Math 4431, Exam 1: Chapters 1-2 Name and section: ________________

1. Let X and Y be topological spaces.

   (a) (10 points) Give a precise definition of what it means for a function $f : X \to Y$ to be a **homeomorphism**. Note: For this definition, you may assume the notion of continuity is known; you do not need to define continuity.

   (b) (10 points) Let $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Find a function $h : \mathbb{S}^1 \to \mathbb{S}^1$ which satisfies

   (i) h is a homeomorphism, and

   (ii) $h(x, y) \neq (x, y)$ for all $(x, y) \in \mathbb{S}^1$.

   (c) (5 points) Let $D = \{(x, y) \subset \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Find a homeomorphism $\bar{h} : D \to D$ such that

   $$\bar{h}\big|_{\mathbb{S}^1} = h$$

   where h is the homeomorphism you found in part (b) above. Hint: Take $\bar{h}(0, 0) = (0, 0)$.

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2. (25 points) A topological space X is called **Hausdorff** if given x and y in X with $x \neq y$, there are disjoint open sets U and V with $x \in U$ and $y \in V$.

   Show that if $f : X \to X$ is continuous and X is a Hausdorff space, then the fixed point set

   $$E = \{x \in X : f(x) = x\}$$

   is closed.

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3. A nonempty collection $\mathcal{B}$ of open sets in a topological space $(X, \mathcal{T})$ is a **base** for $\mathcal{T}$ if

For each open set $U \in \mathcal{T}$ and each $x \in U$, there is some $B \in \mathcal{B}$ with

$$x \in B \subset U.$$

Be careful: This is not the same as the definition given in the lecture.

(a) (10 points) Let X be a set, and let $\mathcal{C}$ be a nonempty collection of subsets of X. Consider the two collections

$$\mathcal{S}_1 = \{\cup_{\alpha \in \Gamma} B_\alpha : B_\alpha \in \mathcal{C} \text{ for } \alpha \in \Gamma, \text{ and } \Gamma \text{ is an indexing set}\},$$

i.e., $\mathcal{S}_1$ is the collection of all unions of sets in $\mathcal{C}$, and

$$\mathcal{S}_2 = \{U \subset X : \text{ for each } x \in U, \text{ there is some } B \in \mathcal{C} \text{ with } x \in B \subset U\}.$$

Determine the sets $\mathcal{S}_1 \cup \mathcal{S}_2$, $\mathcal{S}_1 \cap \mathcal{S}_2$, $\mathcal{S}_1 \backslash \mathcal{S}_2$, and $\mathcal{S}_2 \backslash \mathcal{S}_1$. Be careful: There are multiple cases to consider.

(b) (15 points) Let $\mathcal{B}$ be a nonempty collection of subsets of a set X satisfying the following: (1) If $B_j \in \mathcal{B}$ for $j = 1, 2, \ldots, k$, then

$$\cap_{j=1}^{k} B_j \in \mathcal{B},$$

and (2)

$$\cup_{B \in \mathcal{B}} B = X.$$

Identify a topology $\mathcal{T}$ for which $\mathcal{B}$ is a base.

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4. (a) (10 points) Define the term **metric space**.

   (b) (15 points) Show that given two (nonempty) disjoint closed sets A_1 and A_2 in a metric space X, there are disjoint open sets U_1 and U_2 (in the metric topology) such that $A_j \subset U_j$ for $j = 1, 2$. Hint: You may use the fact that for each $x \notin A_j$ one has

$$d(x, A_j) = \inf\{d(x, a) : a \in A_j\} > 0.$$