Math 4431, Exam 3: Ch. 4 Name and section: ____________________

1. (a) (10 points) State the definition of a **topological space**.

Armstrong's definition of an **identification space** of a topological space $(X, \mathcal{T})$ may be expressed as follows: An identification space of X is a partition $\mathcal{P}$ of X with the topology

$$\mathcal{T}_{\mathcal{P}} = \{V \subset \mathcal{P} : \{x \in X : \text{there is some } P \in V \text{ with } x \in P\} \in \mathcal{T}\}.$$

(b) (10 points) Show that $\mathcal{T}_{\mathcal{P}}$ is a topology.

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(c) (10 points) Show $p : X \to \mathcal{P}$ by $p(x) = P$ where $x \in P$ is well-defined, continuous, and surjective.

(d) (10 points) Show $\mathcal{T}_{\mathcal{P}} = \{V \subset \mathcal{P} : p^{-1}(V) \in \mathcal{T}\}$.

Solution:

(a) This definition should be known by now. A topological space X is a set with a designated collection $\mathcal{T}$ of subsets of X (called open sets) such that

(1) $\phi, X \in \mathcal{T}$,

(2) $\mathcal{T}$ is closed under arbitrary unions: If $U_\alpha \in \mathcal{T}$ for $\alpha \in \Gamma$, then

$$\cup_{\alpha \in \Gamma} U_\alpha \in \mathcal{T},$$

(3) $\mathcal{T}$ is closed under finite intersections: If $U_1, U_2, \ldots, U_k \in \mathcal{T}$, then

$$\cap_{j=1}^{k} U_j \in \mathcal{T}.$$

(b) **(1)** Taking $V = \phi$, we note $\phi \subset \mathcal{P}$, and $\{x \in X : \text{there is some } P \in \phi \text{ with } x \in P\} = \phi \in \mathcal{T}$. Thus, $\phi \in \mathcal{T}_{\mathcal{P}}$.

Similarly, taking $V = \mathcal{P}$, since $\mathcal{P}$ is a partition, $\{x \in X : \text{there is some } P \in \mathcal{P} \text{ with } x \in P\} = X \in \mathcal{T}$.

(2) Next, if $V_\alpha \in \mathcal{T}_{\mathcal{P}}$ for $\alpha \in \Gamma$, then

$$\{x \in X : \text{there is some } P \in \cup_{\alpha \in \Gamma} V_\alpha \text{ with } x \in P\}$$
$$= \cup_{\alpha \in \Gamma} \{x \in X : \text{there is some } P \in V_\alpha \text{ with } x \in P\} \in \mathcal{T}.$$

Therefore, $\cup V_\alpha \in \mathcal{T}_{\mathcal{P}}$.

(3) Similarly, if $V_j \in \mathcal{T}_{\mathcal{P}}$ for $j = 1, 2, \ldots, k$, then

$$\{x \in X : \text{there is some } P \in \cap_{j=1}^{k} V_j \text{ with } x \in P\}$$
$$= \cap_{j=1}^{k} \{x \in X : \text{there is some } P \in V_j \text{ with } x \in P\} \in \mathcal{T}.$$

Therefore, $\cap V_j \in \mathcal{T}_{\mathcal{P}}$.

As an alternative, the fact that $\mathcal{T}_{\mathcal{P}}$ is a topology is somewhat easier to verify in terms of the **generalized projection** $p : X \to \mathcal{P}$ given by $p(x) = P$ where $x \in P$. We show that p is well-defined below. But notice that

$$\{x \in X : \text{there is some } P \in V \text{ with } x \in P\} = p^{-1}(V).$$

This is also verified in detail below. But assuming this form, we wish to show

$$\mathcal{T}_{\mathcal{P}} = \{V \subset \mathcal{P} : p^{-1}(V) \in \mathcal{T}\} \quad \text{is a topology.}$$

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In fact, taking $\phi \subset \mathcal{P}$, we have $p^{-1}(\phi) = \phi \in \mathcal{T}$. Therefore, $\phi \in \mathcal{T}_{\mathcal{P}}$. Also, $p^{-1}(\mathcal{P}) = X \in \mathcal{T}$, so $\mathcal{P} \in \mathcal{T}_{\mathcal{P}}$.

If $V_\alpha \in \mathcal{T}_{\mathcal{P}}$ for $\alpha \in \Gamma$, then $p^{-1}(V_\alpha) \in \mathcal{T}$ for $\alpha \in \Gamma$. Therefore, $\cup_{\alpha\in\Gamma} p^{-1}(V_\alpha) \in \mathcal{T}$. But $\cup_{\alpha\in\Gamma} p^{-1}(V_\alpha) = p^{-1}(\cup_{\alpha\in\Gamma} V_\alpha)$, so

$$\cup_{\alpha\in\Gamma} V_\alpha \in \mathcal{T}_{\mathcal{P}}.$$

Similarly, if $V_1, \ldots, V_k \in \mathcal{T}_{\mathcal{P}}$, then $p^{-1}(V_j) \in \mathcal{T}$ for $j = 1, \ldots, k$, so $\cap_{j=1}^k p^{-1}(V_j) \in \mathcal{T}$. But $\cap_{j=1}^k p^{-1}(V_j) = p^{-1}(\cap_{j=1}^k V_j)$, so

$$\cap_{j=1}^k V_j \in \mathcal{T}_{\mathcal{P}}.$$

(c) Since $\mathcal{P}$ is a partition, we know that for each $x \in X$, there is exactly one $P \in \mathcal{P}$ such that $x \in P$. Thus, p is well-defined.

Let V be open in $\mathcal{P}$. Then $\{x \in X : \text{there is some } P \in V \text{ with } x \in P\} \in \mathcal{T}$. But

$$\begin{aligned}\{x \in X : \text{there is some } P \in V \text{ with } x \in P\} &= \{x \in X : p(x) \in V\} \\ &= p^{-1}(V).\end{aligned}$$

Thus, $p^{-1}(V)$ is open in X, and p is continuous.

Finally, each set $P \in \mathcal{P}$ is a nonempty subset of X. Therefore, there is some $x \in X \cap P$. This means $p(x) = P$, so p is surjective.

(d) Our observation above that

$$\{x \in X : \text{there is some } P \in V \text{ with } x \in P\} = p^{-1}(V). \tag{1}$$

gives that

$$\begin{aligned}\mathcal{T}_{\mathcal{P}} &= \{V \subset \mathcal{P} : \{x \in X : \text{there is some } P \in V \text{ with } x \in P\} \in \mathcal{T}\} \\ &= \{V \subset \mathcal{P} : p^{-1}(V) \in \mathcal{T}\}.\end{aligned}$$

Let us justify this observation in detail: If there is some $P \in V$ with $x \in P$, then $p(x) = P$, so $x \in p^{-1}(V)$. On the other hand, if $x \in p^{-1}(V)$, then $p(x) \in V \cap \mathcal{P}$. That is, $P = p(x)$ is a set in V with $x \in P$. This establishes (1).

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2. Armstong defines an **identification map** to be a continuous surjective function $q : X \twoheadrightarrow Y$ (from a topological space X onto a topological space Y) having the property that V is open in Y if and only if $q^{-1}(V)$ is open in X.

   Consider $q : \mathbb{R}^2 \to \mathbb{R}^3$ by

$$q(x, y) = \left(1 + \frac{\cos y}{2}\right)(\cos x, \sin x, 0) + \frac{\sin y}{2}(0, 0, 1)$$

   and the set

$$W = W(x_0, y_0, \epsilon) = \left\{\left(1 + \frac{\cos y}{2}\right)(\cos x, \sin x, 0) + \frac{\sin y}{2}(0, 0, 1) : |x - x_0|, |y - y_0| < \epsilon\right\}$$

   where $\epsilon \in (0, \pi/10]$.

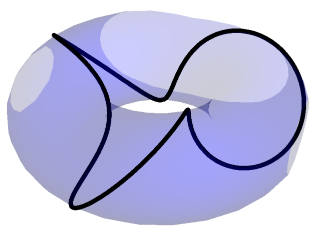
   (a) (10 points) Draw $T^2 = q(\mathbb{R}^2) \subset \mathbb{R}^3$ along with the subset $W(0, 0, \pi/10)$.

   (b) (10 points) Show that if $q(x, y) = q(x_0, y_0)$, then there are some integers k and ℓ such that $x = x_0 + 2\pi k$ and $y = y_0 + 2\pi\ell$.

(c) (10 points) Show that $q : \mathbb{R}^2 \to T^2$ is an identification map. You may assume the sets $W(x_0, y_0, \epsilon)$ are open in T^2.

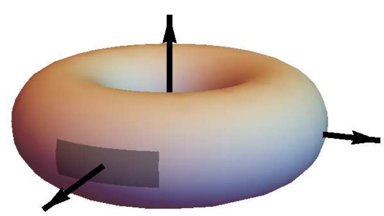
(d) (10 points) Consider $\gamma : [0,1] \to T^2$ by

$$\gamma(t) = \left(1 + \frac{\cos 6\pi t}{2}\right)(\cos 2\pi t, \sin 2\pi t, 0) + \frac{\sin 6\pi t}{2}(0,0,1).$$



If $\Gamma : [0,1] \to \mathbb{R}^2$ is continuous and satisfies $\Gamma(0) = (0,0)$ and $q \circ \Gamma(t) = \gamma(t)$ for $0 \le t \le 1$, then find $\Gamma(1)$.

Solution:



(a)

(b) If

$$\left(1 + \frac{\cos y}{2}\right)(\cos x, \sin x, 0) + \frac{\sin y}{2}(0,0,1) = \left(1 + \frac{\cos y_0}{2}\right)(\cos x_0, \sin x_0, 0) + \frac{\sin y_0}{2}(0,0,1)$$

then

$$\left(1 + \frac{\cos y}{2}\right)\cos x = \left(1 + \frac{\cos y_0}{2}\right)\cos x_0$$
$$\left(1 + \frac{\cos y}{2}\right)\sin x = \left(1 + \frac{\cos y_0}{2}\right)\sin x_0$$
$$\sin y = \sin y_0.$$

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Since $1 + \cos y/2$ is nonzero, we see from the first two equations that $\cos x = \cos x_0$ and $\sin x = \sin x_0$. This means $x = x_0 + 2\pi k$ for some $k \in \mathbb{Z}$. On the other hand, since at least one of $\cos x = \cos x_0$ or $\sin x = \sin x_0$ is nonzero, we conclude that $\cos y = \cos y_0$ and (the last equation gives) $\sin y = \sin y_0$. Therefore, $y = y_0 + 2\pi\ell$ for some $\ell \in \mathbb{Z}$.

(c) The function q is clearly continuous and onto. It remains to show that $\{V \subset T^2 : q^{-1}(V) \text{ is open in } \mathbb{R}^2\}$ is (a subset of) the subspace topology on the torus $T^2 \subset \mathbb{R}^3$.

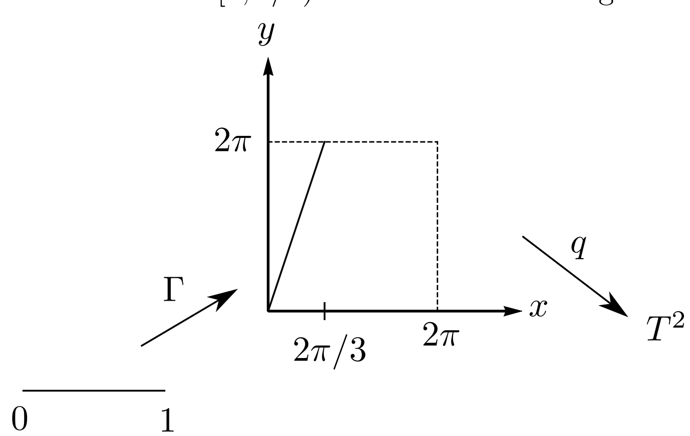
Let $V \subset T^2$ such that $q^{-1}(V)$ is open in $\mathbb{R}^2$. Then for each $Q \in V$, there is a unique $(x_0, y_0) \in [0, 2\pi) \times [0, 2\pi)$ such that $q(x_0, y_0) = Q$. Because $q^{-1}(V)$ is open in $\mathbb{R}^2$, there is some $\epsilon > 0$ (with $\epsilon \leq \pi/10$) such that $U = (x_0 - \epsilon, x_0 + \epsilon) \times (y_0 - \epsilon, y_0 + \epsilon) \subset q^{-1}(V)$. Furthermore, if we restrict q to the set U, then q is one-to-one and $q(U) = W(x_0, y_0, \epsilon)$. This implies $q(U)$ is an open set in T^2 with $Q \in q(U) = W(x_0, y_0, \epsilon) \subset V$. And this means V is open in T^2.

Therefore, $q : \mathbb{R}^2 \to T^2$ is an identification map.

(d) The identification map q is one-to-one on each square $[x_0, x_0+2\pi) \times [y_0, y_0+2\pi) \subset \mathbb{R}^2$. Thus, for $0 \leq t < 1/3$, we have $0 \leq 6\pi t < 2\pi$ and $0 \leq 2\pi t < 2\pi/3$, and

$$\Gamma(t) = q^{-1}\Big|_{[0,2\pi)\times[0,2\pi)} \circ \gamma(t) = (2\pi t, 6\pi t).$$

Note $(2\pi t, 6\pi t)$ is a point in $\mathbb{R}^2$ and not an interval. The image of Γ restricted to the interval $[0, 1/3)$ is indicated in the figure with $\Gamma(1/3) = (2\pi/3, 2\pi)$.



Repeating this reasoning for $1/3 \leq t < 2/3$ and $2/3 \leq t < 1$, we find

$$\Gamma(1) = \lim_{t \nearrow 1} \Gamma(t) = (2\pi, 6\pi).$$

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3. Billl lives in $\mathbb{R}^4$ and wears a knit cap having the topological shape of a familiar surface—familiar to residents of $\mathbb{R}^4$ that is. Billl's hat is homeomorphic to

$$\Sigma = \{(x^2 - y^2, xy, xz, yz) : (x, y, z) \in \mathbb{S}^2\}.$$

Complete the following to determine the identity of the surface Σ.

(a) (10 points) Show that if $(x_0, y_0, z_0) \in \mathbb{S}^2$, then there are exactly two points $(x, y, z) \in \mathbb{S}^2$ such that

$$x^2 - y^2 = x_0^2 - y_0^2, \tag{2}$$
$$xy = x_0 y_0, \tag{3}$$
$$xz = x_0 z_0, \tag{4}$$
$$yz = y_0 z_0. \tag{5}$$

Hint: Square the last two equations and subtract one from the other (then look at the first equation and consider several cases); there is more room on the next page).

(b) (10 points) Armstrong proves that if $q : X \to Y$ is an identification map and $\mathcal{P} = \{q^{-1}(y) : y \in Y\}$, then Y is homemorphic to $\mathcal{P}$ (considered as an identification space). Use this result to identify Σ topologically. Hint: What are the two solutions to the algebraic system in part (a)?

Solution:

(a) Following the hint, we see $(x^2 - y^2)z^2 = (x_0^2 - y_0^2)^2 z_0$. In view of the first equation, this implies $z^2 = z_0^2$ unless $x^2 - y^2 = 0 = x_0^2 - y_0^2$.

In the first case, if $z_0 \neq 0$ then $z \neq 0$ and $z = \pm z_0$. Also, by (4) and (5) $x = \pm x_0$ and $y = \pm y_0$ (with the signs correlated). Thus, there are two solutions

$$(x, y, z) = \pm(x_0, y_0, z_0). \tag{6}$$

Note that these are (always) both solutions, and since $(x_0, y_0, z_0) \neq (0, 0, 0)$, they are distinct. Therefore, we do not need to show existence, but only need to show **there are at most two solutions**.

Another general observation is that if we can show

$$x = \pm x_0 \neq 0, \quad \text{or} \quad y = \pm y_0 \neq 0, \quad \text{or} \quad z = \pm z_0 \neq 0, \tag{7}$$

then equations (3),(4), and (5) will imply all the equalities $x = \pm x_0$, $y = \pm y_0$ and $z = \pm z_0$ with correlated signs, and it follows that there are exactly two solutions (6).

Still in the first case, if $z_0 = 0$, then $z = 0$, and we are reduced to equations (2) and (3) with $x^2 + y^2 = 1 = x_0^2 + y_0^2$. Ignoring the 1 and adding equation (2) we get $x^2 = x_0^2$ or $x = \pm x_0$. Subtracting equation (2) we get $y^2 = y_0^2$ or $y = \pm y_0$. In this case, we cannot have $x = y = 0$, so we must have one of the conditions of (7). This completes the first case.

In the second case, $x_0^2 - y_0^2 = 0$ and $x_0 = \pm y_0$. By (3) we conclude $xy = \pm x_0^2$. Since $|x| = |y|$, we conclude $y = \pm x$ (with correlated signs). Substituting $y = \pm x$ into $xy = \pm x_0^2$ and canceling the correlated signs, we get $x^2 = x_0^2$. The reverse substitution gives $y^2 = y_0^2$. If both of the quantities $x^2 = x_0^2$ and $y^2 = y_0^2$ vanish, then $z = \pm 1$ and $z_0 = \pm 1$, so we are done. If one of the quantities $x^2 = x_0^2$ and $y^2 = y_0^2$ is nonzero, then we get one of the conditions of (7). Thus, in all cases, there are exactly two solutions as in (6).

(b) Take $X = \mathbb{S}^2$ and $Y = \Sigma$ with $q(x, y, z) = (x^2 - y^2, xy, xz, yz)$. This function q is clearly continuous and onto. Furthermore, the domain of q is a compact space and $\Sigma \subset \mathbb{R}^3$ is a Hausdorff space. Therefore q is an identification map, and the theorem above gives that Σ is homeomorphic to the identification space $\mathcal{P}$ on $\mathbb{S}^2$ induced by q. We only need to identify the partition $\mathcal{P}$ of $\mathbb{S}^2$. The previous part of this problem shows that for each $Q \in \Sigma$ we have

$$q^{-1}(Q) = \{(x_0, y_0, z_0), -(x_0, y_0, z_0)\} = \{(x_0, y_0, z_0), \mathrm{an}(x_0, y_0, z_0)\}$$

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where $q(x_0, y_0, z_0) = Q$ and an is the antipodal map. This means Σ is homeomorphic to the quotent space of $\mathbb{S}^2$ induced by the antipodal bijection, that is, Σ is (homemorphic to) the projective plane.