Only the beginning

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December 2018

As we come to the end of all things MATH 4431 (Fall Semester 2018) it occurs to me that some final remarks are in order. There are a number of results I think everyone in an elementary topology course should see and think about (and probably see how to prove). For various reasons, I will not have time to prove all these results. I may not have time to state and discuss all of them in a lecture. For this reason, I will try to write down the statements here.

A small number of you are do not seem to have a really solid understanding of some basic notions like that of an open set, a continuous function, a compact or connected set. At this point, it will be up to those of you who are in this situation to do something about that. Some aspects of identification spaces/maps are still a bit fuzzy for many of you. Again, it will be up to you to put in the effort to fully understand these concepts and the related results.

1 Covering spaces and liftings

In Chapter 10 Armstrong takes up these topics in some generality. Munkres tackles them in Chapter 8 (page 331) of his book. The impression I get is that they both consider them somewhat "optional" or at least somehow advanced. I think of them as pretty basic.

Definition 1 *Given an identification map $\phi : X \to Y$, we say ϕ is a* **covering map** *if for each $x \in X$, there is an open set V (in Y) with*

$$\phi(x) \in V \subset Y$$

such that $\phi^{-1}(V)$ is given by a partition (of disjoint open sets)

$$\phi^{-1}(V) = \cup_{\alpha\in\Gamma} U_\alpha$$

with

$$\phi\Big|_{U_\alpha} : U_\alpha \to V \quad \text{a homeomorphism for each } \alpha \in \Gamma.$$

Note: The complicated part of the identification map structure can be ignored here; all that is needed is for ϕ to be continuous and surjective.

Exercise 1 *Show that if $\phi : X \to Y$ is continuous and surjective and for each $x \in X$, there is an open set V (in Y) with*

$$\phi(x) \in V \subset Y$$

such that $\phi^{-1}(V)$ is given by a partition (of disjoint open sets)

$$\phi^{-1}(V) = \cup_{\alpha\in\Gamma} U_\alpha$$

with

$$\phi\Big|_{U_\alpha} : U_\alpha \to V \quad \text{a homeomorphism for each } \alpha \in \Gamma,$$

then ϕ is an identification map (and hence a covering map).

Exercise 2 *$\phi : \mathbb{R} \to \mathbb{S}^1$ by $\phi(t) = (\cos t, \sin t)$ is a covering map.*

Exercise 3 *$\phi : \mathbb{R} \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ by $\phi(x, y) = (\cos x, \sin x, \cos y, \sin y)$ is a covering map.*

Definition 2 *Given $\phi : X \to Y$ and $f : X_0 \to Y$ both continuous, a third continuous function $\hat{f} : X_0 \to X$ is a* **lifting** *of f if $\phi \circ \hat{f} = f$.*

$$\begin{array}{ccc} X_0 & \xrightarrow{\hat{f}} & X \\ & & \downarrow \phi \\ X_0 & \xrightarrow{f} & Y \end{array}$$

Lemma 1 [path lifting lemma] *If $\phi : X \to Y$ is a covering map and $\gamma : [0, 1] \to Y$ is a path, then given $x_0 \in X$ with $\phi(x_0) = \gamma(0)$, there is a unique lifting $\hat{\gamma}$ of γ with $\hat{\gamma}(0) = x_0$.*

$$\begin{array}{ccc} [0,1] & \xrightarrow{\hat{\gamma}} & X \\ & & \downarrow \phi \\ [0,1] & \xrightarrow{\gamma} & Y \end{array}$$

Exercise 4 *A lifting of a loop in Y may not be a loop in X.*

Lemma 2 [homotopy lifting lemma] *If $\phi : X \to Y$ is a covering map and $H : [0,1] \times [0,1] \to Y$ is a path homotopy, then given $x_0 \in X$ with $\phi(x_0) = H(0,0)$, there is a unique lifting $\hat{H}$ of H with $\hat{H}(0,0) = x_0$.*

$$\begin{array}{ccc} [0,1] \times [0,1] & \xrightarrow{\hat{H}} & X \\ & & \downarrow \phi \\ [0,1] \times [0,1] & \xrightarrow{H} & Y \end{array}$$

Example Remember the loop $\gamma : [0,1] \to \mathbb{T}^2$ by

$$\gamma(t) = \left(1 + \frac{\cos 6\pi t}{2}\right)(\cos 2\pi t, \sin 2\pi t, 0) + \frac{\sin 6\pi t}{2}(0,0,1)$$

from Exam 3. There we had $\gamma(0) = (3/2, 0, 0)$. We also had a covering map $\phi : \mathbb{R}^2 \to \mathbb{T}^2$ by

$$\phi(x,y) = \left(1 + \frac{\cos y}{2}\right)(\cos x, \sin xt, 0) + \frac{\sin y}{2}(0,0,1),$$

and $\phi(0,0) = (3/2, 0, 0)$. Note that

$$\phi\big|_{(-\pi,\pi)\times(-\pi,\pi)} : (-\pi,\pi) \times (-\pi,\pi) \to \mathbb{T}^2$$

is a homeomorphism. This means there is only one possible path $\hat{\gamma} : [0, t_0] \to \mathbb{R}^2$ with

$$\phi \circ \hat{\gamma}(t) = \gamma(t) \tag{1}$$

namely,

$$\hat{\gamma}(t) = \left(\phi\big|_{(-\pi,\pi)\times(-\pi,\pi)}\right)^{-1} \circ \gamma(t),$$

where (1) holds as long as $\gamma(t) \in V_0 = \phi((-\pi,\pi) \times (-\pi,\pi))$ and t_0 can be taken as large as possible consistent with this condition. We see the first time $\gamma(t)$ exist V_0 is when $t_0 = 1/6$ and $\gamma(1/6) = (\cos \pi/3, \sin \pi/3, 0)/2$. Looking down from above, we see Figure 1(right). The portion of the curve corresponding to $y_0 = 6\pi t_0 = \pi$ and $x_0 = 2\pi t_0 = \pi/3$ reaches the boundary of $\phi((-\pi,\pi) \times (-\pi,\pi))$ on the inner circle of $\mathbb{T}^2$ with $t_0 = 1/6$. Since $\gamma(t) = \phi(2\pi t, 6\pi t)$, the lifting must be

$$\hat{\gamma}(t) = (2\pi t, 6\pi t) \qquad \text{for} \quad 0 \le t \le t_0 = 1/6.$$

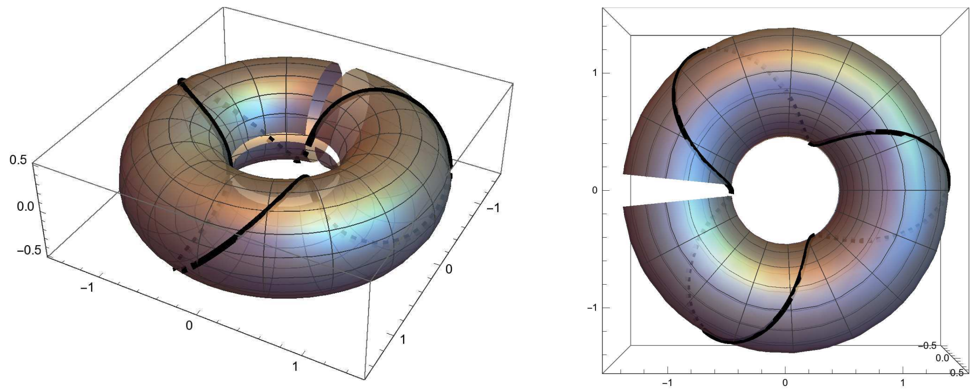

Figure 1: curve on the torus starting at $(3/2, 0, 0)$

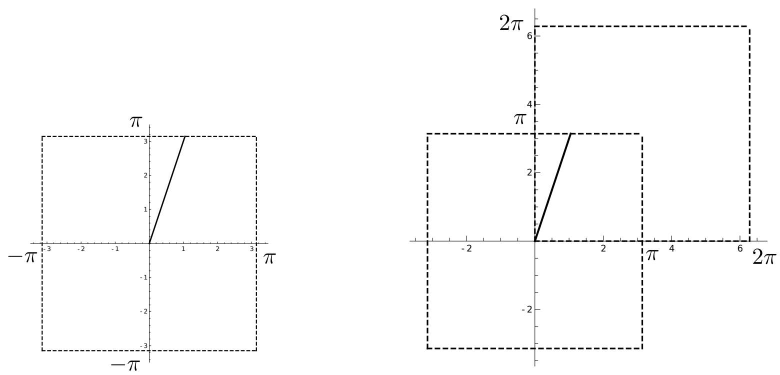

Figure 2: neighborhoods in $\mathbb{R}^2$ on which ϕ is a homeomorphism

At this point, we need to take another neighborhood in $\mathbb{T}^2$ containing $\phi(\pi/3, \pi) = (1/2)(\cos\pi/2, \sin\pi/3, 0)$ with a corresponding homeomorphic neighborhood in $\mathbb{R}^2$ containing $(\pi/3, \pi)$. A natural choice is $(0, 2\pi) \times (0, 2\pi)$ as indicated in Figure 2(right). Notice the lift of this loop is not a loop.

Theorem 1 *Given any loop γ in the torus $\mathbb{T}^2$ with $\gamma(0) = (3/2, 0, 0)$, the lift $\hat{\gamma} : [0, 1] \to \mathbb{R}^2$ satisfies $\hat{\gamma}(1) = (2\pi k, 2\pi\ell)$ for some $k, \ell \in \mathbb{Z}$.*

Theorem 2 *If there is a homotopy of a loop γ to the identity loop $\mathrm{id}(t) \equiv x_0$, then the lift of γ to a covering space must be a loop as well, and the lift $\hat{\gamma}$ must also be homotopic to the identity loop in the covering space.*

Here is a related theorem:

Theorem 3 *If a space Y is path connected and locally path connected and Y admits a simply connected covering space $\mathcal{U}$, then $\mathcal{U}$ is uniquely determined (up to homeomorphism).*

A simply connected covering space $\mathcal{U}$ is called a **universal covering space**. Remember that **simply connected** means the fundamental group if trivial, or every loop can be contracted to a point. I want to state a result on the existence of universal covering spaces, which is the last result in Munkres' book. We need one more definition. A space is **semilocally simply connected** if every point has a simply connected neighborhood. Most spaces we encounter satisfy the stronger condition that there is a basis at each point consisting of simply connected open sets. Such spaces are **locally simply connected**.

Theorem 4 *If a space Y is path connected, locally path connected, and semilocally simply connected, then there exists a (unique) universal covering space $\mathcal{U}$ for Y.*

Thus the space $\mathbb{R}^1$ for $\mathbb{S}^1$ and $\mathbb{R}^2$ for $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ are the unique universal covering spaces for the circle and the torus.

The fact that the lift of some loops on the circle or the torus to the respective universal cover are not loops shows that these spaces are not simply connected.

Exercise 5 *Give examples of loops in $\mathbb{T}^2$ showing the map $\psi : \pi_1(\mathbb{T}^2) \to \mathbb{Z}^2$ by $\langle\gamma\rangle \mapsto (k, \ell)$ where the lift $\hat{\gamma}$ of γ satisfies $\hat{\gamma}(1) = (2\pi k, 2\pi\ell)$.*

Theorem 5 *ψ is a group isomorphism and*

$$\pi_1(\mathbb{T}^2) = \pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}.$$

Theorem 6 *$\pi_1(\mathbb{S}^1) = \mathbb{Z}$.*

2 Fundamental group of spheres

Theorem 7 [**Theorem 5.12 in Armstrong**] *If $X = X_1 \cup X_2$ where X_1 and X_2 are simply connected and $X_1 \cap X_2$ is nonempty and path connected, then X is simply connected.*

The proof of this result uses Lebesgue's covering lemma.

Corolloary 1 $\mathbb{S}^2$ *(and $\mathbb{S}^3$, $\mathbb{S}^4$, etc.) are all simply connected.*

3 Fundamental group of a product space

Theorem 8 [Theorem 5.14 in Armstrong] *If X and Y are path connected, then $X \times Y$ is path connected and*

$$\pi_1(X \times Y) \approx \pi_1(X) \times \pi_1(Y).$$

Example: $\pi_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{Z} \times \mathbb{Z}$.

4 Deformation retraction

Recall that a **homotopy of a set** is just a homotopy of the identity: $X \subset \mathcal{U}$ and $H : X \times [0,1] \to \mathcal{U}$ with

$$X(x,0) = (x) \quad \text{for } x \in X.$$

A **deformation retraction** is a special case where $\mathcal{U} = X$ and we require also that

$$H(x,1) \in A \quad \text{for } xinX$$

where A is some specified subset/subspace of X. Here is a rather amazing result:

Theorem 9 *If A is a deformation retraction of X, then $\pi_1(X) = \pi_1(A)$.*

Example:

$$H(t,\theta,s) = \left(1 + \frac{t(1-s)}{2}\cos\frac{\theta}{2}\right)(\cos\theta, \sin\theta, 0) + \frac{t(1-s)}{2}\sin\frac{\theta}{2}(0,0,1)$$

gives a deformation retraction of the Möbius strip M onto $\mathbb{S}^1$. Therefore,

$$\pi_1(M) = \mathbb{Z}.$$

Note: The situation where A is a deformation retraction of the space X is a special case of two spaces having the same **homotopy type**. This notion of having the same homotopy type is treated nicely in section 5.4 of Armstrong. Two spaces of the same homotopy type have the same fundamental group.