Math 4431, Assignment 10: Chapter 4 Name and section: ________________

§ 4.1-4 Armstrong

Let X be a topological space. We have talked about constructing a quotient space of X with respect to a subset (identifying the subset to a point). We have talked about a quotient of X with respect to an equivalence relation, which is just another way to define a partition. We have talked about a quotient of X with respect to a bijection.

Finally, we wish to describe the quotient of X by a **group action**. Here are the relevant definitions:

A group G is, first of all, a set with an "operation" $* : G \times G \to G$. We have used the symbol "$*$" to denote the operation, but many other "binary operator symbols" are used, and the image of a pair is expressed in terms of this symbol in the following way:

$$*(g,h) = g * h \quad \text{for } g,h \in G.$$

In order for G to be a **group**, we require the operation to have the following properties:

1. $(g * h) * k = g * (h * k)$ for $g,h,k \in G$. (We say the operation is **associative**.)
2. There is some $e \in G$ (called the identity element) with $e * g = g * e = g$ for all $g \in G$.
3. For each $g \in G$, there is some $h \in G$ (called the inverse of g) such that $h*g = g*h = e$. (The element h is often denoted by g^{-1}, but may also be denoted by $-g$, especially if the symbol "+" is used for the operation.)

Given a group G and a set X, we say G **acts** on X if there is a **pairing**, i.e., a function $a : G \times X \to X$ (whose images we denote by $a(g,x) = gx$) such that

(a) $ex = x$ for all $x \in S$, and

(b) $(g * h)x = g(hx)$ for all $g,h \in G$ and $x \in X$.

Given a topological space X and a group G acting on X, the topological space X/G is defined to be the indentification space determined by the partition

$$\mathcal{P}_G = \{\{gx : g \in G\} : x \in X\}.$$

1. (5 points) Show that the inverse elements in a group are unique. (To do this, assume there are elements h_1 and h_2 satisfying $h_i g = g h_i = e$ for $i = 1,2$ and g some fixed element in the group. Then show $h_1 = h_2$.)
2. (5 points) Show that a **left inverse** in a group is an inverse, i.e., if $hg = e$ for some g in a group, then $h = g^{-1}$.
3. (5 points) Show the identity element in a group is unique.
4. (5 points) Let G be a set with an associative operation. Assume also that there is an element e with $eg = g$ for all $g \in G$, i.e., e is a **left identity**. Assume also that for each $g \in G$, there is some $h \in G$ with $hg = e$, i.e., there are **left inverses**. Show G is a group, i.e., the left identity is an identity and the left inverses are (two sided) inverses. Hint: First show that a left inverse is a right inverse.

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5. (5 points) Show that the integers $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ are a group under addition.

6. (10 points) If $\phi : X \to X$ is a bijection, then show $\mathbb{Z}$ acts on X by $jx = \phi^j(x)$. If X is a topological space, what is the relation between X/ϕ and $X/\mathbb{Z}$?

7. Let $\Sigma = [\pi/4, 3\pi/4] \times \mathbb{R}$ and $q_1 : \Sigma \to \mathbb{R}^3$ by

$$q_1(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi).$$

   (a) (5 points) Identify $q_1(\Sigma)$. Prove your assertion.

   (b) (10 points) Consider $\mathrm{an} : \mathbb{R}^3 \to \mathbb{R}^3$ by $\mathrm{an}(x) = -x$. Identify $X_1/\,\mathrm{an}$ and prove your assertion.

8. (10 points) Express $X_1/\,\mathrm{an}$ from the previous problem as X_1/G where G is an appopriate group acting on X_1. Hint: $G = \{\mathrm{id}, \mathrm{an}\}$ is a group.

9. Consider $q : \mathbb{R}^1 \twoheadrightarrow Y = \{0, \pm 1\}$ by

$$q(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

   (a) (10 points) What is the identification topology on Y as a set?

   (b) (5 points) Is Y Hausdorff?

   (c) (5 points) Is Y as a set with the identification topology homeomorphic to Y as a topological subspace of $\mathbb{R}$?

10. (5 points) Show $\mathbb{Z}^2 = \{(n, m) : n, m \in \mathbb{Z}\}$ is a group under addition.

11. (5 points) Show $\mathbb{Z}^2$ acts on $\mathbb{R}^2$ by $(n, m)(x, y) = (x + n, y + m)$. What is $\mathbb{R}^2/\mathbb{Z}^2$ where $\mathbb{Z}^2$ is considered as a group with this action?

12. (10 points) What is $\mathbb{R}^2/\mathbb{Z}^2$ where $\mathbb{Z}^2$ is considered as a subspace of $\mathbb{R}^2$? Is this the same space as considered in the last problem? Can you prove your assertion?