) = g(p) + g(q)$ is therefore,

$$\sqrt[3]{(r\cos\theta + \alpha\cos t)^3 + (r\sin\theta + \alpha\sin t)^3} = r\sqrt[3]{\cos^3\theta + \sin^3\theta} + \alpha\sqrt[3]{\cos^3 t + \sin^3 t}.$$

There is some cancellation if we cube both sides, and cubing should not introduce any extraneous roots. Precisely, we can say $g(p + q) = g(p) + g(q)$ if and only if

$$r\cos^2\theta\cos t + \alpha\cos\theta\cos^2 t + r\sin^2\theta\sin t + \alpha\sin\theta\sin^2 t = r\mu^2\nu + \alpha\mu\nu^2$$

where

$$\mu = \sqrt[3]{\cos^3\theta + \sin^3\theta} \qquad \text{and} \qquad \nu = \sqrt[3]{\cos^3 t + \sin^3 t}.$$

We can further simplify this condition as

$$ur + v\alpha = 0 \tag{2}$$

where

$$u = u(\theta, t) = \cos^2\theta\cos t + \sin^2\theta\sin t - \mu^2\nu$$

and

$$v = v(\theta, t) = \cos\theta\cos^2 t + \sin\theta\sin^2 t - \mu\nu^2.$$

This looks pretty good. Notice that u and v depend only on the angles θ and t. This means, if we fix $p = (x, y) = r(\cos\theta, \sin\theta)$, then r and θ are fixed. We can also assume $r \neq 0$.

Furthermore, we can fix t with $t \neq \theta + k\pi$ for $k \in \mathbb{Z} = \{0, \pm1, \pm2, \pm3, \ldots\}$. This corresponds to looking for the point $q = (z, w) = \alpha(\cos t, \sin t)$ in a subspace different from span$\{p\}$. With this in mind (2) looks like a simple linear relation "determining" the radius α. Roughly speaking (if all things go well) you'd think there must be precisely one nonzero element q in each subspace for which additivity holds.

So far so good, but here is where things get a bit difficult: In order to solve for α, you need to know first of all that $v(\theta, t) \neq 0$, and v turns out to be a rather complicated quantity in terms of the angles θ and t. If you want to have $\alpha \neq 0$, furthermore, then you sort of also need the similar looking (and similarly complicated but yet different) quantity $u = u(\theta, t)$ to be nonzero.

I'll add to this that I don't think it's true generally (from what I've seen computationally) that there is only one point in most other subspaces where additivity holds. I'll try to indicate some examples below.

For particular cases, things should work out pretty well in priniple. Let's take $\theta = 0$ and $r = 1$ corresponding to Leo's choice $p = (1, 0)$. Then $\mu = 1$ and

$$u = \cos t - \nu$$

while

$$v = \cos^2 t - \nu^2.$$

Thus our equation (2) for α becomes

$$(\cos^2 t - \nu^2)\alpha = \nu - \cos t \qquad \text{or} \qquad (\cos t - \nu)(\cos t + \nu)\alpha = -(\cos t - \nu).$$

To get Leo's solution, we can take $t = \pi/2$ and then

$$\nu = \sqrt[3]{\cos^3 t + \sin^3 t} = 1$$

while $\cos t = 0$. Thus, the equation becomes $-\alpha = 1$ or $\alpha = -1$. In fact, we have shown $q = (0, -1)$ is the only nonzero point $q \in \text{span}\{(0, 1)\}$ for which $g((1, 0) + q) = g(1, 0) + g(q)$.

On the other hand, the expression

$$\cos t - \nu = \cos t - \sqrt[3]{\cos^3 t + \sin^3 t}$$

vanishes precisely when $\sin t = 0$. We can't take $t = \pi k$ for $k \in \mathbb{Z}$ since that would put us in the same subspace as $p = (1, 0)$. Therefore, the equation simplifies in this case to

$$(\cos t + \nu)\alpha = -1.$$

The vanishing of

$$\cos t + \nu = \cos t + \sqrt[3]{\cos^3 t + \sin^3 t}$$

is equivalent to

$$2\cos^3 t = -\sin^3 t \qquad \text{or} \qquad \tan t = -\sqrt[3]{2}.$$

Therefore, it looks like in this case, you get a particular subspace

$$\text{span}\left\{\left(1, -\tan^{-1}\left(\sqrt[3]{2}\right)\right)\right\}$$

for which there is **no nonzero** point where additivity holds. In all other subspaces you get precisely one nonzero point

$$q = -\frac{1}{\cos t + \sqrt[3]{\cos^3 t + \sin^3 t}}(\cos t, \sin t)$$

depending on the angle $t \neq -\tan^{-1}\left(\sqrt[3]{2}\right)$ for which

$$g((1, 0) + q) = 1 + g(q).$$

What if we take, for example, $p = (1, 1)$. Then $\theta = \pi/4$ and $r = \sqrt{2}$. We have then (if I can compute correctly)

$$\mu = \sqrt[3]{\cos^3 \theta + \sin^3 \theta} = \frac{\sqrt[3]{2}}{\sqrt{2}}.$$

Therefore,

$$u = \frac{1}{2}\cos t + \frac{1}{2}\sin t - \frac{\sqrt[3]{4}}{2}\sqrt[3]{\cos^3 t + \sin^3 t}$$

and

$$v = \frac{1}{\sqrt{2}} - \frac{\sqrt[3]{2}}{\sqrt{2}} \sqrt[3]{(\cos^3 t + \sin^3 t)^2}.$$

So the equation $v\alpha = ur$ becomes

$$\left(1 - \sqrt[3]{2}\ \sqrt[3]{(\cos^3 t + \sin^3 t)^2}\right) \alpha = \cos t + \sin t - \sqrt[3]{4}\ \sqrt[3]{\cos^3 t + \sin^3 t}.$$

If we plot the constant term on the right we find that it vanishes at four points on $[0, 2\pi)$. See Figure 1. Two of those points are $t = \pi/4$ and $t = 5\pi/4$ corresponding

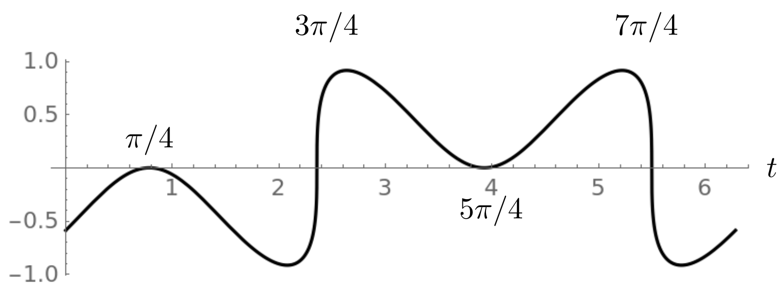

Figure 1: The quantity $\cos t + \sin t - \sqrt[3]{4}\ \sqrt[3]{\cos^3 t + \sin^3 t}$.

to the subspace spanned by $(1, 1)$ as should be expected (since g is linear on that subspace). The other two zeros, are at $t = 3\pi/4$ and $t = 7\pi/4$ corresponding to the subspace $y = -x$. We need to check the coefficient of α to see what happens on this subspace. As an aside, we note that this particular subspace is mapped to zero (the entire subspace). Thus, we can check

$$g((1, 1) + \alpha(1/\sqrt{2}, -1/\sqrt{2})) = \sqrt[3]{\left(1 + \frac{1}{\sqrt{2}}\right)^3 + \left(1 - \frac{1}{\sqrt{2}}\right)^3} = \sqrt[3]{5}.$$

On the other hand, we know $g(1, 1) = \sqrt[3]{2}$ and $g(\alpha(1/\sqrt{2}, -1/\sqrt{2}) = 0$. This means there is no nonzero point q in this subspace for which the additivity condition holds. In particular, this means the coefficient of α must not vanish at $t = 3\pi/4$ and $t = 7\pi/4$. Let's check that.

Figure 2 indicates shows that the coefficient of α has six zeros corresponding to three distinct subspaces. Two of these zeros are at $t = \pi/4$ and $t = 5\pi/4$ corresponding to the subspace $y = x$ as expected. (If this coefficent were nonzero at these agles, then $\alpha = 0$ would be the only solution leading to a point q, but we know every point q in this subspace satisfies $g(p+q) = g(p) + g(q)$ because p is in this subspace. We see from the plot that this coefficient, while continuous across the interval $[0, 2\pi)$ has two singularities corresponding to non-differentiability at $t = 3\pi/4$ and $t = 7\pi/4$. At any rate, these are not zeros. Therefore, the subspace $y = -x$ has no nonzero points $q = \alpha(1/\sqrt{2}, -1/\sqrt{2})$ for which the additivity relation holds with $p = (1, 1)$, as we verified above. The other four zeros t_1, t_2, $t_1 + \pi$, and $t_2 + \pi$ correspond to distinct subspaces, and since the quantity ur on the right does not vanish at t_1 and t_2, we obtain two more subspaces devoid of nonzero points q for which $g(p+q) = g(p) + g(q)$ holds. In Figure 3 we have plotted the coefficient of α and the constant term together.

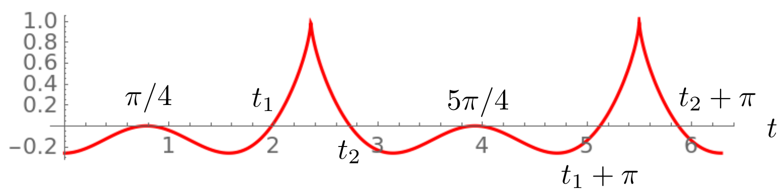

Figure 2: The quantity $1 - \sqrt[3]{2}\ \sqrt[3]{(\cos^3 t + \sin^3 t)^2}$.

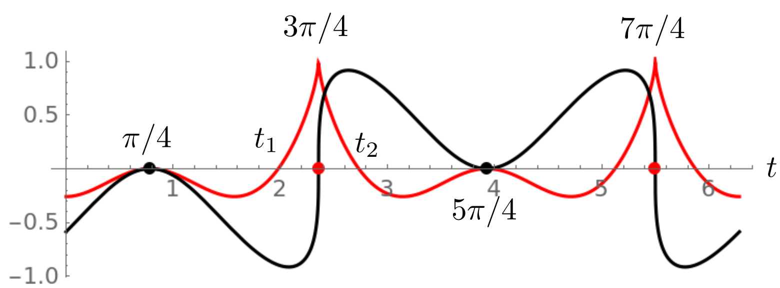

Figure 3: The coefficient of α and the constant term.

2 An Associated Function

Generally, we know a variety of homogeneous functions $G : \mathbb{R}^2 \to \mathbb{R}^2$ can be obtained by scaling in each subspace $\text{span}\{(\cos\theta, \sin\theta)\}$ by a particular constant $\sigma(\theta)$. There needs to be no correlation among the values of $\sigma(\theta)$ for $0 \le \theta < \pi$. This observation along with the scaling property (1) of Russell's function g suggests the consideration of the particular function $G : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$G(r(\cos\theta, \sin\theta)) = \begin{cases} r\sqrt[3]{\cos^3\theta + \sin^3\theta}(\cos\theta, \sin\theta), & \text{for } -\pi/4 \le \theta \le 3\pi/4 \\ -r\sqrt[3]{\cos^3\theta + \sin^3\theta}(\cos\theta, \sin\theta), & \text{for } 3\pi/4 \le \theta \le 7\pi/4. \end{cases}$$

Figure 4 shows the image of this mapping on the unit circle.

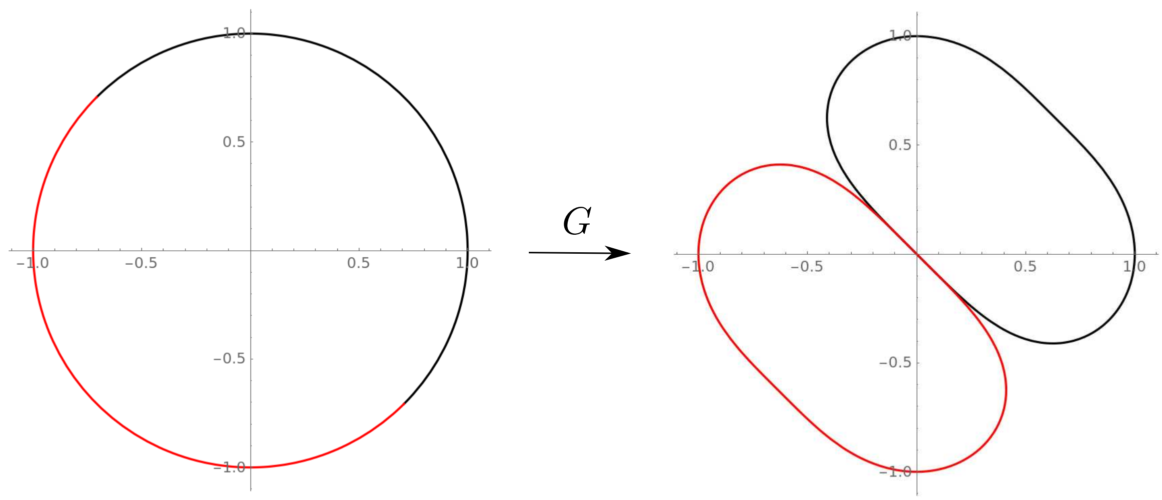

Figure 4: The mapping G.

If you take the black image points and rotate them to the positive x-axis and rotate the red image points to the negative x-axis, then you get g. In a certain sense, this gives one a pretty good picture of what the function g "does," but I'm not seeing the additivity properties.