Math 4431, Assignment 12a: Chapter 5 Name and section: ________________

§ 5.1-2 Armstrong: The Fundamental Group

1. (10 points) Let X and Y be path connected topological spaces, and assume $f : X \to Y$ is a continuous function. Show that given any two loops γ_0 and γ_1 in the equivalence class of loops $\langle\gamma\rangle \in \pi_1(X, p)$, we have
$$\langle f \circ \gamma_0 \rangle = \langle f \circ \gamma_1 \rangle \in \pi_1(Y, f(p)).$$
In other words, given a homotopy between γ_0 and γ_1 in X (with p fixed), show that $f \circ \gamma_0$ and $f \circ \gamma_1$ are homotopic in Y (with $f(p)$ fixed).

2. (10 points) Under the assumptions of the previous problem, the function $f : X \to Y$ induces a function $\phi : \pi_1(X) \to \pi_1(Y)$ by $\phi\langle\gamma\rangle = \langle f \circ \gamma\rangle$. Show ϕ is a group homomorphism.

3. (10 points) If $f, g : X \to Y$ are homotopic (with a given set $A \subset X$ fixed), and $h, k : Y \to Z$ are homotopic (with the set $f(A)$ fixed), show $h \circ f$ is homotopic to $k \circ g$ (with A fixed).

4. (10 points) If $f : \mathbb{S}^1 \to \mathbb{S}^1$ is **not** homotopic to the identity, then there is some $p \in \mathbb{S}^1$ with $f(p) = -p$. (Prove it.)

5. (10 points) (5.1.3) Consider $B_1(0) \subset \mathbb{R}^2$ and the function
$$h : \overline{B_1(0)} \to \overline{B_1(0)} \text{ by}$$
$$h(x, y) = r(\cos(\theta + 2\pi r), \sin(\theta + 2\pi r))$$
where $r = \sqrt{x^2 + y^2}$ and $(x, y) = r(\cos\theta, \sin\theta)$ with $0 \le \theta < 2\pi$. Show h is a homeomorphism.

6. (10 points) (5.1.3) Let $\overline{h} : \overline{B_1(0)} \to \overline{B_1(0)}$ be the function defined in the previous problem. Find a homotopy H of h to the identity such that $h_s(p) = H(p, s)$ is a homeomorphism for each $s \in [0, 1]$.

7. (10 points) (5.1.4) If your solution H to the previous problem did not satisfy $H(p, s) \equiv p$ for each $p \in \mathbb{S}^1$, then find another homotopy of h to the identity which leaves $\mathbb{S}^1$ fixed.

8. (10 points) If $\phi : G \to H$ is a group homomorphism and $\phi^{-1}(\{e\}) = \{\text{id}\}$ where e is the identity in H and id is the identity in G, then show ϕ is one-to-one.

9. (10 points) A convex subset of Euclidean space is simply connected, i.e., has fundamental group with one element.

10. (10 points) There is a path in $\mathbb{S}^1$ which is not homotopic to the identity.