Math 4431, Assignment 4a: compactness Name and section: ________________

§ 3.1-4 Armstrong

1. (2.1.12) Here are some definitions:

   A topological space X is called **first countable** if for each point $x \in X$ there is a countable collection of open sets $\mathcal{B}$ such that for each open set U with $x \in U$, there is some $B \in \mathcal{B}$ with $x \in B \subset U$, i.e., there exists a countable "base" of open sets "at each point."

   A topological space X is called **second countable** if there is a countable base.

   A topological space X is said to be **separable** if it contains a countable dense subset.

   (a) (2 points) Show that a second countable space is always first countable.

   (b) (2 points) Show that a first countable space need not always be second countable.

   (c) (3 points) Show that a first countable space which is separable is second countable. (Correction: Apparently, this is not correct; I'll need to check. Please ignore this part.)

   (d) (3 points) Show that a second countable space is separable.

2. (10 points) (3.2.1) Show that the closure of the unit ball in $L^2(0,1)$ is not compact. Recall that $L^2(0,1)$ is the collection of all square integrable functions, i.e., $f \in L^2$ if $\int f^2 < \infty$, under the metric topology determined by

$$d(f,g) = \left[\int (f-g)^2\right]^{1/2}.$$

3. (10 points) (3.2.2) Let $A_1 \supset A_2 \supset A_3 \supset \cdots$ be a nested sequence of nonempty closed sets in a complete metric space. Assume also that the diameters of these sets satisfy

$$\lim_{j \nearrow \infty} \operatorname{diam}(A_j) = 0.$$

   Show that the intersection

$$\cap_{j=1}^{\infty} A_j$$

   consists of precisely one point. Remember that a metric space is called **complete** if every Cauchy sequence in the space converges (to a point in the space), and the diameter of a set A in a metric space is defined by

$$\operatorname{diam}(A) = \sup_{x_1, x_2 \in A} d(x_1, x_2).$$

4. (10 points) (3.3.5) Recall that the **unit sphere** in R^{n+1} is defined by

$$\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

   Let $p_1, \ldots, p_k \in \mathbb{S}^n$. Show that $X = \mathbb{S}^n \backslash \{p_1, \ldots, p_k\}$ is not compact.

5. (10 points) (3.3.8, Lindelöf's theorem) Show that every open cover of a second countable space has a countable subcover.