Math 4431, Assignment 9: Chapter 4 Name and section: ________________

§ 4.1-2 Armstrong

Let X be a topological space and $A \subset X$. Recall that the topological space X/A is the identification space associated with the partition $\mathcal{P} = \{\{x\} : x \in X \backslash A\} \cup \{A\}$. This is the "one point quotient" of X by A.

We can also talk about a quotient of a topological space by a relation $\sim$ or by a map.

A **relation** is a subset $\sim$ of the cross-product of a set, say a topological space X with the properties

1. The diagonal is in the relation, i.e., $(x, x) \in\sim$ for every $x \in X$.
2. The relation is symmetric, i.e., if $(x, y) \in\sim$, then $(y, x) \in\sim$.
3. The relation is transitive, i.e., if $(x, y), (y, z) \in\sim$, then $(x, z) \in\sim$.

Whenever there is a relation on a set X, there is a partition of X into equivalence classes. The topological space associated with this partition is denoted by $X/\sim$.

Given a bijection $\phi : X \to X$, the **orbit** of an element is defined by $[x] = \{\phi^j(x) : j \in \mathbb{Z}\}$. The orbits of a homeomorphism partition X, and the topology associated with $\mathcal{P} = \{[x] : x \in X\}$ is denoted by X/ϕ.

1. (20 points) Given a relation $\sim$, the condition $(x, y) \in\sim$ is usually written $x \sim y$. Reexpress the three conditions required of a relation in this notation, and show the following:
    1. The equivalence classes $[x] = \{y \in X : x \sim y\}$ partition X.
    2. Given any partition $\mathcal{P}$ of X, there is a unique equivalence relation $\sim$ on X with the sets in $\mathcal{P}$ as equivalence classes.
2. (20 points) Show that the **antipodal map** $\mathrm{an} : \mathbb{S}^n \to \mathbb{S}^n$ by $\mathrm{an}(x) = -x$ is a homeomorphism.
3. (20 points) (4.2.1) Let $\mathbb{P}^n = \mathbb{S}^n/\mathrm{an}$ be n-dimensional projective space. Let $\mathbb{S}^{n-1}$ denote the hypersurface boundary of the closed upper hemisphere $\mathbb{S}^n_+$, i.e.,

    $$\mathbb{S}^n_+ = \{x = (x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : |x| = 1, x_{n+1} \geq 0\}$$

    and

    $$\mathbb{S}^{n-1} = \{x \in \mathbb{S}^n_+ : x_{n+1} = 0\}.$$

    Consider $\phi : \mathbb{S}^n_+ \to \mathbb{S}^n_+$ by

    $$\phi(x) = \begin{cases} x, & x \in \mathbb{S}^n_+ \backslash \mathbb{S}^{n-1} \\ \mathrm{an}(x), & x \in \mathbb{S}^{n-1} \end{cases}$$

    Show that ϕ is a bijection and X/ϕ is (homeomorphic to) $\mathbb{P}^n$.

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4. (20 points) (4.2.5) Let $X \subset \mathbb{R}^2$ be the union of circles

$$X = \cup_{j=1}^{\infty}\{(x,y) : (x-1/j)^2 + y^2 = 1/j^2\}$$

(with the subspace topology induced from $\mathbb{R}^2$. Let $Y = \mathbb{R}/z$ where $z : \mathbb{R} \to \mathbb{R}$ by

$$z(x) = \begin{cases} x, & x \notin \mathbb{Z} \\ x+1, & x \in \mathbb{Z}. \end{cases}$$

Show that z is a bijection so that Y is well defined, and show that X and Y are not homeomorphic.

5. (20 points) (4.2.7) Recall that $\mathbb{T}^2$ is a square, say $X = [0,2\pi]\times[0,2\pi]$ with opposite sides identified without twisting: $(x,0) \sim (x,2\pi)$ and $(0,y) \sim (2\pi,y)$. What topological space do you get if you further identify the union of a meridian circle $\{(x_0,y) : 0 \leq y < 2\pi\}$ and a lattitudinal circle $\{(x,y_0) : 0 \leq x < 2\pi\}$ to a single point? Can you prove your assertion?