Basic Topology - M.A.Armstrong

Answers and Solutions to Problems and Exercises
Gaps (things left to the reader)
and Study Guide

1987/2010 editions

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A Note to the (Potential) Reader of M.A. Armstrong

I think this is a great book. But from a user's perspective, I'm afraid a lot of people never get through chapter one. At issue are Armstrong's seemingly casual approach and his initial and confusing defnition of topology using "neighborhoods". Doing things that way gets confusing real fast, particularly (1.3) (d) on page 13. I imagine some readers are turned away at this point, frustrated by the apparent complexity of the subject.

But happily things become much simpler and clearer starting in chapter 2. Chapter 2 basically starts over, doing things the "right" way (as opposed to the "intuitive" way of chapter 1). The author's intention was obviously to introduce the subject by somewhat informally developing some motivation and intuition. And if you're able to overcome the initial confusions relating to "neighborhoods", then he definitely does achieve a decent overview, which serves as a nice starting point for studying topology. The reader should be patient and try to get the general concepts without buckling under the weight of the details of (1.3).

This won't cost you anything, because for the most part everything starts back over in chapter 2 where the concept of a "topology" is defined anew in much simpler terms. There are a few exercises to show the two approaches are equivalent (you can read their solutions here in these notes), but they're not that critical. Just read chapter one as best you can, after reading a few chapters you can go back and review chapter 1 and it will make a lot more sense.

If you find any mistakes in these notes, please do let me know at one of these email addresses:

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Chapter 1 - Introduction

Notes

Page 8. Armstrong presents a formula for the area of Q as $\alpha_1 + \alpha_2 + \cdots + \alpha_k - (n-2)\pi$ as if it is obvious. This is *NOT* actually supposed to be obvious.

Page 14, Example 2. He says the axioms for a topology are easily checked. This is Chapter 1, Problem 19.

Page 14, Example 3. It is stated that the "inverse is *not* continuous" and we are asked "Why not?" This is Chapter 1, Problem 17.

Page 14, Example 4. We are asked to "Check that radial projection π gives a homeomorphism" in Fig. 1.8. This is Chapter 1, Problem 20.

Page 14, Example 6. We are asked to why the finite complement topology on $\mathbb{R}$ cannot be given by a metric. Suppose $\mathbb{R}$ (with finite complement topology) is a metric space with metric d. Let $x, y \in \mathbb{R}$ and let $D = d(x, y)$ be the distance between x and y. Let B_x be the set of points that are strictly less than $D/2$ units ditsance from x and let B_y be the set of points that are strictly less than $D/2$ units ditsance from y. (All distances mentioned are with respect to d, not to the typical distance in $\mathbb{R}$.) Then B_x and B_y must be non-empty and disjoint. But in the finite complement topology no two non-empty open sets can be disjoint.

Problems

Problem 1. Prove that $v(T) - e(T) = 1$ for any tree T.

Solution: Any tree can be obtained by starting with a single edge and then attaching edges one at a time so that the graph is connected at each step. We will denote a partial tree by T'. Since it is a tree, each attachment adds one edge and one vertex. Therefore each attachment after the first does not change $v(T') - e(T')$ (where T' is the partial tree as built so far). Only the first edge is different. The first edge introduces one edge and two vertices. Therefore at step one $v(T') - e(T') = 1$ and adding each additional edge does not change it further. Therefore $v(T) - e(T) = 1$ for the whole tree.

Problem 2. Even better, show that $v(\Gamma) - e(\Gamma) \leq 1$ for an graph Γ, with equality precisely when Γ is a tree.

Solution: We assume the graph is connected, otherwise it is obviously false. In the same way as before we can build the graph edge by edge. The first edge gives $v(T') - e(T') = 1$. Since we add edges so that T' is always connected, each additional edge adds exactly one edge and at most one additional vertex. Therefore $v(T') - e(T')$ is adjusted at each step by either 0 or -1. So the final sum $v(T) - e(T)$ cannot be greater than one.

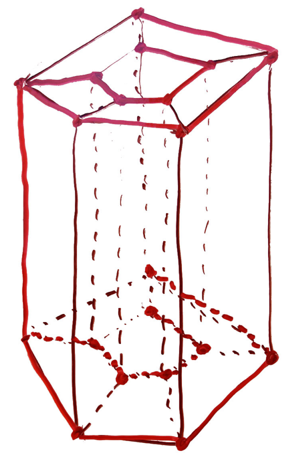
If the graph is not a tree then some edge must connect two existing vertices as we build T. Therefore at some step we will add -1 and the final sum can not be greater than zero.

Problem 3. Show that inside any graph we can always find a tree which contains all the vertices.

Solution: If a graph has a cycle, then any edge in the cycle can be removed without causing it to become disconnected. To see this suppose an edge in the cycle connects vertices A and B. Suppose C and D are any two nodes. Connect them by a path in the original graph. Now if that path passes through the edge connecting A and B, it can be diverted around the cycle to get from A to B the long way. This path is still valid in the graph with the edge connecting AB removed. Therefore there is a path connecting C and D in the modified graph. So the graph remains connected. Now as long as there are cycles, continue to remove edges until there are no more cycles. The remaining graph is connected with no cycles and therefore must be a tree.

Problem 4. Find a tree in the polyhedron of Fig. 1.3 which contains all the vertices. Construct the dual graph Γ and sohw that Γ contains loops.

Solution:



The tree that hits every vertex is shown in red.

The dual graph Γ is shown in blue with the loops being shown in green. There are two loops that connect at a point.

Problem 5. Having done Problem 4, thicken both T and Γ in the polyhedron. T is a tree, so thickening it gives a disc. What do you obtain when you thicken Γ?

Solution: Γ is basically two loops connected at a point, with some other edges connected that do not make any more loops. So thickening should procduce something homeomorphic to what is shown in Problem 11 (b) (right). What is called "Two cylinders glued together over a square patch".

Problem 6. Let P be a regular polyhedron in which each face has p edges and for which q faces meet at each vertex. Using Euler's formula prove that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}.$$

Solution: Every edge in a polyhedron has two faces. Each face has p edges so fp is twice the total number of edges. So $fp = 2e$. Therefore $f = \frac{2e}{p}$. Now each face has p vertices and each vertex is in q faces. So $v = \frac{pf}{q}$. Euler's formula says $v - e + f - 2$. Therefore

$$\frac{2e}{q} - e + \frac{2e}{p} = 2.$$

Divide through by $2e$ to get the result.

Problem 7. Deduce from Problem 6 that there are only five regular polyhedra.

Solution: Clearly $p \geq 3$. Suppose $p \geq 6$. Then

$$\frac{1}{2} + \frac{1}{e} = \frac{1}{p} + \frac{1}{q} \leq \frac{1}{6} + \frac{1}{q}.$$

Therefore

$$\frac{1}{3} < \frac{2}{6} + \frac{1}{e} \leq \frac{1}{q}.$$

Therefore $q < 3$. But q must be at least 3 to make a polyhedron. Therefore it is impossible that $p \geq 6$.

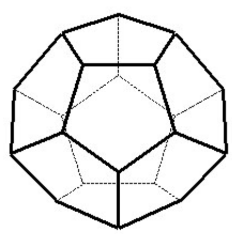
We will show only five cases satisfy $p \leq 5$. Suppose $p = 5$. Then

$$\frac{1}{5} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}.$$

Therefore

$$\frac{1}{q} = \frac{3}{10} + \frac{1}{e} \geq \frac{3}{10}.$$

Therefore $q \leq \frac{10}{3}$. Since q is an integer that must be at least 3 to make a coherent polyhedron, it must be that q is exactly 3. In this case $e = 30$. This is the dodecahedron, the faces are pentagons and each vertex touches 3 faces. There are 12 faces.



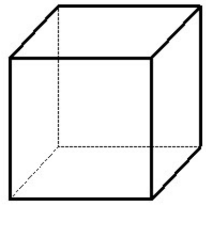
Now suppose $p = 4$. Then

$$\frac{1}{4} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}.$$

So

$$\frac{1}{q} = \frac{1}{4} + \frac{1}{e} > \frac{1}{4}.$$

Therefore $q < 4$. Therefore $q = 3$. So the only regular polyhedron with $p = 4$ is the cube.



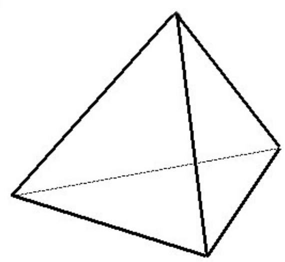
Now suppose $p = 3$. In this case

$$\frac{1}{3} + \frac{1}{q} = \frac{1}{2} + \frac{1}{e}.$$

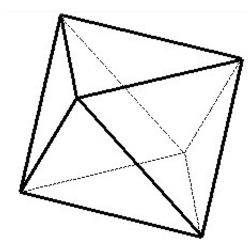
So

$$\frac{1}{q} = \frac{1}{6} + \frac{1}{e} > \frac{1}{6}.$$

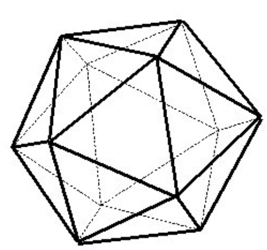
Therefore $q = 3$, 4 or 5. If $q = 3$ then $e = 6$ and $f = 4$. This is a tetrahedron, a pyramid with triangular base.



If $q = 4$ then $e = 12$ and $f = 8$. This is the octahedron, two pyramids with square bases, connected by their bases.

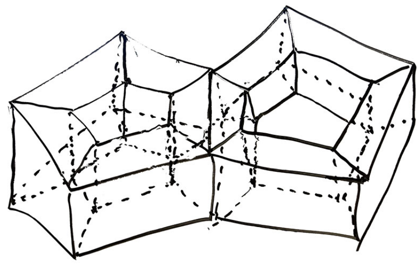


If $q = 5$ then $e = 30$ and $f = 20$. This is the regular icosahedron.



Problem 8. Check that $v - e + f = 0$ for the polyhedron shown in Fig. 1.3. Find a polyhedron which can be deformed into a prezel (see Fig. 1.23c) and calculate its Euler number.

Solution: For the polyhedron in Fig. 1.3, $v = 20$, $e = 40$, $f = 20$. Therefore $v - e + f = 0$. The following is basically a donut with two holes, i.e. a ”pretzel”:



It has 38 faces (10 on the top, 10 on the bottom, 10 inside the hole, and 8 around the outside sides). It has 76 edges (29 on the top, 29 on the bottom, 10 vertical ones inside the holes and 8 vertical ones around the outside sides), and it has 36 verticies (18 on the top, 18 on the bottom). Therefore $v - e + f = -2$.

Problem 9. Borrow a tennis ball and observe that its surface is marked out as the union of two discs which meet along their boundaries.

Solution: Ok, this is obvious.

Problem 10. Find a homeomorphism from the real line to the open interval $(0, 1)$. Show that any two open intervals are homeomorphic.

Solution: Let $y = \frac{2x-1}{x(1-x)}$. Then $y : (0, 1) \to \mathbb{R}$ is a continuous function. We will show it is a homeomorphism by finding a continuous inverse. Solving for x we get

$$x = \begin{cases} \frac{y-2-\sqrt{y^2-4}}{2y}, & y < 0 \\ 1/2, & y = 0 \\ \frac{y-2+\sqrt{y^2-4}}{2y}, & y > 0 \end{cases}$$

Clearly x is a continuous function of y at $y \neq 0$. The left and right limits of y at $y = 0$ both converge to $1/2$ by L'Hospital's rule. x and y are inverses of each other. Therefore they are homeomorphisms. Note that x restricted to $(1/2, 1)$ is a homemorphism from $(1/2, 1)$ to $(0, \infty)$. The map $x \mapsto -x$ is a homemorphism from $(0, \infty)$ to $(-\infty, 0)$.

Now let (a, b) be any interval with $-\infty < a, b < \infty$. Then $y = (b - a)x + a$ is a homeomorphism from $(0, 1)$ to (a, b). Therefore $(0, 1)$ is homemorphic to (a, b) for any finite a and b and also to $(0, \infty)$ and $(-\infty, 0)$. Then for any finite b, $x \mapsto x + b$ is a homeomorphism from $(0, \infty)$ to (b, ∞). And for finite a, $x \mapsto x - a$ is a homeomorphism from $(-\infty, 0)$ to $(-\infty, a)$.

Thus in all cases all intervals (a, b) are homemorphic.

Problem 12. 'Stereographic projection' π from the sphere minus the north pole to the plane is shown in Fig. 1.24. Work out a formula for π and check that π is a homeomorpism. Notice that π provides us with a homeomorphism from the sphere with the north and south poles removed to the plane minus the origin.

Working this out in two dimensions is a good way to get the idea. In general the following formulas give the projection and its inverse:

$$(X, Y) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

and

$$(x, y, z) = \left(\frac{2X}{1 + X^2 + Y^2}, \frac{2Y}{1 + X^2 + Y^2}, \frac{-1 + X^2 + Y^2}{1 + X^2 + Y^2}\right)$$

Problem 13. Let x and y be points on the sphere. Find a homeomorphism of the sphere with itself which takes x to y. Work out the same Problem with the sphere replaced by the plane and by the torus.

Solution:
The Sphere The special orthogonal group SO(n) are rotations of n space given by square $n \times n$ matrices M with $\det(M) = 1$ and $MM^T = I$. These are length (and angle) preserving transformations. An orthogonal matrix therefore takes the unit sphere to itself. The columns are orthogonal unit vectors. Let x be any point on the unit sphere. Extend $\{x\}$ to an orthonormal basis and make a matrix R from these vectors with x as the first column. Then this matrix takes the first standard basis vector e_1 to x. Likewise there is an orthogonal matrix S that takes e_1 to y. Then SR^{-1} is a transformation that takes x to y. Linear transformations are continuous. Things in SO(n) are invertible, so give homeomorphisms.

The Plane Let $(x, y), (z, w) \in \mathbb{R} \times \mathbb{R}$. Consider the map $(a, b) \mapsto ((x + z) - a, (y + w) - b)$. Then $(x, y) \mapsto (z, w)$. Since the map is linear, it is continuous with continous inverse. Therefore it is a homeomorphism that takes an arbitrary point in the plane to another arbitrary point in the plane.

The Torus Consider the torus as $T = S^1 \times S^1$. Let $(x, y), (z, w) \in T$. We know from *The Sphere* result above that there's a homemorphism f of S^1 that takes x to z and another homeomorphism g that takes y to w. Then $f \times g : T \to T$ is a homeomorphism that takes (x, y) to (z, w).

Problem 14. Make a Möbius strip out of a rectangle of paper and cut it along its central circle. What is the result?

Solution: The result is a strip with one full twist, so it now has two sides and the boundary is homemorphic to two separate loops.

Problem 15. Cut a Möbius strip along the circle which lies halfway between the boundary of the strip and the central circle. Do the same for the circle which lies one-third of the way in from the boundary. What are the resulting spaces?

Solution: The first exercise gives two interlocking loops, one (the wider and shorter one) is a Möbius strip and the other (narrower and twice as long) is a loop with two full twists. The second exercise seems to have done basically the same thing, it also gives two interlocking loops, one (wider and shorter) is a Möbius strip, the other (narrower and two times as long) is a loop with two full twists.

Problem 16. Now take a strip which has one full twist in it, cut along its central circle and see what happens.

Solution: I get two interlocking loops, each with one full twist.

Problem 17. Define $f : [0, 1) \to C$ by $f(x) = e^{2\pi ix}$. Prove that f is one-to-one, onto and continuous. Find a point in $[0, 1)$ and a neighborhood N of x in $[0, 1)$ such that $f(N)$ is not a neighborhood of $f(x)$ in C. Deduce that f is not a homeomorphism.

Solution: $2\pi x$ is the angle of the complex number $e^{2\pi ix}$. Since there are 2π radians in a full 360^o angle, if $x, y \in [0, 1)$ and $e^{2\pi ix} = e^{2\pi iy}$, then $x = y$. Therefore f is one-to-one. Similarly, as x ranges from 0 to 1, $2\pi x$ ranges from 0 to 2π. Therefore f is onto. Take an open ball B in the complex plane $\mathbb{C}$. Then $B \cap C$ is either the whole circle or an open ended arc of the circle. Therefore $f^{-1}(B)$ is either the whole interval, or an open subinterval, or possibly $[0, \alpha) \cup (\beta, 1)$. In all cases an open subset of $[0, 1)$. Therefore f is continuous. Now $[0, 1/2)$ is an open neighborhood of 0 in $[0, 1)$. But $f([0, 1/2))$ equals C intersected with the upper half plane minus the single point $z = -1$. This is not a neighborhood of $z = 1 \in C$ because any open ball around $z = 1$ must contain numbers in the lower half plane. Therefore f cannot be a homeomorphism.

Problem 19. Let X be a topological space and let Y be a subset of X. Check that the so-called subspace topology is indeed a topology on Y.

Solution: We must show it satisfies the four axoms of **(1.3)** (page 13). The neighborhoods of Y are $N \cap Y$ where N is a neighborhood of X. If $y \in Y$ then the neighborhoods of y are $N \cap Y$ where N is a neighborhood of y in X. Since N is a neighborhood of y in X, it satisfies (a) and therefore $y \in N$. Also $y \in Y$. Therefore $y \in N \cap Y$. So the neighborhoods of Y satisfy (a). Now suppose $N \cap Y$ and $M \cap Y$ are two neighborhoods of y in Y, where N and M are neighborhoods in X. Then

$(N \cap M) \cap Y = (N \cap Y) \cap (M \cap Y)$ is a neighborhood of y in Y. Therefore the neighborhoods of Y satisfy (b). If $N \cap Y$ is a neighborhood of y (N a neighborhood in X) and U is a subset of Y which contains $N \cap Y$ then $N \cup U$ is a subset of X that contains N so $N \cup U$ is a neighborhood of X, so $(N \cup U) \cap Y = (N \cap Y) \cup (U \cap Y) = U \cap Y = U$. Therefore U is a neighborhood of y in Y. Therefore the neighborhoods satisfy (c). Now suppose $N \cap Y$ is a neighborhood of y in Y (N a neighborhood in X). Let $(N \overset{\circ}{\cap} Y)$ be the interior of $N \cap Y$ in Y. We must show $(N \overset{\circ}{\cap} Y)$ is a neighborhood of y in Y. Now $a \in (N \overset{\circ}{\cap} Y) \Leftrightarrow N \cap Y$ is a neighborhood of a in $Y \Leftrightarrow N$ is a neighborhood of a in $X \Leftrightarrow a \in (\overset{\circ}{N})$ in X. So $(N \overset{\circ}{\cap} Y) = \overset{\circ}{N} \cap Y$. So $(N \overset{\circ}{\cap} Y)$ is a neighborhood of y in Y. Therefore the neighborhoods satisfy (d).

Problem 20. Prove that the radial projection shown in Fig. 1.8 is a homeomorphism from the surface of the tetrahedron to the sphere. (Both spaces are assumed to have the subspace topology from $\mathbb{E}^3$.

Solution: Assume the polyhedron P is cenetered at the origin in $\mathbb{E}^3$. The map $f : v \mapsto \frac{1}{\|v\|} \cdot v$ takes points on the polyhedra to the surface of the unit sphere S^2. If $v_n \to v$ in P, then v_n is never zero, and $\frac{1}{\|v_n\|} \cdot v_n \to \frac{1}{\|v\|} \cdot v$. Therefore f is continuous. Every ray starting at the origin passes exactly once through the polyhedron and the sphere. Therefore f is one-to-one and onto. If U is an open set in S^2 then let $\tilde{U}$ be the set of points p in $\mathbb{E}^3$ such that p is not $(0, 0, 0)$ and p lies on a ray that passes through the origin and whose point of intersection with S^2 is in U. Then $\tilde{U}$ is an open set in $\mathbb{E}^3$ and $\tilde{U} \cap S^2 = U$. Now $\tilde{U} \cap P = f^{-1}(U)$ and therefore $f^{-1}(U)$ is open in P. Therefore f is continuous. The exact same argument, but switching the roles of P and S^2 show that f^{-1} is continuous. Therefore, f is a homeomorphism.

Problem 21. Let C denote the unit circle in the complex plane and D the disc which it bounds. Given two points $x, y \in D - C$, find a homeomorphism from D to D which interchanges x and y and leaves all points of C fixed.

Solution: This is more or less intuitively obvious. But writing down an explicit function is not so easy. First note that for any $a \in \mathbb{C}$, the function $f(z) = \frac{z-a}{1-\bar{A}z}$ takes S^1 to itself. To see this suppose $|z| = 1$. then

$$\begin{aligned} \left|\frac{z-a}{1-\bar{a}z}\right| &= \left|\frac{z-a}{\bar{z}z-\bar{a}z}\right| \\ &= \left|\frac{z-a}{(\bar{z}-\bar{a})z}\right| \\ &= \left|\frac{z-a}{\overline{z-a}}\right| \frac{1}{|z|} \\ &= 1 \end{aligned}$$

If $|a| < 1$ then the denominator never vanishes so this is a continuous function on D. Also, $f(0) = -a$, so if $|a| < 1$ then since f takes C to itself and 0 maps to $-a \in (D - C)$, f must take all of D to itself. The inverse f^{-1} is therefore also a continuous function from D to D that takes C to C. Now suppose $x, y \in \mathbb{C}$ with $|x| < 1$ and $|y| < 1$, let

$$f_1(z) = \frac{z - x}{1 - \bar{x}z}$$

and

$$f_2(z) = \frac{z - ty_1}{1 - t\bar{y}_1 z}$$

where $y_1 = f_1(y)$ and $t = \frac{1-\sqrt{1-|y_1|^2}}{|y_1|^2}$. As shown above, both f_1 and f_2 take C to itself. Finally let

$$g(z) = ze^{i(1-|z|)\pi/(1-|x_2|)}$$

where $x_2 = f_2(0)$. Then $g(x_2) = -x_2$ and $g(-x_2) = x_2$ and $g(z) = z \;\forall\; z \in C$. Since $z \mapsto |z|$ is continuous, g is built up from sums, products and compositions of continuous functions and therefore g is continuous.

The function $f_1^{-1} f_2^{-1} g f_2 f_1$ is therefore a continuous function from D to D that switches x and y and fixes C.

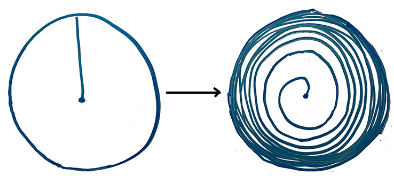
Problem 22. With C, D as above (in Problem 21), define $h : D - C \to D - C$ by

$$h(0) = 0$$

$$h(re^{i\theta}) = r \exp\left[i\left(\theta + \frac{2\pi r}{1-r}\right)\right]$$

Show that h is a homeomorphism, but that h cannot be extended to a homeomorphism from D to D. Draw a picture which shows the effect of h on a diameter of D.

Solution: The function h restricted to a circle of radius r acts by rotation of $2\pi r/(1-r)$ radians. Now $2\pi r/(1-r) \to \infty$ as $r \to 1$, so as the circle radius grows towards one, it gets rotated to greater and greater angles approaching infinity. Thus intuitively it's pretty obvious this could not be extended to the boundary.



Now, we can think of (r, θ) as polar coordinates in $\mathbb{R}^2$. And the topology on $\mathbb{C}$ is the same as that on $\mathbb{R}^2$. Thus as a function of two variables r and θ this is just a combination of continuous functions by sums, products, quotients and composition. Since the only denominator involved does not vanish for $|r| < 1$, this is a continuous function of r and θ on D which is clearly onto. Since it is a simple rotation on each circle of radius r, it is also clearly one-to-one. The inverse is evidently

$$h^{-1}(0) = 0$$

$$h^{-1}(re^{i\theta}) = r \exp\left[i\left(\theta - \frac{2\pi r}{1-r}\right)\right]$$

Now, let $r_n = \frac{n}{n+2}$ for n odd and $r_n = \frac{n-1}{n}$ for n even. Then for n odd, $\frac{r}{1-r} = \frac{n}{2}$, and for n even $\frac{r}{1-r} = n - 1$. Therefore $\exp\left[i\left(\frac{2\pi r_n}{1-r_n}\right)\right]$ equals 1 if r is even and -1 if r is odd. Now, $r_n \to 1$. So if h could be extended to all of D we must have $h(1) = \lim h(r_n)$. But $h(r_n)$ does not converge, it alternates between 1 and -1. Therefore, there is no way to extend h to C to be continuous.

Problem 23. Using the intuitive notion of connectedness, argue that a circle and a circle with a spike attached cannot be homeomorphic (Fig. 1.26.)

Solution: In the circle, if we remove any one point what remains is still connected. However in the circle with a spike attached there is one point we can remove that renders the space not-connected. Since this property of being able to remove a point and retain connectedness must be a topological property preserved by homeomorphism, the two spaces cannot be homeomorphic.

Problem 24. Let X, Y be the subspace of the plane shown in Fig. 1.27. Under the assumption that any homeomorphism from the annulus to itself must send the points of the two boundary circles among themselves, argue that X and Y cannot be homeomorphic.

Solution: The two points that connect the two spikes to the two boundary circles in X must go to the two points that connect the two spikes to the boundary circles in Y, because those are the only two points on the boundary circles that can be removed to result in a disconnected space, and because by assumption the circles go to the circles. Since the two poins lie on the same circle in Y but on different circles in X, some part of the outer circle in Y must go to the outer circle in X and the rest must go to the inner circle in X. But then some part of the outer circle in Y must go to the interior of X. I'm not sure exactly how Armstrong expects us to prove this but it basically follows from the intermediate value theorem, applied to the two coordinates thinking of these shapes as embedded in $\mathbb{R}^2$.

Problem 25. With X and Y as above, consider the following two subspaces of $\mathbb{E}^3$:

$$X \times [0,1] = \{(x,y,z) \mid (x,y) \in X, 0 \leq z \leq 1\},$$

$$Y \times [0,1] = \{(x,y,z) \mid (x,y) \in Y, 0 \leq z \leq 1\}.$$

Convince yourself that if these spaces are made of rubber then they can be deformed into one another, and hense that they are homeomorphic.

Solution: With the extra dimension, the squareness can be continuously deformed so that it is a solid torus, with two flat rectangular shapes sticking off. One has both rectangles pointing out and one has one pointing out and the other pointing in. Since the torus is round, the first space made from X can be rotated at the location where the inner rectangle is a full half turn to point the rectangle out, and as parallel slices (discs) of the torus move away from where the rectangle is attached, the rotation gradually gets less and less until it becomes zero before reaching the other rectangle. In this way the inner rectangle can be rotated to point out without affecting the other rectangle and with a gradual change in rotation angle between them guaranteeing the operation is continuous.

Problem 26: Assuming you have done Problem 14, show that identifying diametrically opposite points on one of the boundary circles of the cylinder leads to the Möbius strip.

Solution: Well, it was exactly diametrically opposite points that were connected before I cut the Möbius strip in half. Therefore, doing that operation backwards would have to restore the Möbius strip. Interestingly, yes, this shape before identifying is homeomorphic to the cylinder. Very cool.

Chapter 2 - Continuity

Section 2.1 - Open and Closed Sets

Notes

Page 31. TYPO: First line of page, should be $x \in B \subseteq N$.

Page 31. TYPO: in Problem 4(b), missing comma after the 2.

Problems 2.1

Problem 1. Verify each of the following for arbitrary subsets A, B of a space X:
(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$; (b) $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$; (c) $\overline{\overline{A}} = \overline{A}$;
(d) $(A \cup B)^\circ \supseteq \mathring{A} \cup \mathring{B}$; (e) $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$; (f) $(\mathring{A})^\circ = \mathring{A}$.
Show that equality need not hold in (b) and (d).

Solution:

(a) $\overline{A}$ and $\overline{B}$ are closed by Theorem 2.3. Thus $\overline{A} \cup \overline{B}$ is closed. Now $A \subseteq \overline{A}$ and $B \subseteq \overline{B}$. Therefore $\overline{A} \cup \overline{B}$ is a closed set containing $A \cup B$. By Theorem 2.3 $\overline{A \cup B}$ is the smallest closed set containing $A \cup B$, thus it must be that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$. Conversely, $\overline{A \cup B}$ is a closed set that contains A, so $\overline{A \cup B} \supseteq \overline{A}$. Similarly $\overline{A \cup B} \supseteq \overline{B}$. Thus $\overline{A \cup B} \supseteq \overline{A} \cup \overline{B}$. Thus $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) $\overline{A}$ is a closed set that contains $A \cap B$, so $\overline{A \cap B} \subseteq \overline{A}$. Likewise $\overline{A \cap B} \subseteq \overline{B}$. Thus $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

To see that equality does not hold, let $A = \mathbb{Q}$ and let $B = \mathbb{R} - \mathbb{Q}$. Then $A \cap B = \emptyset$, so $\overline{A \cap B} = \emptyset$. But $\overline{A} = \mathbb{R}$ and $\overline{B} = \mathbb{R}$, so $\overline{A} \cap \overline{B} = \mathbb{R}$.

(c) $\overline{\overline{A}}$ is the smallest closed set containing $\overline{A}$ by Corrolary 2.4, and $\overline{A}$ is closed by Theorem 2.3, that contains $\overline{A}$. Thus $\overline{A} = \overline{\overline{A}}$.

(d) Let $x \in \mathring{A} \cup \mathring{B}$. Assume wlog that $x \in \mathring{A}$. Then there is an open set $U \subseteq A$ s.t. $x \in A$. But then $x \in U \subseteq A \cup B$. So $x \in \mathring{A} \cup \mathring{B}$. Thus $\mathring{A} \cup \mathring{B} \subseteq (A \cup B)^\circ$.

To see that equality does not hold, let $A = \mathbb{Q}$ and let $B = \mathbb{R} - \mathbb{Q}$. Then $\mathring{A} = \emptyset$ and $\mathring{B} = \emptyset$. And $A \cup B = \mathbb{R}$, so $(A \cup B)^\circ = \mathbb{R}$. Therefore $(A \cup B)^\circ = \mathbb{R}$ but $\mathring{A} \cap \mathring{B} = \emptyset$.

(e) Let U be an open set in $A \cap B$. Then $U \subseteq A$ and $U \subseteq B$. Thus $(A \cap B)^\circ \subseteq \mathring{A} \cap \mathring{B}$. Conversely suppose $x \in \mathring{A} \cap \mathring{B}$. Then $\exists$ open sets U and V s.t. $x \in U \subseteq A$ and $x \in V \subseteq B$. Then $U \cap V$ is open and $x \in U \cap V \subseteq A \cap B$. Thus $(A \cap B)^\circ \supseteq \mathring{A} \cap \mathring{B}$. Thus $(A \cap B)^\circ = \mathring{A} \cap \mathring{B}$.

(f) Clearly $(\mathring{A})^\circ \subseteq \mathring{A}$. Let $x \in \mathring{A}$. Then $\exists$ an open set U s.t. $x \in U \subseteq A$. Now $\mathring{A}$ is a union of open sets so is open. Let $V = \mathring{A} \cap U$. Then $x \in V \subseteq \mathring{A}$. Therefore $x \in (\mathring{A})^\circ$. Thus $\mathring{A} \subseteq (\mathring{A})^\circ$. Thus $\mathring{A} = (\mathring{A})^\circ$.

Problem 2. Find a family of closed subsets of the real line whose union is not closed.

Solution: For each $n \in \mathbb{N}$ let $A_n = [0, 1 - \frac{1}{n}]$. Then $\cup_n A_n = [0, 1)$ which is not closed.

Problem 3. Specify the interior, closure, and frontier of each of the following subsets of the plane:
(a) $\{(x, y) \mid 1 < x^2 + y^2 \leq 2\}$; (b) $\mathbb{E}^2$ with both axes removed;

(c) $\mathbb{E}^2 - \{(x, \sin(1/x)) \mid x > 0\}$.

Solution:

(a) The interior is $\{(x, y) \mid 1 < x^2 + y^2 < 2\}$. The closure is $\{(x, y) \mid 1 \leq x^2 + y^2 \leq 2\}$. The fronteir is $\{(x, y) \mid x^2 + y^2 = 1\} \cup \{(x, y) \mid x^2 + y^2 = 2\}$.

(b) The set is already open so it is equal to its interior. The closure is all of $\mathbb{E}^2$. The frontier is exactly the union of the two axes.

(c) This is similar to (b). The set is open so equal to its interior. The closure is all of $\mathbb{E}^2$. The frontier is exactly the curve $\{(x, \sin(1/x)) \mid x > 0\}$.

Problem 4. Find all limit points of the following subsets of the real line:
(a) $\{(1/m) + (1/n) \mid m, n = 1, 2, \ldots\}$; (b) $\{(1/n) \sin n \mid n = 1, 2, \ldots\}$.

Solution:

(a) Zero is the limit of $\frac{1}{n} + \frac{1}{n}$ as $n \to \infty$. And $\frac{1}{n}$ is the limit of $\frac{1}{n} + \frac{1}{m}$ as $m \to \infty$. Thus $A = \{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}\}$ are all limit points. We will show these are the only limit points.

Note that A^c can be written as a union of open intervals. Thus A^c is open. Let $x \notin A$. Then there is an $\epsilon > 0$ such that the interval $[x - \epsilon, x + \epsilon] \cap A = \emptyset$ (since we can find an open interval $(x - \delta, x + \delta)$ that satisfies this, we can also find a closed interval, just let $\epsilon = \delta/2$).

Therefore, the sum $\frac{1}{n} + \frac{1}{m}$, with least one of $\frac{1}{n}$ or $\frac{1}{m}$ less than $\epsilon/2$, must be at a distance of at least $\epsilon/2$ from x. So the only such numbers in the interval $[x - \epsilon/2, x + \epsilon/2]$ must have $\frac{1}{n} \geq \epsilon/2$ and $\frac{1}{m} \geq \epsilon/2$. So $n \leq 2/\epsilon$ and $m \leq 2/\epsilon$. Hence there are only a finite number of numbers of the form $\frac{1}{n} + \frac{1}{m}$ in the interval $[x - \epsilon/2, x + \epsilon/2]$. Thus x is not a limit point.

(b) A basic calculus result says $\frac{1}{n} \to 0$. Since $-1 \leq \sin(x) \leq 1 \ \forall \ x$, $\frac{1}{n} \sin(n) \to 0$. Also if a sequence a_n converges to L then all subsequences converge to L, thus the limit L can be the only limit point of the sequence, when considered as a set.

Problem 5. If A is a dense subset of a space X, and if O is open in X, show that $O \subseteq \overline{A \cap O}$.

Solution: Suppose $x \in O$, $x \notin \overline{A \cap O}$. Since $\overline{A \cap O}$ is closed, $\exists$ open U s.t. $x \in U$ and $U \cap (\overline{A \cap O}) = \emptyset$. But then $U \cap (A \cap O) = \emptyset$. And $x \in U \cap O \Rightarrow U \cap O \neq \emptyset$. It is also open, so $A \cap (U \cap O) = \emptyset \Rightarrow A$ not dense.

Problem 6. If Y is a subspace of X, and Z a subspace of Y, prove that Z is a subspace of X.

Solution: The open sets in Y are exactly the sets $O \cap Y$ where O is open in X. The open sets in A as a subspace of Y are therefore sets of the form $A \cap (Y \cap O)$ where O is open in X. But $A \subseteq Y$, so $A \cap (Y \cap O) = A \cap O$. Therefore the open sets in A as a subspace of X are exactly the same as the open sets of A as a subspace of Y.

Problem 7. Suppose Y is a subspace of X. Show that a subset of Y is closed in Y if it is the intersection of Y with a closed set in X. If A is a subset of Y, show that we get the same answer whether we take the closure of A in Y, or intersect Y with the closure of A in X.

Solution: In the following, all complements are taken in X. Suppose $A \subseteq Y$ is closed in Y. Then $Y - A$ is open in Y so $Y - A = Y \cap A^c = Y \cap O$ for some O open in X. Now since $A \subseteq Y$, $x \in A \Leftrightarrow x \in Y$ and $x \notin Y \cap A^c \Leftrightarrow x \in Y$ and $x \notin Y \cap O \Leftrightarrow x \in Y$ and $x \notin O \Leftrightarrow x \in Y \cap O^c$. Therefore $A = Y \cap O^c$.

Now let $\overline{A}_Y$ be the closure of A in Y and let $\overline{A}_X$ be the closure of A in X. We know by the above that $\overline{A}_Y = Y \cap C$ where C is closed in X. Since C is a closed set in X contianing A, $\overline{A}_X \subseteq C$. So $\overline{A}_Y = Y \cap C \supseteq Y \cap \overline{A}_X$. Now by the first part of this problem, $Y \cap \overline{A}_X$ is a closed set in Y. It also contains A, so it must contain $\overline{A}_Y$. So $\overline{A}_Y \subseteq Y \cap \overline{A}_X$. Thus we have shown $\overline{A}_Y = Y \cap \overline{A}_X$.

Problem 8. Let Y be a subspace of X. Given $A \subseteq Y$, write $\mathring{A}_Y$ for the interior of A in Y, and $\mathring{A}_X$ for the interior of A in X. Prove that $\mathring{A}_X \subseteq \mathring{A}_Y$, and give an example to show the two may not be equal.

Solution: Let $x \in \mathring{A}_X$. Then $x \in U \subseteq A$ where U is open in X. Since $A \subseteq Y$, $x \in U \cap Y \subseteq A$, and $U \cap Y$ is open in Y, it follows that $x \in \mathring{A}_Y$. Therefore $\mathring{A}_X \subseteq \mathring{A}_Y$.

Let $X = \mathbb{R}$ and let $Y = \mathbb{Z}$ and $A = \{0\}$. Then $\mathring{A}_X = \emptyset$. But every point of Y is open in the subspace topology of Y. Therefore $\mathring{A}_Y = \{0\}$. Therefore $\mathring{A}_X \subsetneq \mathring{A}_Y$.

Problem 9: Let Y be a subspace of X. If A is open (closed) in Y, and if Y is open (closed) in X, show that A is open (closed) in X.

Solution: Suppose $A \subseteq Y \subseteq X$ and A open in Y. Then $A = Y \cap U$ where U is open in X. The intersection of two open sets is open, so if Y is open in X then A is open in X. Similarly, if $A \subseteq Y \subseteq X$ and A closed in Y, then $A = Y \cap C$ where C is closed in X (by exercise 7). The intersection of two closed sets is closed, so if Y is closed in X then A is closed in X

Problem 10: Show that the frontier of a set always contains the frontier of its interior. How does the frontier of $A \cup B$ relate to the frontiers of A and B?

Solution: Let x be in the frontier of $\mathring{A}$. Then by definition $x \in \overline{\mathring{A}} \cap \overline{\mathring{A}^{c}}$. We need to show $x \in \overline{A} \cap \bar{A}^c$. Since $\mathring{A} \subseteq A$, $x \in \overline{\mathring{A}} \Rightarrow x \in \overline{A}$. It remains to show $x \in \bar{A}^c$. Suppose $x \notin A^c$, then we just need to show x is a limit point of A^c. Let U be an open set with $x \in U$. Then $U \cap A^c \neq \emptyset$ because otherwise $x \in U \subseteq \mathring{A}$ which would imply x is not a limit point of $\mathring{A}^c$; but since $x \in \overline{\mathring{A}^c}$ it would then have to be that $x \in \mathring{A}^c$ which would imply $x \notin \mathring{A}$, but $x \in U \subseteq \mathring{A}$. Therefore $U \cap A^c \neq \emptyset$ which implies x is a limit point of A^c. Thus $x \in \bar{A}^c$.

For part two of the problem, let $\mathrm{Fr}(A)$ be the frontier of A.

Claim: $\mathrm{Fr}(A \cup B) \subseteq \mathrm{Fr}(A) \cup \mathrm{Fr}(B)$.

Let $x \in \mathrm{Fr}(A \cup B)$. Then

$$\begin{aligned}
x \in \overline{A \cup B} \cap \overline{(A \cup B)^c} &= (\overline{A} \cup \overline{B}) \cap \overline{A^c \cap B^c} && \text{(by Problem 1a and DeMorgan's Law)} \\
&\subseteq (\overline{A} \cup \overline{B}) \cap (\bar{A}^c \cap \bar{B}^c) && \text{(by Problem 1b)} \\
&= (\overline{A} \cap \bar{A}^c \cap \bar{B}^c) \cup (\overline{B} \cap \bar{A}^c \cap \bar{B}^c) && \text{(intersection distributes across union)} \\
&\Rightarrow \; x \in \overline{A} \cap \bar{A}^c \cap \bar{B}^c \text{ or } x \in \overline{B} \cap \bar{B}^c \cap \bar{A}^c \\
&\Rightarrow \; x \in \overline{A} \cap \bar{A}^c \text{ or } x \in \overline{B} \cap \bar{B}^c \\
&\Rightarrow \; x \in \mathrm{Fr}(A) \text{ or } x \in \mathrm{Fr}(B) \\
&\Rightarrow \; x \in \mathrm{Fr}(A) \cup \mathrm{Fr}(B)
\end{aligned}$$

Let $X = \mathbb{R}$ and $A = \mathbb{Q}$. Then $\overline{A} = \mathbb{R}$ and $\bar{A}^c = \mathbb{R}$. So $\mathrm{Fr}(A) = \mathbb{R}$ and $\mathrm{Fr}(A^c) = \mathbb{R}$. So $\mathrm{Fr}(A) \cup \mathrm{Fr}(A^c) = \mathbb{R}$. But $A \cup A^c = \mathbb{R}$ and the $\mathrm{Fr}(R) = \emptyset$. So $\mathrm{Fr}(A) \cup \mathrm{Fr}(B)$ contains but does not necessarily equal $\mathrm{Fr}(A) \cup \mathrm{Fr}(B)$.

Problem 11: Let X be the set of real numbers and β the family of all subsets of the form $\{x \mid a \leq x < b \text{ where } a < b\}$. Prove that β is a base for a topology on X and that in this toplogy each member of β is both open and closed. Show that this topology does not have a countable base.

Solution: The intersection of two elements of β is clearly another element of β. By induction the intersection of a finite number of elements of β is in β. Also clearly $\cup_{b\in\beta} b = \mathbb{R}$. Thus β is a base for a topology by Theorem 2.5.

Let $a \in \mathbb{R}$. Then $[a,\infty) = \cup_{n\in\mathbb{N}}[a,n)$ and $(-\infty,a) = \cup_{n\in\mathbb{N}}[-n,a)$. Thus $[a,\infty)$ and $(-\infty,a)$ are open sets in the toplogy generated by β. For $a,b \in \mathbb{R}$. Then $[a,b)$ is open by the definition of base. And if $A = (-\infty,a) \cup [b,\infty)$, then A is open because it is the union of two open sets. But $A^c = [a,b)$. Therefore $[a,b)$ is also closed.

Suppose $\{B_i\}_{i=1}^{\infty}$ is a countable base. Define $N : \mathbb{R} \to \mathbb{N}$ as follows. For each $x \in \mathbb{R}$ pick an n s.t. $x \in B_n \subseteq [x, x+1)$. We can do this because $\{B_i\}_{i=1}^{\infty}$ is a base and $[x, x+1)$ is open. Suppose $N(x) = N(y)$. If $x < y$ then $x \in B_{N(y)} \subseteq [y, y+1)$ which is impossible. Likewise $y \geq x$ is impossible. Thus $N(x) = N(y) \Rightarrow x = y$. Thus N is a one-to-one function from $\mathbb{R}$ to $\mathbb{N}$. But $\mathbb{R}$ is uncountable and $\mathbb{N}$ is countable. So this is a contradiction.

Problem 12: Show that if X has a countable base for its topology, then X contains a countable dense subset. A space whose topology has a countable base is called a *second countable* space. A space which contains a countable dense subset is said to be *separable.*

Solution: Let $\{B_i\}_{i=1}^{\infty}$ be a countable base. For each i choose $x_i \in B_i$. Then $\{x_i\}$ is dense. To see this let U be any open set. Then since $\{B_i\}$ is a base, $\exists\ i$ s.t. $B_i \subseteq U$. Since $x_i \in B_i$ it follows that $x_i \in U$. Therefore every open set intersects $\{x_i\}$. Thus $\{x_i\}$ is dense.

Section 2.2 - Continuous Functions

Notes

Page 31. Theorem 2.5. It is worth noting that not all bases satisfy the conditions of this theorem, for example the set of all open discs in $\mathbb{R}^2$.

Page 32. Theorem 2.6, converse is left to the reader. Suppose the inverse image of each open set of Y is open in X. Let $x \in X$ and N be a neighborhood of $f(x)$. Let $\mathring{N}$ be the set of points a in N for which N is neighborhood of a. Then $\mathring{N}$ is an open set containing $f(x)$. Thus $f^{-1}(\mathring{N})$ is an open set containing x. and thus is a neighborhood of x and $f^{-1}(\mathring{N}) \subseteq f^{-1}(N)$. Thus $f^{-1}(N)$ is a nieghborhood of x. Thus f is continiuous.

Page 33. Three of the implications are left to the reader.

(a) $\Rightarrow$ (b): Asasume $f : X \to Y$ is a map. Let β be a base for the topology on Y. We must show $f^{-1}(B)$ is open $\forall\ B \in \beta$. This is obvious since the definition of map is that $f^{-1}(U)$ is open $\forall\ U$ open and every element of β *is* open.

(c) $\Rightarrow$ (d): Assume $f(\overline{A}) \subseteq \overline{f(A)}\ \forall\ A \subseteq X$. We must show $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})\ \forall\ B \subseteq Y$. Let $B \subseteq Y$. Let $x \in \overline{f^{-1}(B)}$. Then $f(x) \in f(\overline{f^{-1}(B)}) \subseteq \overline{f(f^{-1}(B))} \subseteq \overline{B}$ (by the assumption). Thus $x \in f^{-1}(\overline{B})$.

(e) $\Rightarrow$ (a): Assume $f(\text{closed}) = \text{closed}$. Let O be open in Y. Then $f^{-1}(O) = (f^{-1}(O^c))^c$. O^c closed $\Rightarrow f^{-1}(O^c)$ closed $\Rightarrow (f^{-1}(O^c))^c$ closed $\Rightarrow f^{-1}(O)$ open.

Page 34. In the example it says "One readily checks that U is open in $\mathbb{E}^{n+1}$. I don't know what he means by "readily" but the argument of Chapter 1, Problem 12 can be extended to general dimension n.

Page 34. In the paragraph following the example, he says "It is intuitively obvious...". I think it's a bit overstated to call it "obvious". I would instead have said something like "As you might suspect ...".

Page 33. TYPO: Two lines above **Example**. $f^{-1}(B)$ should be $\overline{f^{-1}(B)}$.

Problems 2.2

Problem 13. If $f : \mathbb{R} \to \mathbb{R}$ is a map (i.e. a continuous function), show that the set of points which are left fixed by f is a closed subset of $\mathbb{R}$. If g is a continuous real-valued function on X show that the set $\{x \mid g(x) = 0\}$ is closed.

Solution: First we show the sum of two continuous functions is continuous. I don't see a real easy way to do this from the theorems we've proven so far. So I prove this the basic characterization of continuity using sequences. Thus we need the following:

Lemma: The following are equivalent: (a) If $a_n \to a$ then $f(a_n) \to f(a)$, and (b) $f^{-1}(U)$ is open $\forall$ open sets U.

Proof: $(a) \Rightarrow (b)$ Let U be open. If $f^{-1}(U)$ is not open then $\exists\, p \in f^{-1}(U)$ s.t. $\forall\, n\ \exists\, a_n \notin f^{-1}(U)$ s.t. $|p - a_n| < \frac{1}{n}$. Then $a_n \to p$ so $f(a_n) \to f(p)$. But $f(a_n) \notin U\ \forall n$ thus $\lim f(a_n) \notin U$ since U^c is closed. Thus $f(p) \notin U$, contradicting $p \in f^{-1}(U)$.
$(b) \Rightarrow (a)$ Suppose $a_n \to a$. Let $B_m = f^{-1}(f(a) - \frac{1}{m}, f(a) + \frac{1}{m})$. B_m is open and contains a. Thus $\exists\, N$ s.t. $a_n \in B_m\ \forall\, n \geq N$. Thus $f(a_m) \in (f(a) - \frac{1}{n}, f(a) + \frac{1}{n})\ \forall\, m > N$. Thus $|f(a_m) - f(a)| < \frac{1}{n}\ \forall\, m \geq N$. Thus $f(a_m) \to f(a)$.

Using the lemma it is easy to see that the sum of continuous functions is continuous. Let $h(x) = f(x) - x$. Then h is continuous. The set $\{0\}$ is closed and the set of points left fixed by f is exactly $h^{-1}(\{0\})$. Since h is continuous, by Theorem 2.9 (e) this set is closed.

For the second part, in the same way, since g is continuous and $\{0\}$ is closed, $\{x \mid g(x) = 0\} = g^{-1}(\{0\})$ and so is closed.

Problem 14. Prove that the function $h(x) = e^x/(1 + e^x)$ is a homeomorphism from the real line to the open interval $(0, 1)$.

Solution: Let $f(x) = \frac{e^x}{1+e^x}$. Let $y \in (0, 1)$. Let $x = \ln\left(\frac{y}{1-y}\right)$, which is defined since $0 < y < 1 \Rightarrow \frac{y}{1-y} > 0$. Then $f(x) = y$, so f is onto. Now suppose $f(x) = f(y)$. Then

$$\frac{e^x}{1 + e^x} = \frac{e^y}{1 + e^y}$$

which implies $e^x + e^{x+y} = e^y + e^{x+y}$. Therefore $e^x = e^y$. Taking the natural log we get $x = y$. Thus f is one-to-one. For $a, b \in \mathbb{R}$, $f^{-1}((a, b)) = \left(\ln \frac{a}{1-a}, \ln \frac{b}{1-b}\right)$ which is open. Since the intervals (a, b) are a basis, it follows from Theorem 2.9 (b) that f is continuous. And $f((a, b)) = (f(a), f(b))$ so f takes open sets in the base to open sets. Therefore f^{-1} is continuous. Thus f is a homeomorphism.

Problem 15. Let $f : \mathbb{E}^1 \to \mathbb{E}^1$ be a map and define its graph $\Gamma_f : \mathbb{E}^1 \to \mathbb{E}^2$ by $\Gamma_f(x) = (x, f(x))$. Show that Γ_f is continuous and that its image (taken with the topology induced from $\mathbb{E}^2$) is homeomorphic to $\mathbb{E}^1$.

Solution: f is one-to-one by the definition of a function, and it's obviously onto. We just have to show f and $f^{-1}|_{\Gamma_f(\mathbb{E}^1)}$ are continuous. The finite open intervals (a, b) form a basis of $\mathbb{E}^1$. Let $U_{a,b} = \left\{(x, y) \in \mathbb{E}^2 \mid x \in (a, b)\right\}$. Then U is open in $\mathbb{E}^2$. Sets of the form $B \cap \Gamma_f(\mathbb{E}^1)$ form a base of open sets in $\Gamma_f(\mathbb{E}^1)$, where B is an open ball in $\mathbb{E}^2$. Note that $B \cap \Gamma_f(\mathbb{E}^1) = U_{a,b} \cap \Gamma_f(\mathbb{E}^1)$ for some $U_{a,b}$. And then $\Gamma_f^{-1}(B \cap \Gamma_f(\mathbb{E}^1)) = \Gamma_f^{-1}(U_{a,b}) = (a, b)$ is open in $\mathbb{E}^1$. Thus Γ_f is continuous. Conversely, $f((a, b)) = \Gamma_f(\mathbb{E}^1) \cap U_{a,b}$. Therefore $f((a, b))$ is open in $\Gamma_f(\mathbb{E}^1)$. Thus Γ_f^{-1} is continuous. Thus Γ_f is a homeomorphism.

Problem 16. What topology must X have if every real-valued function defined on X is continuous?

Solution: X must have the discrete topology where every subset is open. To show this it suffices to show points in X are open. Let $x \in X$. Define $f : X \to \mathbb{R}$ by $f(x) = 0$ and $f(y) = 1\ \forall\, y \neq x$. Then $f^{-1}((-1/2, 1/2)) = \{x\}$. Thus if f is continuous, since $(-1/2, 1/2)$ is open, $\{x\}$ is open.

Problem 17. Let X denote the set of all real numbers with the finite-complement topology, and define $f : \mathbb{E}^1 \to X$ by $f(x) = x$. Show that f is continuous, but is not a homeomorphism.

Solution: Let U be open in X. Then $U^c = \{x_1, \ldots, x_n\}$ a finite set. Thus the set U is also open in the usual topology on $\mathbb{R}$ because points are closed. Thus f is continuous. Conversely, let $U = (0, 1)$. Then U is open in the usual topology but not in X. Thus $f(U)$ is not open in X. Thus f^{-1} is not continuous. Thus f is not a homeomorphism.

Problem 18. Suppose $X = A_1 \cup A_2 \cup \ldots$, where $A_n \subseteq \mathring{A}_{n+1}$ for each n. If $f : X \to Y$ is a function such that, for each n, $f|A_n : A_n \to Y$ is continuous with respect to the induced topology on A_n, show that f is itself continuous.

Solution: Let $x \in X$. Then $x \in A_n$ for some n and $A_n \subseteq \mathring{A}_{n+1}$. Thus $x \in \mathring{A}_{n+1}$. Thus $X = \cup\mathring{A}_n$. Now let U be open in Y. Since f is continuous on A_n, by Theorem 2.8 f is continuous on $\mathring{A}_n$. Thus $f^{-1}(U) \cap \mathring{A}_n = f|_{\mathring{A}_n}^{-1}(U)$ is open in $\mathring{A}_n$. Thus $\exists$ open sets V_n s.t. $V_n \cap \mathring{A}_n = f^{-1}(U) \cap \mathring{A}_n$. Thus

$$f^{-1}(U) = f^{-1}(U) \cap X = f^{-1}(U) \cap \cup_n(\mathring{A}_n) = \cup_n(f^{-1}(U) \cap \mathring{A}_n) = \cup_n(V_n \cap \mathring{A}_n)$$

which is a union of open sets and so is open.

Problem 19. The *characteristic function* of a subset A of a space X is the real-valued function on X which assigns the value 1 to points of A and 0 to all other points. Describe the frontier of A in terms of this function.

Solution:

Definition: Let $f : X \to Y$. Then f is continuous at x if $\forall\ U \subseteq Y$ with $f(x) \in U$, $\exists$ an open set V in X s.t. $x \in V \subseteq f{-}1(U)$.

Now let f be the characteristic function of A. We first show that f is continuous on $\mathring{A} \cup \mathring{A^c}$. Let $x \in \mathring{A}$. Let U be open with $f(x) \in U$. Since $x \in \mathring{A} \subseteq A$, $1 \in U$. So $f^{-1} = A$ or X. If $f^{-1}(U) = X$ then $f^{-1}(U)$ is open. If $f^{-1}(U) = A$ then since $x \in \mathring{A}$ $\exists$ V s.t. $x \in V \subseteq \mathring{A} \subseteq A = f^{-1}(U)$. Thus f is continuous at x. Now suppose $x \in \mathring{A^c}$. Let $U \subseteq Y$ be open s.t. $f(x) \in U$. Since $x \in \mathring{A^c} \subseteq A^c$, $0 \in U$. So $f^{-1}(U) = X$ or A^c. If $f^{-1}(U) = X$ then $f^{-1}(U)$ is open. If $f^{-1}(U) = A^c$ then since $x \in \mathring{A^c}$ $\exists$ $V \subseteq X$ open s.t. $x \in V \subseteq \mathring{A^c} \subseteq A^c = f^{-1}(U)$. So f is continuous at x.

Claim: x is in the frontier of $A \Longleftrightarrow f$ is not continuous at x.

Proof: ($\Rightarrow$) We are given that $x \in \overline{A} \cap \overline{A^c}$. Let x be in the frontier of A. First suppose $f(x) = 1$. Let $U = (\frac{1}{2}, \frac{3}{2})$. Suppose $f^{-1}(U)$ were open. Then $x \in f^{-1}(U)$ and $x \in \overline{A^c}$ implies $f^{-1}(U) \cap A^c \neq \emptyset$ which implies $0 \in U$, a contradiction. Thus $f^{-1}(U)$ cannot be open. Thus f cannot be continuous at x. Now suppose $f(x) = 0$. Let $U = (-\frac{1}{2}, \frac{1}{2})$. Suppose $f^{-1}(U)$ were open. Then $x \in f^{-1}(U)$ and $x \in \overline{A}$ implies $f^{-1}(U) \cap A \neq \emptyset \Rightarrow 1 \in U$. A contradiction, thus $f^{-1}(U)$ cannot be open. Thus in this case also f cannot be continuous at x. Since $f(x)$ equals 0 or 1, this covers all cases.
($\Leftarrow$) We assume f is not continuous at x. Let $x \in U$ open. Suppose $f(U) = 1$. Then $x \in U \subseteq A$. Thus $x \in \mathring{A}$. But f is continuous on $\mathring{A}$. Thus $0 \in f(U)$. Suppose $f(U) = 0$. Then $x \in U \subseteq A^c$. So $x \in \mathring{A^c}$. But f is continuous on $\mathring{A^c}$. Therefore, $f(U) = \{0, 1\}$. Thus $\exists$ $y \in U$ s.t. $f(y) = 0$. So $y \in A^c$ and $y \in U$. Thus $U \cap A^c \neq \emptyset$. Thus $x \in \overline{A^c}$. And $\exists$ $y \in U$ s.t. $f(y) = 1$. So $y \in A$ and $y \in U$. Thus $U \cap A \neq \emptyset$. Thus $x \in \overline{A}$. Thus x is in $x \in \overline{A^c} \cap \overline{A}$. Thus x is in the frontier of A.

Problem 20. An *open map* is one which sends open sets to open sets; a *closed map* takes closed sets to closed sets. Which of the following maps are open or closed?

(a) The exponential map $x \mapsto e^{ix}$ from the real line to the circle.
(b) The folding map $f : \mathbb{E}^2 \to \mathbb{E}^2$ given by $f(x, y) = (x, |y|)$.

(c) The map which winds the plane three times on itself given, in terms of complex numbers, by $z \mapsto z^3$.

Solution:

(a) Call the map f. Then f is an open map and not a closed map.

Open: Let $C = f(\mathbb{R})$. Since $f(A \cup B) = f(A) \cup f(B)$, it suffices to check that f is open on a base of open sets. Let B be an open interval in $\mathbb{R}$. If B has length greater than 2π then $f(B)$ is all of C. So $f(B)$ is open in C. Otherwise $f(B)$ is an open arc of the circle. Also open. Thus in all cases f maps B to an open set. Since open balls are a base f must be an open map.

Not Closed: To show f is not closed, for each $n \in \mathbb{N}$ let $E_n = [2n\pi + 1/n, (2n+1)\pi - 1/n]$. Let $E = \cup_n E_n$. Suppose x is a limit point of E. Then $(x - 1/2, x + 1/2)$ intersects at most one E_n. Thus x is a limit point of E_n. Thus $x \in E_n$ since E_n is closed. Thus E is closed. But $f(E)$ is equal to C intersected with the upper half plane $\mathrm{im}(z) > 0$. This is an open set in C, and not closed since $z = 1$ is a limit point of $f(E)$ not in $f(E)$. Thus $f(E)$ is not closed.

(b) f is a closed map and not an open map.

Not Open: To see f is not open, let D be the open unit disc. Let $H^+ = \{(x, y) \mid y \geq 0\}$ and $H^- = \{(x, y) \mid y \leq 0\}$. Then $f(D) = D \cap H^+$. Therefore $z = 0 \in f(D)$ and every open ball containing z intersects $H^{+c} \subset f(D)^c$. Thus $f(D)$ is not open.

Closed: Now, suppose E is closed in $\mathbb{E}^2$. Let $E' = E \cap H^-$ and let E'' be E' reflected about the x-axis. Then $f^{-1}(E) = (E \cap H^+) \cup (E'')$. Now E and H^+ are closed so $E \cap H^+$ is closed. And E'' is closed because H^- is closed, so E' is closed, and reflection is a homeomorphism. Thus $f^{-1}(E)$ is closed.

(c) Call the map f. Then f is an open map and a closed map.

Open: We first show it is open. Let $A_{\theta_1,\theta_2,r_1,r_2} = \{re^{i\theta} \in \mathbb{C} \mid \theta_1, \theta_2, r_1, r_2 \in [0, \infty) \text{ and } \theta_1 < \theta < \theta_2, r_1 < r < r_2\}$. Then the set $\beta = \{A_{\theta_1,\theta_2,r_1,r_2} \mid \theta_2 - \theta_1 < 2\pi \text{ and } 0 \leq r_1 < r_2\}$ form a base for the usual topology on $\mathbb{C}$. The sets are clearly open and the intersection of any two of them is another one. Also for any $z \in A_{\theta_1,\theta_2,r_1,r_2}$ there is an open disc D s.t. $z \in D \in A_{\theta_1,\theta_2,r_1,r_2}$ and for any open disc D with $z \in D$ we can find $\theta_1, \theta_2, r_1, r_2$ s.t. $z \in A_{\theta_1,\theta_2,r_1,r_2} \in D$. Now consider what happens to $A_{\theta_1,\theta_2,r_1,r_2}$ under the function f. If $\theta_2 - \theta_1 > 2\pi/3$ then $f(A_{\theta_1,\theta_2,r_1,r_2})$ is a full open annulus. Otherwise $f(A_{\theta_1,\theta_2,r_1,r_2}) = A_{3\theta_1,3\theta_2,r_1,r_2}$ which is open. Thus f takes basic open sets in β to open sets. Since $f(A \cup B) = f(A) \cup f(B)$ for any sets A and B, it suffices to check open-ness on the base β. Thus f is an open map.

Closed: We now show f is a closed map. Let E be any closed set in $\mathbb{C}$. Let Q_1 be the first quadrant $\{x + iy \mid x, y \geq 0\}$. Let Q_2, Q_3, Q_4 be the other (closed) quadrants. Then Q_1 is closed, so $E \cap Q_1$ is closed. The map f restricted to Q_1 is a homeomorphism from Q_1 to its image $f|_{Q_1} : Q_1 \to f(Q_1) = Q_1 \cup Q_2 \cup Q_3$ because its inverse $re^{i\theta} \mapsto re^{i\theta/3}$ is continuous. Thus $f(E \cap Q_1)$ is closed in Q_1. Likewise $f(E \cap Q_i)$ is closed for all $i = 1, 2, 3, 4$. Functions respect unions, thus since $E = \cup_i(E \cap Q_i)$ it follows that $f(E) = \cup_i f(E \cap Q_i)$. But each $f(E \cap Q_i)$ is closed. Thus $f(E)$ is closed.

Problem 21. Show that the *unit ball* in $\mathbb{E}^n$ (the set of points whose coordinates satisfy $x_1^2 + \cdots + x_n^2 \leq 1$) and the *unit cube* (points whose coordinates satisfy $|x_i| \leq 1$, $1 \leq i \leq n$) are homeomorphic if they are both given the subspace topology from $\mathbb{E}^n$.

Solution: Note that nowhere in the proof of Lemma 2.10 is it used that we are in two dimensions. The proof goes through basically without change to any finite dimension where we replace "disc" with "ball". Now let $f : \mathbb{E}^n - 0 \to \mathbb{E}^n - 0$ be given by $f(\mathbf{v}) = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$. Then f is continuous and the image of f is the unit sphere. Let g be f restricted to *the surface* of the unit cube. Then g is one-to-one continuous. The intersection of an open ball with the surface of the cube maps to the intersection of an open ball with the sphere. Thus g is an open map. Therefore the inverse of g must be continuous. Therefore g is a homeomorphism. By the generalization of Lemma 2.10 g may be extended from the boundaries to a homeomorphism from the whole cube to the whole ball.

Section 2.3 - A space-filling curve

Notes

Page 37 Proof of the space-filling curve theorem. Where it says ”we can find” (line four of the page) I think it would be helpful to say ”The triangle which contains $f_m(t)$ in the m-th iteration contains $f_n(t)$ $\forall$ $n \geq m$.

Problems 2.3

Problem 22. Find a Peano curve which fills out the unit square in $\mathbb{E}^2$.

Solution: The boundary of the unit square is homeomorhpic to the circle S^1 and the boundary of the unit triangle is also homeomorphic to the circle S^1. Therefore by Lemma 2.10 there is a homeomorphism from the unit disk to the unit triangle. Let T be the unit triangle. Let B be the unit cube. Let $h : T \to B$ be a homeomorphism. Let $f : [0, 1] \to T$ be a space-filling curve. Then $h \circ f$ is a continuous onto map from $[0, 1]$ to the unit cube and therefore a space-filling curve.

Problem 23. Find an onto, continuous function from $[0, 1]$ to S^2.

Solution: By Chapter 1, Problem 12, the $\mathbb{E}^2$ is homeomorpic to S^2 with the north pole removed. From Chapter 2, Problem 14 it easily follows that $\mathbb{E}^2$ is homeomorphic to $(0, 1) \times (0, 1)$. Find a homeomorphism $f : (0, 1) \times (0, 1) \to S^2 - (0, 0, 1)$. Extend f to a function $g : [0, 1] \times [0, 1] \to S^2$ by sending any point on the boundary (i.e. points with at least one coordinates equal to 0 or 1) to $(0, 0, 1)$. Then if B is an open ball in $\mathbb{E}^3$ containing $(0, 0, 1)$ then $g^{-1}(B \cap S^2)$ is either the entire square $[0, 1] \times [0, 1]$ or it can be obtained by thickening the boundary inside the square and taking its interior, together with the boundary itself. This is open in $[0, 1] \times [0, 1]$ so g is continuous. Now let $h : [0, 1] \to [0, 1] \times [0, 1]$ be a space filing curve (which exists by Problem 22). Then $g \circ h$ is a continuous onto map from $[0, 1]$ to S^2.

Problem 24. Can a space-filling curve fill out all of the plane?

Solutoin: Suppose there was such a curve $f : [0, 1] \to \mathbb{E}^2$. Let $x_n \in [0, 1]$ s.t. $f(x_n) = n$. Then x_n is a bounded sequence in $\mathbb{R}$. We know from basic calculus (The Bolzano-Weierstrass Theorem, which has an elementary proof) that x_n has a convergent subsequence $x_{n_k} \to L \in [0, 1]$. But then $f(x_{n_k})$ must converge to $f(L)$. But $f(x_{n_k}) = n_k \to \infty$. Thus no such curve can exist.

Problem 25. Can a space-filling curve fill out all of the unit cube in $\mathbb{E}^3$.

Solution: Let $I = [0, 1]$. Let $f : I \to I \times I$ be a space filling curve. Let $g : I \times I \to I \times I \times I \times I$ be the map $(x, y) \mapsto (f(x), f(y))$ with the natural identifications $(I \times I) \times (I \times I)$ with $I \times I \times I \times I$. Let p be projection onto the first three coordinates of $I \times I \times I \times I$. Then $p \circ g \circ f$ is a continuous function from I onto I^3.

Problem 26. Do you think a Peano curve can be one-one? (See Theorem (3.7).)

Solution: Well, clearly from Theorem 3.7 if there were such a curve and if I is compact (which according to Theorem 3.3 it is - since we're looking ahead), then $[0, 1]$ and $[0, 1] \times [0, 1]$ would be homeomorphic. Arguing as in the fourth paragraph of Section 1.6, page 19, that is impossible since one can remove a single point from $[0, 1]$ to result in a disconnected space, while no single point can do that in $[0, 1] \times [0, 1]$.

Section 2.4 - The Tietze extension theorem

Notes

Page 39 The end of the paragraph below Figure 2.4 it says ”We leave it to the reader to work out the general case.” There's nothing more to this than noting that inside a hypersphere (i.e. a sphere in $\mathbb{E}^n$) we can always find a hypercube (i.e. a cube in

$\mathbb{E}^n$) and conversely. And same with a hypersphere and a hyper-diamond.

Page 39 ERROR: Just above Lemma 2.13, he defines the distance from a point to a set. However, he does not handle the case where the set may be empty. This may seem trivial but it matters for the proof of theorem 2.15. The fix is to define $d(x, \emptyset) = 1$ $\forall$ x. This then works for Lemma 2.14 and Theorem 2.15.

Page 40 Lemma 2.14. Note that with the definition $d(x, \emptyset) = 1$ $\forall$ x the statement and proof go through without modification.

Page 40 Theorem 2.15. Paragraph 2 of proof. Here is where A_1 or B_1 really can be empty. Thus we need the modification to the definition and Lemma 2.14 discussed above.

Page 40 Theorem 2.15. Paragraph 2 of the proof. He says "therefore A_1 must be closed in X. This was proven in Chapter 2, Problem 9, page 31.

Page 41 TYPO: Theorem 2.15. Last line of the second to last paragraph of the proof. $|g(x)|$ should be $|g_n(x)|$.

Problems 2.4

Problem 27. Show $d(x, A) = 0$ iff x is a point of $\overline{A}$.

Solution:

($\Rightarrow$) Suppose $d(x, A) = 0$. Then $\inf_{a \in A} d(x, a) = 0$. Thus $\forall$ $\epsilon > 0$ $\exists$ $a \in A$ s.t. $d(x, a) < \epsilon$. Let U be an open set with $x \in U$. Find $\epsilon > 0$ s.t. $d(x, b) < \epsilon \Rightarrow b \in U$. Choose $a \in A$ s.t. $d(x, a) < \epsilon$. Then $a \in U \cap A$. Thus $U \cap A \neq \emptyset$ $\forall$ open sets U with $x \in U$. Thus $x \in \overline{A}$.

($\Leftarrow$) Suppose $x \in \overline{A}$. For $n \in \mathbb{N}$ let $B_n = \{y \mid d(x, y) < 1/n\}$. Then B_n is open $\forall n$ so $B_n \cap A \neq \emptyset$. Let $a_n \in B_n \cap A$. Then $d(x, a_n) < 1/n$ thus $\inf_{a \in A} d(x, a) < 1/n$ $\forall$ n. Thus $\inf_{a \in A} d(x, a) = 0$.

Problem 28. If A, B are disjoint closed subsets of a metric space, find disjoint open sets U, V such that $A \subseteq U$ and $B \subseteq V$.

Solution: Let X be the metric space. By Lemma 2.14 there is a continuous function $f : X \to [-1, 1]$ s.t. $f(A) = 1$, $f(B) = -1$. Let $O_1 = [-1, 0)$ and $O_2 = (0, 1]$. Then O_1 and O_2 are open sets in $[-1, 1]$. Thus $f^{-1}(O_i)$ is open, for $i = 1, 2$. And $f^{-1}(O_1) \cap f^{-1}(O_2) = \emptyset$. And $A \subseteq f^{-1}(O_1)$ and $B \subseteq f^{-1}(O_2)$.

Problem 29. Show one can define a distance function on an arbitrary set X by $d(x, y) = 1$ if $x \neq y$ and $d(x, x) = 0$. What toplogy does d give to X?

Solution: We first show d is a metric. It is real-valued. And clearly $d(x, y) \geq 0 \Leftrightarrow x = y$. Also clearly $d(x, y) = d(y, x)$. Finally, the only way we could have $d(x, y) + d(y, z) < d(x, z)$ is if the LHS is 0. But then $x = y = z$ and so the RHS is also zero. Thus d is a metric. Since $\{x\} = \{y \mid d(x, y) < 1/2\}$, the sets $\{x\}$ are open. Since every set is a union of its points, every set is open. Thus this metric gives the discrete topology.

Problem 30. Show that every closed subset of a metric space is the intersection of a countable number of open sets.

Solution: Let A be closed in the metric space X with distance d. Let $A_n = \{x \in X | d(x, A) < 1/n\}$. Then A_n is the union of open balls, so is open. We claim $A = \cap_n A_n$. Clearly $A \subseteq A_n$ $\forall$ n, so $A \subseteq \cap_n A_n$. Suppose $x \notin A$. Then $\inf_{a \in A} d(a, x) = \epsilon > 0$ since otherwise x would be in A by Problem 27. Let $n \in \mathbb{N}$ s.t. $\frac{1}{n} < \epsilon$. Then $x \notin A_n$. Thus $\cap_n A_n \subseteq A$. Thus $A = \cap_n A_n$.

Problem 31. If A, B are subsets of a metric space, their *distance apart* $d(A, B)$ is the infinum of the numbers $d(x, y)$ where $x \in A$ and $y \in B$. Find two disjoint closed subsets of the plane which are zero distance apart. The *diameter* of A is the supremum of the numbers $d(x, y)$ where $x, y \in A$. Check that both of the closed sets which you have just found have infinite diamter.

Solution: Let A be the x-axis. Let B be the set $\{(x, 1/x) \mid x > 0\}$. The functions $x \mapsto 0$ and $x \mapsto \frac{1}{x}$ are continuous on $(0, \infty)$ so A and B are closed by Chapter 2, Problem 15. Let $a_n = (n, 0)$ and $b_n = (n, 1/n)$. Then $a_n \in A$ and $b_n \in B$ and $d(a_n, b_n) = \frac{1}{n}$. Thus $d(A, B) < \frac{1}{n}\ \forall\ n$. Thus $d(A, B) = 0$. Both sets clearly have infinite diamter.

Problem 32. If A is a closed subset of a metric space X, show that any map $f : A \to \mathbb{E}^n$ can be extended over X.

Solution: Let p_i be the projection onto the i-th coordinate. Each p_i is continuous because the inverse of an open interval is clearly open. Let $f_i = p_i \circ f$. By Theorem 2.15, each f_i can be extended to a map $g_i : X \to \mathbb{E}$. Then $g = (g_1, \ldots, g_n)$ extends f. We must show g is continuous. The sets $I_1 \times \cdots \times I_n$, where I_i are open intervals, form a base for the topology on $\mathbb{E}^n$. And $g^{-1}(I_1 \times \cdots \times I_n) = g_1^{-1}(I_1) \cap \cdots \cap g_n^{-1}(I_n)$, which is therefore open. $g^{-1}(U)$ is open for all U in a base of open sets, therefore g is continuous by Theorem 2.9 (b).

Problem 33. Find a map from $\mathbb{E}^1 - \{0\}$ to $\mathbb{E}^1$ which cannot be extended over $\mathbb{E}^1$.

Solution: Let $f(x) = 1/x$. Then f is continuous on $\mathbb{E}^1 - \{0\}$ by Theorem 2.9 (b) because f^{-1} of an open interval is an open interval, or the union of two open intervals. Now suppose g extends f to all of $\mathbb{E}$. Let $a_n = \frac{1}{n}$. Then $a_n \to 0$. Thus $g(a_n) \to g(0)$. But $g(a_n) = f(a_n) = n \to \infty$. Thus no such g can exist.

Problem 34. Let $f : C \to C$ be the identity map of the unit circle in the plane. Extend f to a map from $\mathbb{E}^2 - \{0\}$ to C. Would you expect to be able to extend f over all of $\mathbb{E}^2$? (For a precise solution to this latter problem see Section 5.5.)

Solution: If we could extend f to all of $\mathbb{E}^2$ then we'd have a continuous map from the closed unit disc to its boundary, that is the identity on the boundary. Apparently from the results to come in Section 5.5 this is impossible, because we could rotate the circle after applying g and get a map from the disc to itself that does not fix any points. It seems obvious that any map from the disc to its boundary that fixes the boundary would have to "tear" a hole somewhere and would therefore have to separate points which are close. But glancing ahead at the proof in Section 5.5, there is apparently no basic way to prove this, we will need heavier machinery. So I'm not 100

Problem 35. Given a map $f : X \to \mathbb{E}^{n+1} - \{0\}$ find a map $g : X \to S^n$ which agrees with f on the set $f^{-1}(S^n)$.

Solution: Let $h : \mathbb{E}^{n+1} - \{0\} \to S^n$ be given by $\mathbf{v} \mapsto \frac{1}{\|\mathbf{v}\|}\mathbf{v}$. Then h is continuous and h is the identity on S^n. Let $g = h \circ f$. Then g is continuous and agrees with f on $f^{-1}(S^n)$.

Problem 36. If X is a metric space and A closed in X, show that a map $f : A \to S^n$ can always be extended over a *neigborhood of* A, in other words over a subset of X which is a neighborhood of each point of A. (Think of S^n as a subspace of $\mathbb{E}^{n+1}$ and extend f to a map of X into $\mathbb{E}^{n+1}$. now use Problem 35.)

Solution: Following the hint we think of S^n as a subspace of $\mathbb{E}^{n+1}$. Then $f = (f_1, \ldots, f_n)$. Each f_i is $p_i \circ f$ where p_i is the i-th projection. The solution to Problem 32 shows p_i is continuous. So each f_i is continuous. By Theorem 2.15 each component f_i can be extended to a function g_i on all of X s.t. g_i agrees with f_i on A. Then $g = (g_1, \ldots, g_n)$ extends f on A to a map from X to $\mathbb{E}^{n+1}$. The same argument as in Problem 32 shows g is continuous. Note that $g^{-1}(0) \cap A = \emptyset$ because f maps A into S^n. And $g^{-1}(0)$ is a closed set in X (Theorem 2.9 (e)). Thus by Problem 28 we can find *disjoint* open sets U and V in X such that $A \subseteq U$ and $g^{-1}(0) \subseteq V$. Let h be the map from Problem 35. Then $h \circ g$ is well-defined as long as $g(x) \neq 0$. Thus $h \circ g$ is well-defined on U. And $h \circ g|_U$ agrees with f on A since h is the identity on S^n.

Chapter 3 - Compactness and Connectedness

Section 3.1 - Compactness and Connectedness

Notes

Page 44. Two lines from the bottom it says“’let O be the member of $\mathcal{F}\ldots$”. More accurately it should say “let O be a member of $\mathcal{F}\ldots$”.

Page 45. Six lines from the bottom it says “The reader should make sure that he can suppy the details for these statements.” The first statement is that x_n converges to p. This is a basic calculus result, a monotonically increasing sequence converges to its supremum. Since for each n the sequence $\{a_i\}_{i=n}^{\infty}$ is entirely contained in I_n, and I_n is closed, it follows that $p \in I_n$. If the intersection $\cap_{n=1}^{\infty} I_n$ contains two points x and y, let $\delta = (x-y)/2$. Then there is an I_n whose diameter is less than δ. Thus both x and y cannot be in I_n. Thus x and y cannot both be in $\cap_{n=1}^{\infty} I_n$.

Page 46. Seven lines before the problems he says “It is an interesting exercise to prove that $\cap_{n=1}^{\infty} S_n$ is exactly one point.” I’m not sure why this is so interesting, maybe I’m missing something. But this is exactly Problem 2 (see below).

Problems 3.2

Problem 1. Find an open cover of $\mathbb{E}^1$ which does not contain a finite subcover. Do the same for $[0, 1)$ and $(0, 1)$.

Solution:

$\mathbb{E}^1$: Let $\epsilon = 1/10$. For each $n \in \mathbb{Z}$ let $I_n = (n-\epsilon, n+1+\epsilon)$. Then $\{I_n\}_{n\in\mathbb{Z}}$ is an open cover of $\mathbb{E}^1$. If we remove any I_n then $n+1/2$ is no longer covered. Thus $\{I_n\}$ cannot have a finite subcover.

$[0, 1)$: For $n \in \mathbb{N}$ let $I_n = [0, 1-1/n)$. Then I_n is open in $[0, 1)$ and $\cup_{n\in\mathbb{N}} I_n = [0, 1)$. If $\{I_{n_1}, \ldots, I_{n_k}\}$ is a finite subcover, let $m = \max_{i=1,\ldots,k} n_i$. Then $I_{n_1} \cup \cdots \cup I_{n_k} = I_m = [0, 1-1/m)$ and $[0, 1) \nsubseteq [0, 1-1/m)$. Thus $\{I_{n_1}, \ldots, I_{n_k}\}$ is not a cover. Thus there cannot be a finite subcover of $\{I_n\}_{n\in\mathbb{N}}$.

$(0, 1)$: Let $I_n = (1/n, 1-1/n)$. Then I_n is open in $(0, 1)$ $\forall$ n and $\cup_{n\in\mathbb{N}} I_n = (0, 1)$. If $\{I_{n_1}, \ldots, I_{n_k}\}$ is a finite subcover, let $m = \max_{i=1,\ldots,k} n_i$. Then $I_{n_1} \cup \cdots \cup I_{n_k} = I_m = (1/m, 1-1/m)$ and $(0, 1) \nsubseteq (1/m, 1-1/m)$. Thus $\{I_{n_1}, \ldots, I_{n_k}\}$ is not a cover. Thus there cannot be a finite subcover of $\{I_n\}_{n\in\mathbb{N}}$.

Problem 2. Let $S \supseteq S_1 \supseteq S_2 \supseteq \cdots$ be a nested sequence of squares in the plane whose diameters tend to zero as we proceed along the sequence. Prove that the intersection of all these squares consists of exactly one point.

Solution: Each S_n is a square so is of the form $I_n \times J_n$ for closed one-dimensional intervals. And $S_n \supset S_{n+1}$ means $I_n \supset I_{n+1}$ and $J_n \supset J_{n+1}$. Thus we can apply the one-dimensional argument given above to the I_n’s and J_n’s. We get a sequence of points x_n converging to p and y_n converging to q. So $(x_n, y_n) \in S_n$ therefore converges to (p, q) which must be in S_n for all n since S_n is closed, so $(p, q) \in \cap_{n=1}^{\infty} S_n$. Similarly the one-dimensional argument shows there can be only a unique x and y coordinate of anything in $\cap_{n=1}^{\infty} S_n$. Thus $\cap_{n=1}^{\infty} S_n = \{(p, q)\}$.

Problem 3. Use the Heine-Borel theorem to show that an infinite subset of a closed interval must have a limit point.

Solution: (Also see Theorem 3.8). Let I be a closed interval. Let $A \subseteq I$ be an infinite subset. Suppose A did not have any limit points. Then for each $x \in I$ there is an open set $U_x \subseteq I$ such that $x \in U_x$ and $U_x \cap A - \{x\} = \emptyset$. Since $x \in U_x$ $\forall$ x, it follows that $\cup_{x\in I} U_x \subseteq I$. Thus $\{U_x\}$ is an open cover of I. By the Heine-Borel theorem there is a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$. It must be that $A \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. But the only element of each U_x that is in A is x. Thus $U_{x_1} \cup \cdots \cup U_{x_n} = \{x_1, \ldots, x_n\}$, a finite set. Thus A must be finite. A contradiction.

Problem 4. Rephrase the definition of compactness in terms of closed sets.

Solution: The following comes from De Morgan's laws. But there is also another equivalent characterization that is less obvious. Below are both.

Claim 1: X is compact $\Leftrightarrow$ for every set of closed sets $\{C_\alpha\}$ with $\cap_\alpha C_\alpha = \emptyset$ has a finite subset $\{C_{\alpha_1}, \ldots, C_{\alpha_n}\}$ s.t. $C_{\alpha_1} \cap \cdots \cap C_{\alpha_n} = \emptyset$.

Proof of Claim 1: ($\Rightarrow$) Suppose X is compact. Let $\{C_\alpha\}$ be a collection of closed sets s.t. $\cap_\alpha C_\alpha = \emptyset$. Then $\{C_\alpha^c\}$ is a collection of open sets and $\cup_\alpha C_\alpha^c = (\cap C_\alpha)^c = \emptyset^c = X$. Thus $\{C_\alpha^c\}$ is an open cover of X. Since X is compact there is a finite subcover $C_{\alpha_1}^c, \ldots, C_{\alpha_n}^c$. Since $C_{\alpha_1}^c \cup \cdots \cup C_{\alpha_n}^c = X$, $\emptyset = X^c = (C_{\alpha_1}^c \cup \cdots \cup C_{\alpha_n}^c)^c = C_{\alpha_1} \cap \cdots \cap C_{\alpha_n}$.

($\Leftarrow$) Let $\{U_\alpha\}$ be an open cover of X. Then $\{U_\alpha^c\}$ is a collection of closed sets s.t. $\cap_\alpha U_\alpha^c = (\cup U_\alpha)^c = X^c = \emptyset$. Thus $\exists$ finite subset $\{U_{\alpha_1}^c, \ldots, U_{\alpha_n}^c\}$ s.t. $U_{\alpha_1}^c \cap \cdots \cap U_{\alpha_n}^c = \emptyset$. But then $(U_{\alpha_1}^c \cap \cdots \cap U_{\alpha_n}^c)^c = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} = X$. So $\{U_\alpha\}$ has a finite subcover.

Definition: Let $\mathcal{F}$ be a collection of sets. Then $\mathcal{F}$ has the finite intersection property (FIP) if whenever $F_1, \ldots, F_n \in \mathcal{F}$, $F_1 \cap \cdots \cap F_n \neq \emptyset$.

Claim 2: Let X be a topological space. Then X is compact $\Longleftrightarrow$ for every collection $\{C_\alpha\}_{\alpha\in A}$ of closed sets in X with the FIP, $\cap_{\alpha\in A}\{C_\alpha\} \neq \emptyset$.

Proof of Claim 2: ($\Leftarrow$) Let $\{U_\alpha\}$ be an open cover of X. Then $\mathcal{F} = \{U_\alpha^c\}$ is a collection of closed sets such that $\cap U_\alpha^c = (\cup U_\alpha)^c = X^c = \emptyset$. Thus $\mathcal{F}$ cannot have the FIP. So $\exists$ a finite set such that $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} = \emptyset$. Thus $U_{\alpha_1} \cup \cdots \cup U_{\alpha_n} = X$. Thus $\{U_\alpha\}$ has a finite subcover. Thus X is compact.

($\Rightarrow$) Let $\{C_\alpha\}$ be a collection of closed sets with the FIP. Then $\{C_\alpha^c\}$ is a collection of open sets. Since $\{C_\alpha\}$ has the FIP, no finite subset of it has empty intersection. Note that $\cap A_i = \emptyset \Leftrightarrow \cup A_i^c = X$. Thus no finite subset of $\{C_\alpha^c\}$ is an open cover. Since X is compact, it follows that $\{C_\alpha^c\}$ cannot be an open cover. In other words $\cup C_\alpha^c \neq X$. Thus $\cap C_\alpha \neq \emptyset$.

Section 3.3 - Properties of compact spaces

Problems 3.3

Problem 5. Which of the following are compact? (a) the space of ratoinal numbers; (b) S^n with a finite number of points removed; (c) the torus with an open disc removed; (d) the Klein bottle; (e) the Möbius strip with its boundary circle removed.

Solution:

(a) No, $\mathbb{Q}$ is not compact. By Theorem 3.9 a compact subset of $\mathbb{R}$ is closed and bounded. $\mathbb{Q}$ is neither.

(b) No, $S^n - \{p_1, \ldots p_n\}$ is not compact. Again, by Theorem 3.9, a compact subset of $\mathbb{R}^{n+1}$ is closed and bounded. Since S^n lives in $\mathbb{R}^{n+1}$, the theorem applies. S^n is bounded, but $S^n - \{p_1, \ldots, p_n\}$ is not closed, because one can find a sequence in S^n that converges to any of the removed points.

(c) Yes, the torus with an open disc removed is compact. The torus can be embedded in $\mathbb{R}^3$ as a bounded subset. And since we are removing an open disc, what remains is a closed subset of the torus and therefore (by Chapter 2, Problem 7, page 31) is a closed subset of $\mathbb{R}^3$. Therefore, by Theorem 3.9 it is compact.

(d) The Klein bottle is compact. It is the continuous image of a closed finite rectangle. By Theorem 3.9 a closed finite rectangle is compact. So by Theorem 3.4 the Klein bottle is compact. Alternatively, the Klein bottle can be embedded into

$\mathbb{R}^4$ as a closed and bounded set. Therefore, it is compact.

(e) The Möbius strip with its boundary circle removed is not compact. Think of the strip as a subset of $\mathbb{R}^3$. Then one can find a sequence of points in the strip that converge to a point on the boundary, which has been removed. Since compact sets must be closed in $\mathbb{R}^3$, the Möbius with boundary removed cannot be compact.

Problem 6. Show that the Hausdorff condition cannot be relaxed in theorem (3.7).

Solution: Let $X = \{a, b\}$ a set with two points. Let X_1 be X with the discrete topology (so every subset is open). Let X_2 be X with the indiscrete topology (in other words the only open sets are X and $\emptyset$. Then X_2 is not Hausdorff. The function $f : X_1 \to X_2$ given by $f(a) = a$, $f(b) = b$ is one-to-one, onto and continuous. But f^{-1} is not continuous, so f is not a homeomorphism.

Problem 7. Show that Lebesgue's lemma fails for the plane.

Solution: Construct an open cover as follows. First let U_0 be the open ball of radius one around the origin. For each point $p \neq (0, 0)$ let U_p be the open disc of radius $\frac{1}{\|p\|}$ centered at p. Now let $\delta > 0$. Find $n \in \mathbb{N}$ such that $\frac{1}{n} < \delta/3$. Then the open ball B of radius $2\delta/3$ is around the point $(n, 0)$ is not contained in any U_p and the diameter of B is less than δ. Thus for any $\delta > 0$ we can always find an open set of diameter less than delta that is not contained in any member of the open cover $\{U_p\}$.

Problem 8. (*Lindelöf's theorem*). If X has a countable base for its topology, prove that any open cover of X contains a countable subcover.

Solution: Let $\{U_n\}_{n\in\mathbb{N}}$ be a countable base for the topology and $\{V_\alpha\}_{\alpha\in A}$ an open cover of X. For each $x \in X$ there is an $n \in \mathbb{N}$ and an α such that $x \in U_n \subseteq V_\alpha$ (first find α then find n). Let $\mathcal{B} = \{U_{n_k}\}$ be the set of all of the open sets in $\{U_n\}$ that appear in these relations as x ranges over all of X. Clearly $\mathcal{B}$ is an open cover of X. And by definition, for each $U \in \mathcal{B}$ there is an α_U s.t. $U \subseteq V_{\alpha_U}$. Thus $\{V_{\alpha_U}\}_{U\in\mathcal{U}}$ is an open subcover of X. Since $\{U_n\}$ is countable, $\mathcal{B}$ is countable, thus it is a countable subcover.

Problem 9. Prove that two disjoint compact subsets of a Hausdorff space always posess disjoint neighborhoods.

Solution Let X be Hausdorff and let A and B be two disjoint compact subsets of X. We know by Theorem 3.6 that for any element $x \in X$ we can find *disjoint* open sets U and V such that $A \subseteq U$ and $x \in V$. Now for each $a \in A$ find open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. Then $\{U_a\}_{a\in A}$ is an open cover of A. So there is a finite subcover $U_{a_1}, \ldots, U_{a_n}$. Let $U = U_{a_1} \cup \cdots \cup U_{a_n}$ and $V = V_{a_1} \cap \cdots \cap V_{a_n}$. Then $U \cap V = \emptyset$ since any element of U is in U_{a_i} for some i and if it is also in V then it is in V_{a_i} but $U_{a_i} \cap V_{a_i} = \emptyset$. Since V is an intersection of a finite number of open sets it is open. And $B \subseteq V$ since $B \subseteq V_a\ \forall\ a$. Thus $A \subseteq U$ and $B \subseteq V$.

Problem 10. Let A be a compact subset of a metric space X. Show that the diameter of A is equal to $d(x, y)$ for some pair of poitns $x, y \in A$. Given $x \in X$, show that $d(x, A) = d(x, y)$ for some $y \in A$. Given a closed subset B, disjoint from A, show that $d(A, B) > 0$.

Solution:

Let D be the diameter of A. We first need to know that $D < \infty$. The proof of boundedness in Theorem 3.9 goes through without change if we replace the origin with any element of A. Now for each $n \in \mathbb{N}$ find $a_n, b_n \in A$ such that $d(a_n, b_n) > N - \epsilon/3$. By Theorem 3.8 there is a convergent subsequence $\{a_{n_i}\}$ of $\{a_n\}$. Since A is compact it is closed by Theorem 3.6 (metric spaces are Hausdorff). Thus $\lim_{i\to\infty} a_{n_i} = a \in A$. Likewise, there is a subsequence $\{b_{n_i}\}$ of $\{b_n\}$ that converges to $b \in A$. Choose n s.t. $d(a, a_n) < \epsilon/3$ and $d(b, b_n) < \epsilon/3$. Then $d(a_n, b_n) \leq d(a_n, a) + d(a, b) + d(b, b_n) \leq \epsilon/3 + d(a, b) + \epsilon/3$. Thus $d(a, b) \geq d(a_n, b_n) - 2\epsilon/3 \geq D - \epsilon/3 - 2\epsilon/3 = D - \epsilon$. Thus $d(a, b) \geq D$. And since $a, b \in A$, $d(a, b) \leq D$. Thus $d(a, b) = D$.

We next show given $x \in X$ that $d(x, A) = d(x, y)$ for some $y \in A$. Let $\epsilon > 0$. For each $n\ \exists\ a_n \in A$ s.t. $d(x, a_n) > d(x, A) - \epsilon$.

Then as in the first part of this exercise, a_n has a subsequence a_{n_i} that converges to $y \in A$. Find n s.t. $d(y, a_n) < \epsilon$. Then $d(x,y) \leq d(x,a_n) + d(a_n,y) < d(x,A) + \epsilon$. Thus $d(x,y) \leq d(x,A)$. Since $y \in A$, $d(x,y) \geq d(x,A)$. Thus $d(x,y) = d(x,A)$.

We now show given a closed subset B disjoint from (compact) A that $d(A,B) > 0$. Suppose $d(A,B) = 0$. First notice that

$$d(A,B) = \inf_{a \in A, b \in B} d(a,b) = \inf_{a \in A} \inf_{b \in B} d(a,b) = \inf_{a \in A} d(a,B).$$

Thus $\exists \{a_n\}$ in A s.t. $d(a_n,B) \to d(A,B) = 0$. Since A is compact, as in the first part $\exists$ a subsequence $\{a_{n_i}\}$ that converges to $a \in A$. Then $d(A,B) \leq d(a,B) \leq d(a,a_n) + d(a_n,B) \to d(A,B)$. Thus $d(a,B) = d(A,B) = 0$. Since $0 = d(a,B) = \inf_{b \in B} d(a,b)$, $\exists$ a sequence $\{b_n\}$ in B s.t. $d(a,b_n) \to 0$. But then $b_n \to a$. Since $b_n \in B$ $\forall$ n and B is closed, if $\lim b_n$ exists it must be in B. Thus $a \in B$. But A and B are disjoint. Thus it is impossible that $d(A,B) = 0$.

Problem 11. Find a topological space and a compact subset whose closure is not compact.

Solution: Let $X = \mathbb{R}$ with the following topology. The only open sets are $\emptyset$, X, and sets of the form $(-r,r)$ where $r \in (0,\infty)$. It's immediate to see that this is a topology. The set $\{0\}$ is a compact set, because it's finite and obviously any finite set is compact in any topological space. But the only closed set contianing $\{0\}$ is X itself. Thus, $\overline{\{0\}} = X$. And X is not compact. To see this let β be the open cover consisting of every open set except X itself. Then β is an open cover since $r \in (-2|r|, 2|r|)$ $\forall$ $r \in \mathbb{R}$. But β has no finite subcover.

Problem 12. Do the real numbers with the finite-complement topology form a compact space? Answer the same question for the *half-open interval toplogy* (see Problem 11 of Chapter 2).

Solution:

The real numbers with finite-complement topology is compact. Let $\beta = \{U_\alpha\}$ be an open cover. Then let U_{α_0} be an element of β. Then there are only a finite number n of points that are not in U_{α_0}. We only need n more elements of β to cover everything.

The real numbers with the half-open interval topology is not compact. Let β be the open cover $\{[n,n+1) \mid n \in \mathbb{Z}\}$. Then β clearly does not have a finite subcover.

Problem 13. Let $f : X \to Y$ be a closed map with the property that the inverse image of each point of Y is a compact subset of X. Show that $f^{-1}(K)$ is compact whenever K is compact in Y. Can you remove the condition that f be closed?

Solution: We use the characterization of compactness from Chapter 3, Problem 4, Claim 2. In other words X is compact if any collection of closed sets with the FIP has non-empty intersection.

Let $K \in Y$ be compact. Let $\mathcal{F}$ be a family of closed subsets of $H = f^{-1}(K)$ with the finite intersection property (FIP). We need to show that $\cap_{F \in \mathcal{F}} F \neq \emptyset$. Let $\mathcal{F}^* = \{F_1 \cap \cdots \cap F_n \mid F_1, \ldots, F_n \in \mathcal{F}\}$. Then $\cap_{F \in \mathcal{F}} F = \cap_{F \in \mathcal{F}^*} F$. And $\mathcal{F}^*$ also has the FIP. Now consider the sets $f(F)$ where $F \in \mathcal{F}^*$. Since $F \in \mathcal{F}^*$ are closed subsets of H, $\exists$ closed sets $C_F \in X$ s.t. $C_F \cap H = F$. The collection $\{C_F\}_{F \in \mathcal{F}^*}$ also has the FIP. It follows that $\{f(C_F)\}_{F \in \mathcal{F}^*}$ has the FIP (since it's always true that $f(A \cap B) \subseteq f(A) \cap f(B)$). Notice that for any finite subset of $\{C_F\}_{F \in \mathcal{F}^*}$ we have $H \cap (C_{F_1} \cap \cdots \cap C_{F_n}) \neq \emptyset$. It follows that $\{f(C_F) \cap K\}_{F \in \mathcal{F}^*}$ has the FIP. Since C_F is closed and f is a closed map, $f(C_F)$ is closed in Y. Since K is compact and $\{f(C_F) \cap K\}_{F \in \mathcal{F}^*}$ is a collection of closed sets in K, it must be that $\cap_{F \in \mathcal{F}^*} f(C_F) \cap K \neq \emptyset$. Next we will show that $f(C_F) \cap K = f(F)$. To see this let $x \in f(C_F) \cap K$. Then $x = f(c)$ for $c \in C_F$. And $f(c) = x \in K$, so $c \in f^{-1}(K) = H$. Thus $c \in C_F \cap H$. Thus $c \in F$. Thus $x = f(c) \in f(F)$. So we have shown $f(C_F) \cap K \subseteq f(F)$. Now let $y \in f(F)$. Then $\exists$ $x \in F$ s.t. $f(x) = y$. Since $F \subseteq f^{-1}(K)$, $y = f(x) \in K$. Also $x \in F \subseteq C_F$. So $y = f(x) \in f(C_F)$. Thus $y \in f(C_F) \cap K$. Thus we can conclude that $f(C_F) \cap K = f(F)$. Thus we have shown that $\{f(F)\}_{F \in \mathcal{F}^*}$ is a collection of closed sets in K that have the FIP. Thus $\cap_{F \in \mathcal{F}^*} f(F) \neq \emptyset$. Let $y \in \cap_{F \in \mathcal{F}^*} f(F)$. Then $C = f^{-1}(y)$ is compact. Now consider $\mathcal{F}^*_C = \{F \cap C : F \in \mathcal{F}^*\}$. Let $F_1, \ldots, F_n \in \mathcal{F}^*$. Then by the way $\mathcal{F}^*$ was defined, $F_1 \cap \cdots \cap F_n = F \in \mathcal{F}^*$. Thus $(F_1 \cap C) \cap \cdots \cap (F_n \cap C) = (F_1 \cap \cdots \cap F_n) \cap C = F \cap C$. Now $y \in f(F) \Rightarrow \exists$ $x \in F$ s.t. $f(x) = y$ $\Rightarrow x \in C$. Thus $F \cap C \neq \emptyset$. Thus $(F_1 \cap C) \cap \cdots \cap (F_n \cap C) \neq \emptyset$. Thus $\{F \cap C \mid F \in \mathcal{F}^*$ has the finite intersection property. Since C is compact, $\cap_{F \in \mathcal{F}^*} F \cap C \neq \emptyset$. Thus $\cap_{F \in \mathcal{F}^*} F \neq \emptyset$. Thus $\cap_{F \in \mathcal{F}} F \neq \emptyset$. Thus H is compact.

Consider the following example to see that the condition that f be closed cannot be removed. Let X be $\mathbb{E}^1$ with the usual topology. And let Y be $\mathbb{E}^1$ with the following topology. The open sets are $\emptyset$, Y and $(-r, r)$ where $r \in \mathbb{N}$. It is pretty obvious that this satisfies the requirements to be a topology. Let f be the identity map. Then f is continuous because $(-r, r)$ is open in the usual topology. For any $p \in Y$, $f^{-1}(p) = p$ is compact in X. Thus f satisfies all properties except being a closed map, which it clearly is not. Now $(-1, 1)$ is compact in Y, in fact any bounded subset of Y is compact. But $f^{-1}((-1, 1)) = (-1, 1)$ is not compact in X.

Problem 14. If $f : X \to Y$ is a one-one map, and if $f : X \to f(X)$ is a homeomorphism when we give $f(X)$ the induced topology from Y, we call f an *embedding* of X in Y. Show that a one-one map from a compact space to a Hausdorff space must be an embedding.

Solution: This follows immediately from the fact that a subspace of a Hausdorff space is Hasusdorff, and Theorem 3.7.

Problem 15. A space is *locally compact* if each of its points has a compact neighborhood. Show that the following are all locally compact: any compact space; $\mathbb{E}^n$; any discrete space; any closed subset of a locally compact space. Show that the space of ratoinals is not locally compact. Check that local compactness is preserved by a homeomorphism.

Solution:

A compact space is locally compact. Take the neighborhood of any point to be the whole space.

A discrete space is locally compact. Take the neighborhood of any point to be the point itself. Any finite set is clearly compact in any space.

Let X be a locally compact space. Let A be closed subset of a X. Let $p \in A$. Then there is a compact neighborhood B of X with $p \in B \in X$. Then $B \cap A$ is a neighborhood of p in A. Let β be an open cover of $B \cap A$. Then $\beta \cup \{A^c\}$ is an open cover of B. Thus there exists a finite subcover β'. Then $\beta' - \{A^c\}$ is a finite subcover of $B \cap A$. Thus $B \cap A$ is a compact neighborhood of p in A. Thus A is locally compact.

Let Suppose $\mathbb{Q}$ were locally compact. Suppose $A \subseteq \mathbb{Q}$ is a neighborhood. Then $\exists$ I an open interval in $\mathbb{R}$ s.t. $I \cap \mathbb{Q} \subseteq A$. Let $x \in I$ be any irrational. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists$ a sequence of rationals in I which converge to x. Therefore, there's a sequence of elements of $A \cap I$ converging to $x \notin A$. Therefore A is not closed. By theorem 3.5 A cannot be compact.

Suppose $f : X \to Y$ is a homeomorphism and X is locally compact. Let $p \in Y$. Then $\exists$ a compact neighborhood A of $f^{-1}(p)$ in X. We will show $f(A)$ is a compact neighborhood of p in Y. Let β be an open cover of $f(A)$. Then $\beta' = \{f^{-1}(B) \mid B \in \beta\}$ is an open cover of A in X. Thus β' has a finite subcover β''. But then $\beta''' = \{f(B) \mid B \in \beta''\}$ is a finite subset of β that is an open cover of A in Y. Thus A is compact in Y.

Problem 16. Suppose X is a locally compact and Hausdorff. Given $x \in X$ and a neighborhood U of x, find a compact neighborhood of x which is contained in U.

Solution: We know $\exists$ a compact neighborhood K of x. Since K and U are neighborhoods of x, $\exists$ a set V open in X s.t. $x \in V \subseteq K \cap U$. By Theorem 3.6 K is closed in X, thus $V^c \cap K$ is closed in X. By Chapter 2, Problem 7, $V^c \cap K$ is closed in K. By Theorem 3.5 $V^c \cap K$ is a compact subset of K. By the remark following the proof of Theorem 3.4, it follows that if $V^c \cap K$ is compact subset of X. By Theorem 3.6 there are disjoint sets W_1 and W_2 both open in X such that $x \in W_1$ and $V^c \cap K \subseteq W_2$. Let $W = W_1 \cap V$. Then W is an open set s.t. $W \subseteq W_2^c$ and $W \subseteq V$. So $W \subseteq W_2^c \cap V \subseteq W_2^c \cap K$. Since $V^c \cap K \subseteq W_2$, we have $W_2^c \subseteq V \cup K^c$. Thus $W_2^c \cap K \subseteq (V \cup K^c) \cap K = V \cap K \subseteq V$. Thus $W \subseteq W_2^c \cap K \subseteq V$. Since $W_2^c \cap K$ is closed, it follows that $\bar{W} \subseteq V$. Thus $x \in W \subseteq \bar{W} \subseteq V \subseteq U$. Thus $\bar{W}$ is a closed neighborhood of x in K with $x \in \bar{W} \subseteq U$. By Thoeorem 3.5 $\bar{W}$ is compact.

Problem 17. Let X be a locally compact Hausdorff space which is not compact. Form a new space by adding one extra point, usually denoted by ∞, to X and taking the open sets of $X \cup \{\infty\}$ to be those of X together with sets of the form $\{X - K\} \cup \{\infty\}$, where K is a compact subset of x. Check the axioms for a topology, and show that $X \cup \{\infty\}$ is a compact Hausdorff space which contains X as a dense subset. The space $X \cup \{\infty\}$ is called the *one-point compactification* of X.

Solution: Let β be the topology on X. Define $\beta' = \beta \cup \{U \cup \{\infty\} \mid U \subseteq X \text{ and } U^c \text{ is compact}\}$. Clearly $\emptyset \in \beta'$. And since $\emptyset$ is compact, $X \cup \{\infty\} \in \beta'$. It remains to show β' is closed w.r.t. arbitrary unions and finite intersection

Let $\{U_\alpha\}_{\alpha \in A} \subseteq \beta'$ be an arbitrary subset of β'. We must show $\cup_{\alpha \in A} U_\alpha \in \beta'$. Write $A = A_1 \cup A_2$ where $\alpha \in A_1 \Rightarrow \infty \notin U_\alpha$ and $\alpha \in A_2 \Rightarrow \infty \in U_\alpha$. Then $\{U_\alpha\}_{\alpha\in A} = \{U_\alpha\}_{\alpha\in A_1} \cup \{U_\alpha\}_{\alpha \in A_2}$. If $A_2 = \emptyset$ then $\cup_{\alpha\in A} U_\alpha = \cup_{\alpha \in A_1} U_\alpha \in \beta \subseteq \beta'$. If $A_2 \neq \emptyset$ then we must show that $(\cup_{\alpha\in A} U_\alpha)^c$ is compact. For $\alpha \in A_2$, U_α^c is compact. Since X is Hausdorff U_α^c is closed (Theorem 3.6). Thus $\cap_{\alpha \in A_2} U_\alpha^c$ is closed. Fix any $\alpha' \in A_2$. Then $\cap_{\alpha\in A_2} U_\alpha^c$ is a closed subset of $U_{\alpha'}^c$ and so is compact (Theorem 3.5). Thus $(\cup_{\alpha \in A_2} U_\alpha)^c$ is compact. Thus $(\cup_{\alpha\in A} U_\alpha)^c = ((\cup_{\alpha\in A_1} U_\alpha) \cup (\cup_{\alpha \in A_2} U_\alpha))^c = (\cup_{\alpha\in A_1} U_\alpha)^c \cap (\cup_{\alpha\in A_2} U_\alpha)^c = (\cap_{\alpha \in A_1} U_\alpha^c) \cap (\cap_{\alpha\in A_2} U_\alpha^c)$, which is a closed subset of the compact set $\cap_{\alpha\in A_2} U_\alpha^c$ and therefore is compact.

Now let $\{U_\alpha\}_{\alpha\in A} \subseteq \beta'$ be a finite subset of β'. As above decompose A into $A_1 \cup A_2$. If $A_1 = \emptyset$ then $\infty \in (\cap_{\alpha\in A} U_\alpha)$. And $(\cap_{\alpha\in A} U_\alpha)^c = (\cap_{\alpha\in A_2} U_\alpha)^c = \cup_{\alpha\in A_2} U_\alpha^c$ is a finite union of compact sets and therefore compact. If $A_1 \neq \emptyset$ then $\infty \notin (\cap_{\alpha \in A} U_\alpha)$. And $(\cap_{\alpha\in A} U_\alpha)^c = ((\cap_{\alpha\in A_1} U_\alpha) \cap (\cap_{\alpha\in A_2} U_\alpha))^c = (\cap_{\alpha\in A_1} U_\alpha)^c \cup (\cap_{\alpha\in A_2} U_\alpha)^c = (\cup_{\alpha\in A_1} U_\alpha^c) \cup (\cup_{\alpha\in A_2} U_\alpha^c)$. These are all finite unoins, thus this last expressoin is the union of a closed and a compact and is therefore closed. Thus $(\cap_{\alpha\in A} U_\alpha)^c$ is closed. Thus $(\cap_{\alpha\in A} U_\alpha)$ is open.

We next show $X \cup \{\infty\}$ is compact. Let $\{U_\alpha\}$ be an open cover of X. Then $\exists\ \alpha_0$ such that $\infty \in U_{\alpha_0}$. Since $U_{\alpha_0}^c$ is compact, and $\{U_\alpha\}$ is an open cover of $U_{\alpha_0}^c$, there are $U_{\alpha_1}, \dots, U_{\alpha_n}$ s.t. $U_{\alpha_0}^c \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$. But then $U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_n}$ is an open cover of $X \cup \{\infty\}$ and so we have found a finite subcover.

We now show $X \cup \{\infty\}$ is Hausdorff. Let $x, y \in X \cup \{\infty\}$. If $x, y \in X$ then since X itself is Hausdorff we know there are open sets that separate them. So suppose without loss of generality that $y = \infty$. Since X is locally compact there is an open set $U \subseteq X$ and a compact set $K \subseteq X$ with $x \in U \subseteq K$. Then $\infty \in K^c$ is an open set in $X \cup \{\infty\}$ and $K^c \cap U = \emptyset$.

It remains to show X is dense in $X \cup \{\infty\}$. The only way this is not true is if $\{\infty\}$ is an open set. But $\{\infty\}^c = X$ which was assumed to be *not* compact.

Problem 18. Prove that $\mathbb{E}^n \cup \{\infty\}$ is homeomorphic to S^n. (Think first of the case $n = 2$. Stereographic projection gives a homeomorphism between $\mathbb{E}^2$ and S^2 minus the north pole, points 'out towards infinity' in the plane becoming points near to the north pole on the sphere. Think of replacing the north pole in S^2 as adding a point at ∞ to $\mathbb{E}^2$.

Solution: We know from the example on page 34 that there is a homeomorphism $g : \mathbb{E}^n \to S^n - \{p\}$ where we think of S^n as a subset of $\mathbb{E}^{n+1}$ and $p = (0, \dots, 0, 1)$. Define $h : \mathbb{E}^n \cup \{\infty\} \to S^n$ by $\infty \mapsto p$ and $x \mapsto g(x)$ for $x \neq \infty$. Then g is clearly one-to-one and onto. Let U be any open set $U \subseteq S^n$. If $p \notin U$ then $h^{-1}(U) = g^{-1}(U)$ is open. If $p \in U$ then $h^{-1}(U) = g^{-1}(U - \{p\}) \cup \{\infty\}$. Since $S^n \subseteq \mathbb{E}^{n+1}$ and U contains p, there is a basic open set of the form $p \in I_1 \times \cdots \times I_{n+1} \cap S^n \subseteq U$ where each I_i is an open interval in $\mathbb{E}^1$. Thus $\exists\ \epsilon > 0$ such that $(x_1, \dots, x_{n+1}) \in U \subseteq S^n$ implies $1 - \epsilon > x_{n+1}$. Now by Chapter 1 Problem 12, for $1 \leq i \leq n$ the X_i coordinate of the stereographic projection of $(x_1, \dots, x_{n+1})$ is $x_i/(1 - x_{n+1})$. Thus $|x_i/(1 - x_{n+1})| < |x_i|/(1 - (1 - \epsilon)) = |x_i|/\epsilon < 1/\epsilon$. This implies $[g^{-1}(U - \{p\})]^c$ is bounded in $\mathbb{E}^n$. Now $U - \{p\}$ is open in $S^n - \{p\}$. Thus $g^{-1}(U - \{p\})$ is open thus $(g^{-1}(U - \{p\}))^c$ is closed. Thus we have shown that $[g^{-1}(U - \{p\})]^c$ is closed and bounded. A closed and bounded subset of $\mathbb{E}^n$ is compact by Theorem 3.1 (Can we use this? I don't see any other way to do it withoug this). Thus $[g^{-1}(U - \{p\})]^c$ is compact. Thus $g^{-1}(U)$ is open in $\mathbb{E}^n \cup \{\infty\}$. Thus $g^{-1}(U)$ is open for all open sets $U \subseteq S^n$. Thus g is one-to-one, onto and continuous. X is hausdorff and from Problems 15 and 17 we know $\mathbb{E}^n \cup \{\infty\}$ is compact. Thus by Theorem 3.7 g must be a homeomorphism.

Problem 19. Let X and Y be locally compact Hausdorff spaces and let $f : X \to Y$ be an onto map. Show f extends to a map from $X \cup \{\infty\}$ onto $Y \cup \{\infty\}$ iff $f^{-1}(K)$ is compact for each compact subset K of Y. Deduce that if X and Y are homeomorphic spaces then so are their one-point compactifications. Find two spaces which are not homeomorphic but which have homeomorphic one-piont compactifications.

Solution: Let $f^* : X \cup \{\infty\} \to Y \cup \{\infty\}$ be given by $x \mapsto f(x)$ for $x \in X$ and $\infty \mapsto \infty$. We want to show that f^* is continuous. Let $U \subseteq Y \cup \{\infty\}$ be open. If $\infty \notin U$ then $(f^*)^{-1}(U) = f^{-1}(U)$ is open. If $\infty \in U$ then $[(f^*)^{-1}(U)]^c = X - f^{-1}(U - \{\infty\}) = f^{-1}(Y - (U - \{\infty\}))$. Now $Y - (U - \{\infty\})$ is compact by the construction of $Y \cup \{\infty\}$ and therefore by assumption $f^{-1}(Y - (U - \{\infty\}))$ is compact. Thus $[(f^*)^{-1}(U)]^c$ is compact. Thus $(f^*)^{-1}(U)$ is open in $X \cup \{\infty\}$. Thus f^* is continuous.

Now we need to find two non-homeomorphic spaces whose one-point compactifications are homeomorphic. Let $X_1 = \{(x, y) \in \mathbb{E}^2 \mid x^2 + y^2 = 1 \text{ and } y \neq 1\}$ (the unit circle with north pole removed) and let $X_2 = \{(x, y) \in \mathbb{E}^2 \mid x = 0 \text{ and } -1 \leq y < 1\}$. Let $X = X_1 \cup X_2$. Now let $Y_1 = \{(x, y) \in \mathbb{E}^2 \mid x^2 + y^2 = 1\}$ and let $Y_2 = \{(x, y) \in \mathbb{E}^2 \mid x = 0 \text{ and either } -1 \leq y < 0 \text{ or } 0 < y \leq 1 \text{ and } y \neq 0\}$. Let $Y = Y_1 \cup Y_2$. Then X and Y are basically the same subsets of $\mathbb{E}^2$ except for two points, the points $(0, 1)$ and $(0, 0)$. In X if we remove $(0, -1)$ the space breaks into a disjoint union of three separate open subsets. But there is no point in Y that we can remove to achieve this - removing any point in Y breaks it into two pieces, never three. Thus X and Y canont be homeomorphic. It remains to show their one point compactifications *are* homeomorphic. Let $Z_2 = \{(x, y) \mid x = 0, -1 \leq y \leq 1\}$. Let $Z = Y_1 \cup Z_2$. We will show both $X \cup \{\infty\}$ and $Y \cup \{\infty\}$ are homeorphic to Z. Define $f : X \cup \{\infty\} \to Z \cup \{\infty\}$ by $f(p) = p$ if $p \neq \infty$. And $\infty \mapsto (0, 1)$. Then f is one-to-one and onto. Z is Hausdorff and by Problem 17 $X \cup \{\infty\}$ is a compact compact Hausdorff space, so if we can show f is continuous, then by Theorem 3.7 it will follow that f is a homeomorphism. Let U be an open set in Z. If $(0, 1) \notin U$ then $f^{-1}(U) = U$ which is open in X. Thus by the definition of the topology on $X \cup \{\infty\}$, it follows that $f^{-1}(U)$ is open in $X \cup \{\infty\}$. Now suppose $(0, 1) \in U$. Then $f^{-1}(U) = (U - \{(0, 1)\}) \cup \infty$. Thus $[f^{-1}(U)]^c = X - U = Z - U$ which is closed and therefore compact in Z (Theorem 3.5) and therefore compact in X. Thus again by definition of the topology on $X \cup \{\infty\}$, it follows that $f^{-1}(U)$ is open. The proof that Y and Z are homeomorphic is essentially the same. Thus X and Y are homeomorphic.

Section 3.4 - Product Spaces

Problems 3.4

Problem 20. If $X \times Y$ has the product topology, and if $A \subseteq X$, $B \subseteq Y$, show that $\overline{A \times B} = \overline{A} \times \overline{B}$, $(A \times B)^\circ = \mathring{A} \times \mathring{B}$, and $\text{Fr}(A \times B) = [\text{Fr}(A) \times \overline{B}] \cup [\overline{A} \times \text{Fr}(B)]$ where Fr( ) denotes the frontier.

Solution:

1) $\overline{A \times B} = \overline{A} \times \overline{B}$: Suppose $(x, y) \in \overline{A} \times \overline{B}$. Then $x \in \overline{A}$ and $y \in \overline{B}$. Let U be an open set containing (x, y). Then by the definition of the product topology U contains $I \times J$ for open sets I and J. Since $x \in \overline{A}$, $I \cap A \neq \emptyset$ and likewise $J \cap B \neq \emptyset$. Let $a \in I \cap A$ and $b \in J \cap B$. Then $(a, b) \in A \times B$ and $(a, b) \in I \times J \subseteq U$. Thus $U \cap A \times B \neq \emptyset$. Thus $(x, y) \in \overline{A \times B}$. Thus $\overline{A \times B} \supseteq \overline{A} \times \overline{B}$. Now suppose $(x, y) \in \overline{A \times B}$. Let I be any open set containing x and J any open set containing y. Then $I \times J$ is an open set in $X \times Y$ that contains (x, y). Thus $\exists\ (a, b) \in A \times B \cap I \times J$. But then $a \in I \cap A \neq \emptyset$ and $b \in J \cap B \neq \emptyset$. Since I and J are arbitrary open sets it follows that $x \in \overline{A}$ and $y \in \overline{B}$. Thus $(x, y) \in \overline{A} \times \overline{B}$. Thus $\overline{A \times B} \subseteq \overline{A} \times \overline{B}$. It follows that $\overline{A \times B} = \overline{A} \times \overline{B}$.

2) $(A \times B)^\circ = \mathring{A} \times \mathring{B}$: $(x, y) \in \mathring{A} \times \mathring{B} \Longleftrightarrow x \in \mathring{A}$ and $y \in \mathring{B} \Longleftrightarrow \exists$ open sets $U \subseteq X$ and $V \subseteq Y$ such that $x \in U \subseteq A$ and $y \in V \subseteq B \Longleftrightarrow (x, y) \in U \times V \subseteq X \times Y$ for open sets $U \subseteq X$ and $V \subseteq Y \Longleftrightarrow (x, y) \subseteq W \subseteq A \times B$ for some open set $W \subseteq X \times Y$ $\Longleftrightarrow (x, y) \in (A \times B)^\circ$.

3) $\text{Fr}(A \times B) = [\text{Fr}(A) \times \overline{B}] \cup [\overline{A} \times \text{Fr}(B)]$: We first note the following (nearly obvious) fact: $(U \cap V) \times W = (U \times W) \cap (V \times W)$. We will use that fact in the following.

$$
\begin{aligned}
&\overline{A \times B} \cap \overline{(A \times B)^c} \\
&= \overline{A \times B} \cap \overline{A^c \times B \cup A \times B^c \cup A^c \times B^c} \\
&= \overline{A \times B} \cap \left(\overline{A^c \times B} \cup \overline{A \times B^c} \cup \overline{A^c \times B^c}\right) \qquad \text{(by Chapter 2, Problem 1)} \\
&= (\overline{A} \times \overline{B}) \cap \left(\overline{A^c} \times \overline{B} \cup \overline{A} \times \overline{B^c} \cup \overline{A^c} \times \overline{B^c}\right) \qquad \text{(by part one of this problem)}
\end{aligned}
$$

$$= [(\overline{A} \times \overline{B}) \cap (\overline{A^c} \times \overline{B})] \cup [(\overline{A} \times \overline{B}) \cap (\overline{A} \times \overline{B^c})] \cup [(\overline{A} \times \overline{B}) \cap (\overline{A^c} \times \overline{B^c})]$$

$$= [(\overline{A} \cap \overline{A^c}) \times \overline{B}] \cup [\overline{A} \times (\overline{B} \cap \overline{B^c})] \cup [(\overline{A} \times \overline{B}) \cap (\overline{A^c} \times \overline{B^c})] \qquad \text{(by the fact stated at the beginning of this proof)}$$

$$= [\mathrm{Fr}(A) \times \overline{B}] \cup [\overline{A} \times \mathrm{Fr}(B)] \cup [\mathrm{Fr}(A) \times \mathrm{Fr}(B)].$$

But $\mathrm{Fr}(A) \times \mathrm{Fr}(B) \subseteq \mathrm{Fr}(A) \times \overline{B}$ (as well as $\overline{A} \times \mathrm{Fr}(B)$), so this last expression is equal to

$$[\mathrm{Fr}(A) \times \overline{B}] \cup [\overline{A} \times \mathrm{Fr}(B)].$$

Problem 21. If A and B are compact, and if W is a neighborhood of $A \times B$ in $X \times Y$, find a neighborhood U of A in X and a neighborhood V of B in Y such that $U \times V \subseteq W$.

Solution: We have $A \times B \subseteq W \subseteq X \times Y$, where A and B are compact. For each $(x, y) \in A \times B$, we know there is a basic open set $U_x \times V_y$ where $x \in U_x$ is open in A and $y \in V_y$ is open in Y, and $(x, y) \in U_x \times V_y \subseteq W$. By Theorem 3.15 $A \times B$ is compact. Thus $\exists$ a finite subset $U_{x_1} \times V_{y_1}, \ldots, U_{x_n} \times V_{y_n}$ of $\{U_x \times V_y \mid x \in A, y \in B\}$ such that $A \times B \subseteq (U_{x_1} \times V_{y_1}) \cup \cdots \cup (U_{x_n} \times V_{y_n})$. Now for each $x \in A$, let $E_x = \cap_{x \in U_{x_i}} U_{x_i}$ and for $y \in B$ let $F_y = \cap_{y \in V_{x_i}} V_{x_i}$. In words E_x is the intersection of all of the U_{x_i} that contain x, and F_y is the intersection of all of the V_{y_i} that contain y. Since there are only a finite number of U_{x_i} and V_{y_i}, $E_x \times F_y$ is open for all $(x, y) \in A \times B$ and the set $\{E_x \times F_y \mid x \in A, y \in B\}$ is a finite set. Let $U = \cup E_x$ and $V = \cup F_y$. Let $(a, b) \in A \times B$. Then $(a, b) \in E_a \times F_b$ so $A \times B \subseteq \cup_{(x,y) \in A \times B} E_x \times F_y = (\cup E_x) \times (\cup F_y) = U \times V$. Now for each $(x, y) \in A \times B$, $\exists\, U_{x_i}, V_{y_j}$ such that $(x, y) \in U_{x_i} \times V_{y_j} \subseteq W$. Thus $E_x \times F_y \subseteq U_{x_i} \times V_{y_j} \subseteq W$. Thus $E_x \times F_y \subseteq W$ for all $(x, y) \in A \times B$. Thus $\cup_{(x,y) \in A \times B} E_x \times F_y \subseteq W$. Since $\cup_{(x,y) \in A \times B} E_x \times F_y = (\cup E_x) \times (\cup F_y) = U \times V$, we have shown $U \times V \subseteq W$. Thus U and V have the required properties $A \times B \subseteq U \times V \subseteq W$.

Problem 22. Prove that the product of two second-countable spaces is second-countable and that the product of two separable spaces is separable.

Fact: First not that if A and B are countable then $A \times B$ is countable. Indeed this follows from an argument very similar to the one for showing $\mathbb{Q}$ is countable.

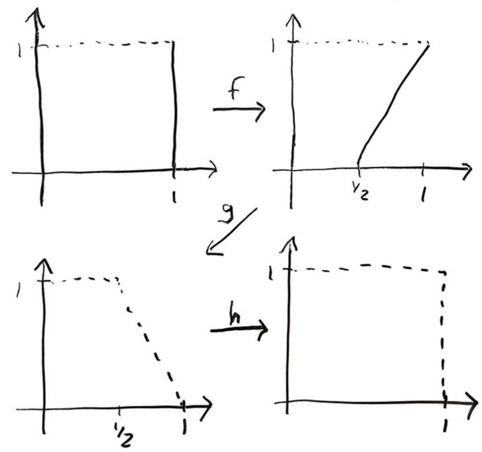
By Chapter 2, Problem 12, page 32, a space is second-countable means it has a countable base. Let α be a countable base for X and β a countable base for Y. Let $\gamma = \{A \times B \mid A \in \alpha, B \in \beta\}$. We claim γ is a base for $X \times Y$. Let $(x, y) \in X \times Y$. Let W be open in $X \times Y$ with $(x, y) \in W$. By the definition of the product topology, $\exists$ open sets $U \subseteq X$ and $V \subseteq Y$ such that $(x, y) \in U \times V \subseteq W$. Since α is a base for X there is an open set $A \in \alpha$ such that $x \in A \subseteq U$ and since β is a base for Y there is an open set $B \in \beta$ such that $y \in B \subseteq V$. Now $(x, y) \in A \times B \subseteq U \times V \subseteq W$. Thus γ is a base for the topology on $X \times Y$. Since α and β are countable, γ is countable by the *Fact* stated above.

By Chapter 2, Problem 12, page 32, a space is separable if it has a countable dense subset. Suppose X and Y are separable. Let $A \subseteq X$ and $B \subseteq Y$ be countable dense subsets. We claim that $A \times B$ is a countable dense subset of $X \times Y$. It is countable by the *Fact* above. Let $(x, y) \in X \times Y$. Let W be an open set contianing (x, y). We must show $W \cap A \times B \neq \emptyset$. We know $\exists$ open sets $U \subseteq X$ and $V \subseteq Y$ such that $(x, y) \in U \times V \subseteq W$. Since A is dense in X there is an $a \in A \cap U$. Likewise there is a $b \in B \cap V$. But then $(a, b) \in U \times V \subseteq W$. Thus $W \cap A \times B \neq \emptyset$.

Problem 23. Prove that $[0, 1) \times [0, 1)$ is homeomorphic to $[0, 1] \times [0, 1)$.

Soluton: We will need the following lemma:
Lemma: Let X, Y be closed in $X \cup Y$. If $f : X \cup Y \to B$ is continuous when restricted to X and when restricted to Y, then f is continuous.
Proof: Let $U \subseteq B$ be open. Then $f^{-1}(U) \cap X$ and $f^{-1}(U) \cap Y$ are closed since the intersection of closed sets is closed and f restricted to X and Y is contiuous. Therefore their union $f^{-1}(U)$ is closed in $X \cup Y$. Thus f is continuous.

We will construct three homeomorphisms f, g and h as shown in the following diagram. The composition $h \circ g \circ f$ will then be the desired homeomorphism from $X = [0, 1) \times [0, 1)$ to $Y = [0, 1] \times [0, 1)$



Let Y_1 be the shape shown in the top right graph in the figure. The function $f : X \to Y_1$ is defined as follows: $f(x, y) = ((\frac{1}{2}y + \frac{1}{2})x, y)$. Then $f(x, 1) = (x, 1)$ so f fixes the top of the square. And $f(0, y) = (0, y)$ so f also fixes the y-axis. $f(x, 0) = (\frac{1}{2}x, 0)$ so f contracts the x axis by a factor of $1/2$. And on any horizontal line between $y = 0$ and $y = 1$, f is a contraction of $\frac{1}{2}y + \frac{1}{2}$ which implies f is one-to-one and onto, $f(X) = Y_1$. Since f is given by polynomials it is continuous. The inverse of f is given by $(x, y) \mapsto (x/(\frac{1}{2}y + \frac{1}{2}), y)$ which is continuous for $y > 0$. Thus f is a homeomorphism from X to Y_1.

Let Y_2 be the shape shown in the bottom left graph in the figure. Define $g : Y_1 \to Y_2$ by $(x, y) \mapsto (x, y)$ if $x \leq 1/2$ and $(x, y) \mapsto (x, y - (2x - 1))$ if $x \geq 1/2$. Note that g fixes the rectangle $[0, 1/2] \times [0, 1/2)$. And g takes the right side ($y = 2x - 1$) to the x-axis. And g takes the line segment $y = 1$, $1/2 \leq x \leq 1$ to the line $y = 2 - 2x$. Thus clearly $g : Y_1 \to Y_2$ is one-to-one and onto. We have defined g by breaking up the domain into two parts. Since g is given by polynomials, it is continuous on each part of the domain. Since g agrees on the line $x = 1/2$, by the lemma proved at the beginning of this problem, g is continuous on all of Y_1. The inverse of g is given by the rule $(x, y) \mapsto (x, y)$ if $x \leq 1/2$ and $(x, y) \mapsto (x, y + (2x - 1))$, which is given by polynomials in the components and is therefore continuous.

Let $h : Y_2 \to Y$ be given by $(x, y) \mapsto ((y + 1)x, y)$. Then h is very similar to f and the proof that it is a homeomorphism from Y_2 to Y is nearly identical.

It follows that $h \circ g \circ f$ is a homeomorphism from X to Y.

Problem 24. Let $x_0 \in X$ and $y_0 \in Y$. Prove that the functions $f : X \to X \times Y$, $g : Y \to X \times Y$ defined by $f(x) = (x, y_0)$, $g(y) = (x_0, y)$ are embeddings (as defined in Problem 14).

Solution: By Theorem 3.13, f is an continuous $\Leftrightarrow$ $p_1 \circ f$ and $p_2 \circ f$ are continuous. Thus since $p_1 \circ f(x) = x$ (the identity function) and $p_2 \circ f(x) = y_0$ (a constant function), it follows that f is continuous. Now f is clearly one-to-one and onto its image $f(X) \subseteq X \times Y$. Note that $p_1|_{f(X)}$ is the invese of f on $f(X)$. Since p_1 is continuous, it follows that $p_1|_{f(X)}$ is continuous on $f(X)$. Thus f is a homeomorphism. The proof that g is a homeomorphism is basically identical.

Problem 25. Show that the diagonal map $\Delta : X \to X \times X$ defined by $\Delta(x) = (x, x)$ is indeed a map, and check that X is Hausdorff iff $\Delta(X)$ is closed in $X \times X$.

Solution:

We first show Δ is a map. Let $U \subseteq X \times X$ be open. Then $\Delta^{-1}(U) = p_1(U) \cap p_2(U)$. Since $p_1(U)$ and $p_2(U)$ are open, $\Delta^{-1}(U)$ is open

Now suppose X is Hausdorff. Let $(x_1, x_2) \notin \Delta(X)$. Then $x_1 \neq x_2$. So $\exists\, U$ and V such that $x_1 \in U$, $x_2 \in V$ and $U \cap V = \emptyset$. Let $W = U \times V$. Then $(x_1, x_2) \in W$. Suppose $p \in \Delta(X) \cap (U \times V)$. Since $p \in \Delta(X)$, $p = (x, x)$ for some x. But then $(x, x) \in U \times V \Rightarrow x \in U$ and $x \in V$ which implies $U \cap V \neq \emptyset$. Thus there can be no p in $\Delta(X) \cap (U \times V)$. Thus $\Delta(X) \cap (U \times V) = \emptyset$. Thus $\forall$ $p \notin \Delta(X)$ $\exists$ an open set W s.t. $p \in W \subseteq \Delta(X)^c$. Thus $\Delta(X)$ is closed.

Conversely, suppose $\Delta(X)$ is closed in $X \times X$. Let $x_1, x_2 \in X$ with $x_1 \neq x_2$. Since $x_1 \neq x_2$, $(x_1, x_2) \notin \Delta(X)$. Since $\Delta(X)$ is closed, $\exists$ a basic open set $U \times V$ such that $U \times V \subseteq \Delta(X)^c$. This implies $(U \times V) \cap \Delta(X) = \emptyset$. Suppose $U \cap V \neq \emptyset$. Then $\exists\, x \in U \cap V$. But that implies $(x, x) \in U \times V$, which contradicts the assumption that $(U \times V) \cap \Delta(X) = \emptyset$. Thus $U \cap V = \emptyset$. Thus X is Hausdorff.

Problem 26. We know that the projections $p_1 : X \times Y \to X$, $p_2 : X \times Y \to Y$ are open maps. Are they always closed?

Solution: The projections are not always closed maps. We will prove this with a counter-example. But first we prove the following lemma we will use to prove the counter-example works:

Lemma: If X is a metric space then $C \subseteq X$ is closed $\Longleftrightarrow$ whenever $\{c_n\}$ is a sequence in C with $c_n \to L$, then $L \in C$.
Proof: ($\Rightarrow$): Suppose $c_n \to L$ and $c_n \in C$ $\forall$ n. If $L \notin C$ then we can put an open ball B of radius $\epsilon > 0$ centered at L s.t. $B \cap C = \emptyset$. Since $c_n \in C$ for all n then $c_n \notin B$ for all $n >> 0$. Thus c_n cannot converge to L, a contradiction.
($\Leftarrow$): Suppose C was not closed. Then $\exists\, x \in C^c$ s.t. $\forall$ open sets U with $x \in U$, we have $U \cap C \neq \emptyset$. For each n let U_n be the open ball centered at x of radius $1/n$. Then for each n there is a $c_n \in C$ s.t. $c_n \in U_n \cap C$. But then $c_n \to x$ and by assumption $x \in C$, contradicting the choice of $x \in C^c$.

Now, consider the subset $C = \{(x, y) \mid y = 1/x\} \subseteq \mathbb{E}^2$. We will prove this is closed using the lemma. Suppose $\{a_n\}$ is a sequence in C s.t. $a_n \to L$. Write $a_n = (x_n, y_n)$ and $L = (x, y)$. Then $(x_n, y_n) \to (x, y) \Rightarrow x_n \to x$ and $y_n \to y$. Thus $x_n \to x$ and $1/x_n \to y$. Since $1/x_n \to y \in \mathbb{E}^1$, x_n cannot converge to zero. Thus $1/x_n \to 1/x$. Thus $y = 1/x$. Thus $(x, y) \in C$. Thus C must be closed. But $p_1(C) = (-\infty, 0) \cup (0, \infty)$ which is not closed (by the lemma again) because $1/n \to 0$.

Problem 27. Given a countable number of spaces $X_1, X_2, \ldots$, a typical point of the product ΠX_i will be written $x = (x_1, x_2, \ldots)$. The *product topology* on ΠX_i is the smallest topology for which all of the projections $p_i : \Pi X_i \to X_i$, $p_i(x) = x_i$, are continuous. Construct a base for this topology from the open sets of the spaces $X_1, X_2, \ldots$.

Solution: Define the base β for a topology on ΠX_i by $\beta = \{\emptyset\} \cup \{U_1 \times U_2 \times \cdots \mid U_i \in X_i$ is open, and $U_i = X_i$ for all but finitely many $i\}$. Clearly $\emptyset \in \beta$ and $\Pi X_i \in \beta$. It is also obvious that β is closed under finite intersection since $\cap_\alpha (A_\alpha \times B) = (\cap_\alpha A_\alpha) \times B$. Now let $U \subseteq X_i$. Then $p_i^{-1}(U) = X_1 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times X_{i+2} \times \cdots \in \beta$. Thus p_i is continuous. Now suppose B is the smallest topology for which p_i is continuous. Then for $U \subseteq X_i$, $p_i^{-1}(U) = X_1 \times \cdots \times X_{i-1} \times U \times X_{i+1} \times X_{i+2} \times \cdots$ must be in B. Since any element of β is a finite intersection of such sets, it follows that $\beta \subseteq B$. Since B is the smallest topology for which p_i is continuous for all i, and the topology generated by β is contained in B, it must be that the topology generated by β equals B.

Problem 28. If each X_i is a metric space, the topology on X_i being induced by a metric d_i, prove that

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$$

defines a metric on ΠX_i which induces the product topology.

Solution: Since $0 \leq \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} \leq 1$, $0 \leq \frac{1}{2^i} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} \leq \frac{1}{2^i}$. And $\sum \frac{1}{2^i} = 1$ so the series converges and is finite for all x, y. No term of the series is negative, so $d(x, y) \geq 0$. And $d_i(x_i, y_i) = d_i(y_i, x_i)$ for all i, so $d(x, y) = d(y, x)$. Since $d_i(x_i, x_i) = 0$ for all i, it follows that $d(x, x) = 0$. Now suppose $d(x, y) = 0$. Then every term of the series must be zero. Thus $d_i(x_i, y_i) = 0$ for all i. Thus $x_i = y_i$ for all i. Thus $x = y$. Thus we have shown $x = y \Leftrightarrow d(x) = d(y)$. It remains to show d satisfies the triangle inequality. To show this we will use the following lemmas:
Lemma 1: Let $a, b, c, d \in \mathbb{R}$ with $a, b \geq 0$ and $c, d > 0$. Then $\frac{b}{d} \geq \frac{a}{c} \Leftrightarrow \frac{a+b}{c+d} \geq \frac{a}{c}$.
Proof: $\frac{b}{d} \geq \frac{a}{c} \Leftrightarrow bc \geq ad \Leftrightarrow ac + bc \geq ac + ad \Leftrightarrow \frac{a+b}{c+d} \geq \frac{a}{c}$.

Lemma 2: Let $f(x) = \frac{x}{1+x}$. Then $0 \le a \le x + y \Rightarrow f(a) \le f(x) + f(y)$.
Proof: If $x = 0$ or $y = 0$ then the statement is equivalent to saying f is an increasing function, which it is. So suppose $xy > 0$. Then

$$f(x) + f(y) = \frac{x}{1+x} + \frac{y}{1+y} = \frac{x(1+y) + y(1+x)}{(1+x)(1+y)} = \frac{x+y+2xy}{1+x+y+xy}.$$

This is greater than $\frac{x+y}{1+x+y}$ by Lemma 1 ($a = x + y, b = 2xy, c = 1 + x + y, d = xy$). Now since f is an increasing function, $\frac{x+y}{1+x+y} \ge \frac{a}{1+a}$. □

Now suppose $x, y, z \in \Pi X_i$. Then

$$d(x,y) + d(y,z) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} + \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)}$$

$$= \sum_{i=1}^{\infty} \frac{1}{2^i} \left(\frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} + \frac{d_i(y_i,z_i)}{1+d_i(y_i,z_i)} \right)$$

Now since $d_i(x_i,z_i) \le d_i(x_i,y_i) + d_i(y_i,z_i)$, by Lemma 2 this last expression is

$$\ge \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i,z_i)}{1+d_i(x_i,z_i)} = d(x,z).$$

Thus we have shown $d(x,y) + d(y,z) \ge d(x,z)$ as required. Thus d is a well-defined metric on $\Pi_i X_i$.

We next show the metric d on ΠX_i induces the product topology. For any $x \in X_i$ let $B_i(x,\epsilon)$ be the open ball of radius ϵ centered at x and similarly for $x \in X$ let $B(x,\epsilon)$ be the open ball of radius ϵ in X. We know that in each space these open balls constitute a bases for the metric topology. Let β be the product topology on X and β' the metric topology on X.

Let A be a finite subset of $\mathbb{N}$. For each $i \in A$ let $U_i \subseteq X_i$ be open. Let $Y_i = U_i$ for $i \in A$ and $Y_i = X_i$ for $i \notin A$. Let $U = \Pi_i Y_i$. We know from Problem 27 that sets of this form constitute a base for the product topology on X. Let $x = (x_i) \in U$. We will find an open ball $B(x,\epsilon) \subseteq U$. From this it will follow that U is open in the metric topology, or in other words that $\beta \subseteq \beta'$. This will get us half way there.

Since there are only finitely many U_i, and each U_i is open in the metric topology of X_i, there exists $\delta > 0$ such that $B_i(x_i,\delta) \subseteq U_i$ for all $i \in A$. Without loss of generality we can assume $\delta < 1$. Let $N = \max_{a \in A} a$. Since A is a finite set, we can choose $\epsilon > 0$ such that $\frac{2^i\epsilon}{1-2^i\epsilon} < \delta \ \forall\ i \in A$. We will show $B(x,\epsilon) \subseteq U$. Since $B_i(x_i,\delta) \subseteq U_i \ \forall\ i \in A$, it suffices to show the implication $d(x,y) < \epsilon \Rightarrow y_i \in B_i(x_i,\delta) \ \forall\ i \in A$, or equivalently that $d(x,y) < \epsilon \Rightarrow d_i(x_i,y_i) < \delta \ \forall\ i \in A$. Now,

$$d(x,y) < \epsilon$$

$$\Longrightarrow$$

$$\sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} < \epsilon$$

$$\Longrightarrow$$

$$\frac{1}{2^i} \frac{d_i(x_i,y_i)}{1+d_i(x_i,y_i)} < \epsilon \text{ for all } i$$

$$\Longrightarrow$$

$$d_i(x_i,y_i) < 2^i\epsilon(1 + d(x_i,y_i)) \text{ for all } i$$

$$\Longrightarrow$$

$$d_i(x_i,y_i)(1 - 2^i\epsilon) < 2^i\epsilon \text{ for all } i$$

$$\Longrightarrow$$

$$d_i(x_i, y_i) < \frac{2^i\epsilon}{1-2^i\epsilon} \text{ for all } i$$

And since $\frac{2^i\epsilon}{1-2^i\epsilon} < \delta \; \forall \; i \in A$, this implies $d_i(x_i, y_i) < \delta$ for all $i \in A$. Thus we have finished showing $\beta \subseteq \beta'$, or that the product topology is contained in the metric topology.

We now show $\beta' \subseteq \beta$. Let $V \in \beta'$, the metric topology on X. Let $x \in V$. Then $x \in B(x, \epsilon) \subseteq V$ for some $\epsilon > 0$. We can assume without loss of generality that $\epsilon < 1$. Let N be such that

$$\sum_{i=N+1}^{\infty} \frac{1}{2^i} < \epsilon/2.$$

For each $i = 1, \ldots, N$ let $U_i = B_i\left(x_i, \frac{\epsilon}{2-\epsilon}\right)$. Let

$$U = U_1 \times \cdots \times U_N \times X_N \times X_{N+1} \times \cdots$$

Then $x \in U$. We claim that $y \in U \Rightarrow d(x, y) < \epsilon$, which will imply we have found $U \in \beta$ such that $x \in U \subseteq B(x, \epsilon)$. It follows from this that $\beta' \subseteq \beta$ and thus we will have shown that $\beta = \beta'$.

Let $y \in U$. Then

$$d(x, y) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$$

$$= \sum_{i=1}^{N} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} + \sum_{i=N+1}^{\infty} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)}$$

$$\leq \sum_{i=1}^{N} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} + \sum_{i=N+1}^{\infty} \frac{1}{2^i}$$

$$\leq \sum_{i=1}^{N} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} + \epsilon/2$$

Now for $i \leq N$, $y \in U$ implies $d_i(x_i, y_i) < \frac{\epsilon}{2-\epsilon}$. Thus $2d_i(x_i, y_i) < \epsilon + \epsilon d_i(x_i, y_i)$, which implies

$$\frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} < \epsilon/2$$

Therefore

$$\sum_{i=1}^{N} \frac{1}{2^i} \frac{d_i(x_i, y_i)}{1 + d_i(x_i, y_i)} + \epsilon/2$$

$$\leq \frac{\epsilon}{2} \sum_{i=1}^{N} \frac{1}{2^i} + \epsilon/2$$

$$< \epsilon.$$

Therefore $y \in U \Rightarrow y \in B(x, \epsilon)$. Thus $x \in U \subseteq V$. Thus $\forall \; x \in V$, $\exists \; U \in \beta$ such that $x \in U \subseteq V$. Thus $V \in \beta$. Thus $\beta' \subseteq \beta$. Thus the two topologies are equal.

Problem 29. The *box topology* on ΠX_i has as base all the sets of the form $U_1 \times U_2 \times \cdots$, where U_i is open in X_i. Show that the box topology contains the product topology, and that the two are equal iff X_i is an indiscrete space for all but finitely many values of i. (X is an *indiscrete space* if the only open sets are $\emptyset$ and X.)

Solution: The box topology contains the product topology since by definition of the product topology the open set are ones of the form $U_1 \times U_2 \times \cdots$, where U_i is open in X_i and $U_i = X_i$ for all but finitely many i. If all but finitely many of the spaces are indiscrete then the any *non-empty* set of the form $U_1 \times U_2 \times \cdots$, where U_i is open in X_i necessarily has $U_i = X_i$ for all but finitely many i.

Section 3.5 - Connectedness

Problems 3.5

Problem 30. Let X be the set of all points in the plane which have at least one rational coordinate. Show that X, with the induced topology, is a connected space.

Solution: For each $r \in \mathbb{Q}$ let $X_r = \{(x, y) \mid x = r \text{ or } y = r\}$ and let $A_r = \{(x, y) \mid x = r\}$ and $B_r = \{(x, y) \mid y = r\}$. Then A_r and B_r are homeomorphic to $\mathbb{E}^1$ so by Theorem 3.18 and Corollary 3.22 are connected. $A_r \cap B_r = \{(r, r)\}$ so A_r and B_r are separated. And $X_r = A_r \cup B_r$. So by Thoerem 3.25 X_r is connected. Now $X = \sum_{r \in \mathbb{Q}} X_r$. Since $X_r \cap X_s = \{(r, s), (s, r)\}$ if $r \neq s$, no two members of $\{X_r\}$ are separated from each other. And X_r is connected for each r. Thus by Theorem 3.25 X is connected.

Problem 31. Give the real numbers the finite-complement topology. What are the connected components of the resulting space? Answer the same question for the half-open interval topology.

Solution: For the finite-complement topology, the entire space is connected, since no set can be both open and closed.

For the half-open interval topology, the space is totally disconnected. Let $x < y$. Then by Chapter 2, Problem 11 every set $[a, b)$ is both open and closed. Thus $[x, y)$ contains x and not y, and $[x, y)$ and $[x, y)^c$ are both open. Thus x and y are not in the same connected component. Since x and y are arbitrary, connected components cannot contain more than one point.

Problem 32. If X has only a finite number of components, show that each component is both open and closed. Find a space none of whose components are open sets.

Solution: $X = C_1 \cup \cdots \cup C_n$ a disjoint union of connected components. By Thereom 3.27 each C_i is closed. Since a finite union of closed sets is closed, $X - C_i = \cup_{j \neq i} C_j$ is closed. Thus $C_i = \left(\cup_{j \neq i} C_j\right)^c$ is open. The space $\mathbb{Q}$ with the natural metric topology is totally disocnnected, but points in $\mathbb{Q}$ are not open sets.

Problem 33. (*Intermediate value theorem*). If $f : [a, b] \to \mathbb{E}^1$ is a map such that $f(a) < 0$ and $f(b) > 0$, use the connectedness of $[a, b]$ to establish the existence of a point c for which $f(c) = 0$.

Solution: By Theorem 3.19 we know $[a, b]$ is connected. By Theorem 3.21 $f([a, b])$ is connected. By Theorem 3.19 again $f([a, b])$ is an interval. Thus $f([a, b])$ contains every point between $f(a)$ and $f(b)$. By assumption zero is between $f(a)$ and $f(b)$. Thus 0 is in the range of f. Thus $\exists\, c$ s.t. $f(c) = 0$.

Problem 34. A space X is *locally connected* if for each $x \in X$ and each neighborhood U of x there is a connected neighborhood V of x which is contained in U. Show that any euclidean space, and therefore any space which is locally euclidean (like a surface), is locally connected. If $X = \{0\} \cup \{1/n \mid n = 1, 2, \dots\}$ with the subspace topology from the real line, show that X is not locally connected.

Solution: Theorems 3.19 and 3.26 imply sets $I_1 \times \cdots \times I_n$ where I_n is an open interval in $\mathbb{E}^1$ are connected. These sets also form a base for $\mathbb{E}^n$. It is clear that if X has a base consisting of connected sets then X is locally connected. Local connectedness is preserved by homeomorphism (see next Problem) thus any space that is locally euclidean is locally connected.

Now suppose $X = \{0\} \cup \{1/n \mid n = 1, 2, \dots\}$. Let U be an open set containing 0. Suppose $V \subseteq X$ is a connected open set with $0 \in V \subseteq U$. Then $\exists\, n \in \mathbb{N}$ such that $1/n \in V$. Since the only connected sets in $\mathbb{E}^1$ are the intervals, it must be that the entire interval $[0, 1/n] \subseteq V$. But $V \subseteq X$, a countable set, so this is impossible.

Problem 35. Show that local connectedness is perserved by a homeomorphism, but need not be preserved by a continuous function.

Solution: Suppose X is locally connected and $f : X \to Y$ is a homeomorphism. Let $U \subseteq Y$ be open with $y \in U$. Then $f^{-1}(U)$ is an open neighborhood of $f^{-1}(y)$. So $\exists$ a connected open set $V \subseteq X$ with $f^{-1}(x) \in V \subseteq f^{-1}(U)$. By Theorem 3.21 $f(V)$ is connected in Y. Since f is a homeomorphism, $f(V)$ is an open neighborhood of y in Y, $y \in f(V) \subseteq U \subseteq Y$. Thus Y is locally connected.

Let $X = \{0\} \cup \mathbb{N}$ with the discrete topology. Let $Y = \{0\} \cup \{1/n \mid n = 1, 2, \dots\}$. By Problem 34 Y is not locally connected. Define a function $f : X \to Y$ by $f(0) = 0$ and $f(n) = 1/n$ for $n \neq 0$. Since X has the discrete topology and single points are connected, X is locally connected. Since X has the discrete topology, f is continuous (any function with discrete domain is continuous). Thus X is locally connected, but its continuous image $f(X)$ is not locally connected.

Problem 36. Show that X is locally connected iff every component of each open subset of X is an open set.

Solution: First suppose every component of each open subset of X is an open set. Let $x \in X$ and let $U \subset X$ be open with $x \in U$. Then the components of U are open. Let V be the component of U containing x. Then V is a connected open set with $x \in V \subseteq U$. Thus X is locally connected. Now suppose X is locally connected. Let $U \subset X$ be open. Let C be a connected component of U. Then for each $x \in C$, there is a connected open set V_x such that $x \in V_x \subseteq U$. Since C is a maximally connected set in U it must be that $x \in V_x \subseteq C$ (otherwise by Theorem 3.25 $V_x \cup C$ would be a connected set properly containing C). Thus $C = \cup_{x \in C} V_x$, a union of open sets. Thus C is open.

Section 3.6 - Joining points by paths

Problems 3.6

Notes

Page 62 After the proof, two lines after the displayed equations, it says "It is easy to check that the closure of Z in $\mathbb{E}^2$ is X. Let $y \in [-1, 1]$. Then $\exists$ w such that $\sin(w) = y$. Thus $\sin(w + 2\pi n) = y$ $\forall$ n. Thus for $x_n = \pi/(w + 2\pi n)$, $\sin(\pi/x_n) = y$ $\forall$ n. Now $(x_n, \sin(\pi/x_n)) \subseteq Z$, and $(x_n, \sin(\pi/x_n)) \to (0, y)$, it follows that $(0, y) \in \overline{Z}$. Y is closed and $Z \cap \{(x, y) \mid x \geq \epsilon\}$ is closed $\forall$ $\epsilon > 0$ it follows that there are no other limit points of Z. It follows that $\overline{Z} = Z \cup Y$.

Problem 37. Show that the continuous image of a path-connected space is path-connected.

Solution: Supposer $f : X \to Y$ is onto and X is path-connected. Let $y_1, y_2 \in Y$. Choose $x_1 \in f^{-1}(y_1)$ and $x_2 \in f^{-1}(y_2)$. Since X is path-connected, there is a path $h : [0, 1] \to X$ s.t. $h(0) = x_1$ and $h(1) = x_2$. Then $g = f \circ h$ is a path in Y and $g(0) = f(h(0)) = f(x_1) = y_1$ and $g(1) = f(h(1)) = f(1) = y_2$. Thus there is a path connecting y_1 to y_2 in Y. Thus Y is path-connected.

Problem 38. Show that S^n is path-connected for $n > 0$.

Solution: Think of S^n as the points of unit distance from the origin in $\mathbb{E}^{n+1}$. Suppose $x, y \in S^n$ and $x \neq -y$. Define $h : [0, 1] \to S^n$ by

$$h(t) = \frac{(1 - t)x + ty}{||(1 - t)x + ty||}.$$

Then $h(0) = x$ and $h(1) = y$. Furthermore since $x \neq -y$, the straight line $(1 - t)x + ty$ does not pass through the origin. Thus $(1 - t)x + ty \neq 0$ for all $t \in [0, 1]$. Thus h is a well defined function joining x and y. Now $z \mapsto ||z||$ is continuous function from $\mathbb{E}^{n+1}$ to $\mathbb{R}$, and the ratio of two non-vanishing continuous functions to $\mathbb{R}$ is continuous (basic calculus results). It follows that

h is continuous. Now if $x = -y$ we can choose any point $z \neq x, y$ and join x to z by a path and then z to y by a path. Thus by the comments after the proof of Theorem 3.29, we can join x to z by a path.

Problem 39. Prove that the product of two path-connected spaces is path-connected.

Solution: Suppose X and Y are path-connected. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then $\exists$ maps $h_1 : [0, 1] \to X$ and $h_2 : [0, 1] \to Y$ such that $h_1(0) = x_1$, $h_1(1) = x_2$, $h_2(0) = y_1$, $h_2(1) = y_2$. Define $h : [0, 1] \to X \times Y$ by $h(t) = (h_1(t), h_2(t))$. Then $h(0) = (x_1, y_1)$ and $h(1) = (x_2, y_2)$. By Theorem 3.13 h is continuous. Thus $X \times Y$ is path-connected.

Problem 40. If A and B are path-connected subsets of a space, and if $A \cap B$ is nonempty, prove that $A \cup B$ is path-connected.

Solution: Let $x, y \in A \cup B$. If $x, y \in A$ then since A is path connected x and y can be connected by a path. Likewise if $x, y \in B$. If $x \in A$ and $b \in B$ then since $A \cap B \neq \emptyset$, $\exists\, c \in A \cap B$. Since A is path-connected we can connect x to c by a path and since B is path-connected we can connect c to y by a path. Thus by the comments after the proof of Theorem 3.29, x can be connected to y by a path. Thus in all cases x can be connected to y by a path. Thus $A \cup B$ is path-connected.

Problem 41. Find a path-connected subset of a space whose closure is not path-connected.

Solution: Let X be the space defined in the example following Theorem 3.30. Decompose $X = Y \cup Z$ as in the example. Since Z is the continuous image of $(0, 1]$ it is path-connected by Problem 37. It is shown in the example that $\overline{Z} = X$ and X is not path-connected.

Problem 42. Show that any indiscrete space is path-connected.

Solution: Since *any* function into an indiscrete space X is continuous, we can define any function we want from $[0, 1]$ to X and it will be continuous. Thus for $x, y \in X$ just choose any function $[0, 1] \to X$ that sends $0 \mapsto x$ and $1 \mapsto y$. Thus X is path-connected.

Problem 43. A space X is *locally path connected* if for each $x \in X$, and each neighborhood U of x, there is a path-connected neighborhood V of x which is contained in U. Is the space shown in Fig. 3.4 locally path-connected? Convert the space $\{0\} \cup \{1/n \mid n = 1, 2, \dots\}$ into a subspace of the plane which is path-connected but not locally path-connected.

Solution: The space shown in Fig. 3.4 is not locally path-connected because any open set containing the origin $(0, 0)$ must contain points from both Y and Z and as shown in the example there are no paths connecting points in Y to points in Z.

Let $X_0 = \{(0, y) \mid y \in [0, 1]\}$. For each $n \in \mathbb{N}$ let $X_n = \{(1/n, y) \mid y \in [0, 1]\}$. Let $Y = \{(x, 0) \mid x \in [0, 1]\}$. Let $X = Y \cup \left(\cup_{n=0}^{\infty} X_n\right)$. Then X is path connected since $Y \cap X_n \neq \emptyset$ for all n. Let $p = (0, 1)$ and let U be the open ball centered at p of radius 1. Let $V \subseteq U$ be open. Then $V \cap Y = \emptyset$. So $V \cap X$ is just a collection of line segments any two of which are separated from each other. So $V \cap X$ is not path connected for any open V with $p \in V \subseteq U$. Thus X is not locally path connected. I believe this space is called the "Comb Space".

Problem 44. Prove that a space which is connected and locally path-connected is path-connected.

Solution: Let X be connected and locally path-connected. Let $x \in X$. Let E be the path component containing x. Suppose E were not closed. Then $\exists\, y \in E^c$ s.t. $y \in \overline{E}$. Since X is locally path-connected $\exists$ an open path-connected set V s.t. $y \in V$. Since $y \in \overline{E}$, $V \cap E \neq \emptyset$. Let $z \in V \cap E$. Then x can be path connected to z since both are in E and z can be path connected to y since $z, y \in V$. Thus by the comments after the proof of Theorem 3.29, x can be connected to y by a path, contradicting the assumption that $y \in E^c$. Thus E is closed. Now suppose E^c were not closed. Then $\exists\, y \in E$ s.t. $y \in \overline{E^c}$. Since X is locally path-connected $\exists$ an open path-connected set V s.t. $y \in V$. Since $y \in \overline{E^c}$, $V \cap E^c \neq \emptyset$. Let $z \in V \cap E^c$. Then x, y can be connected by a path since both are in E and y, z can be connected by a path since both are in V. Thus x and z can be connected by a path. Thus $z \in E$ contradicting the assumptoin that $z \in E^c$. Thus E^c is closed. Thus E and E^c are both open and both closed. By Theorem 3.20 (c), since X is connected and $E \neq \emptyset$, it must be that $E^c = \emptyset$. It follows that every point in X can be connected to x by a path. Thus X is path-connected.

Chapter 4 - Identification Spaces

Section 4.2 - The identification topology

Notes

Page 66. End of the first paragraph he says "it is easy to check that the identification topology coincides with that induced from $\mathbb{E}^3$. I don't recall anywhere that the Möbius strip was given as embedded in $\mathbb{E}^3$." In fact in the first paragraph of this section he points out that in Chapter 1 he introduced the Möbius informally as an "identification space" - so not as a subset of $\mathbb{E}^3$. Unless he means that since he showed how to construct one out of paper and we live in $\mathbb{E}^3$ then it is naturally embedded in $\mathbb{E}^3$ in the same way that I am. In any case, I think he means it will be easy to show the topologies are the same once we have Theorem 4.2 and 4.3. Since it can shown easily using those theorems, I don't think it's worth doing here in any more basic manner.

Page 67. Theorem 4.3. It's worth noting that a closed map is not necessarily an open map and an open map is not necessarily a closed map.

Page 68. In the torus example he defines the identification map f. It's worth noting that this is a closed map, but not an open map.

Page 68. The end of the B^n/S^{n-1} example he says "The continuity of f is easy to check." Intuitively it is pretty clear. But I'm not sure how easy it is to prove rigorously. Any open set in S^n that does not contain p clearly has open inverse image since h_1 and h_2 are continuous. If an open set in S^n contains p then obviously it contains the entire boundary S^{n-1} in B^n (the whole boundary maps to p) and since we start with an open neighborhood of p in S^n if we remove p we get an open set in S^n whose inverse is open in $B^n - S^{n-1}$. Adding back S^{n-1} we should get an entire neighborhood of S^{n-1}. The rigorous proof of this should be basically the same as that given in Chapter 3, Problem 18 where we show all points far enough from the origin lie in the set.

Page 68. Lemma 4.5. The statement of this lemma should say X must be compact.

Page 69. Comment after the Glueing lemma. It says the lemma is true also if X and Y are open. In this case we could also allow infinitely many open subsets, not just two. This is not the case for closed subsets.

Page 70. Line 3. He talks about the "disjoint union" without defining it. If X and Y are subspaces of Z we can take the disjoint union $X + Y \equiv \{(v, i) \in X \cup Y \times \{0, 1\} \mid v \in x \text{ if } i = 0, v \in Y \text{ if } i = 1\}$. And for the disjoint union of infinitely many subspaces $X_\alpha \subseteq X$ indexed by A we give A the discrete topology, and then $\oplus X_\alpha = \{(x, \alpha) \in X \times A \mid x \in X_\alpha\}$.

Page 70. Theorem 4.8. He says to observe that "$Fj : \oplus X_\alpha \to Z$ is continuous if and only if each f_α is continuous..." But I believe we only need the "if" part of that to do this proof.

Page 71. Top of page it says the "identification topology gives a space homeomorphic to the nonnegative part of the real line...". The thing to note is that $\oplus_{i=1}^{\infty}[i-1, i] \to [0, \infty)$ is an identification map.

Page 71. Projective spaces. It says "... theorem (4.2) and corollary (4.4) can be used to show that all three lead to the same space." This is done in Problem 1 of this section.

Page 72. At the end of the first paragraph he says he leaves it to "the reader to reconcile this description with those listed in Projetive spaces' above". I'll have to circle back to this because I'm really not sure right now what he has in mind or how to do this. (XXX)

Page 72. At the end of the text he says "We invite the reader to check that the corresponding identification space is an *indiscrete* space." This follows from the fact that given any open interval $I \subseteq \mathbb{R}$ and any real number $x \in \mathbb{R}$, $\exists\, r \in \mathbb{Q}$ such that $x - r \in I$. An open set in the identification space must contain (the classes of) an open interval of real numbers and therefore by what was said in the previous sentence, it must contain every real number.

Problem 1. Check that the three descriptions (a), (b), (c) of $\mathbb{P}^n$ listed in 'Projective spaces' above do all lead to the same space.

Solution: Let X be the space in (a), Y the space in (b) and Z the space in (c).

We first show Y is homeomorphic to X. Let $h : \mathbb{E}^{n+1} - \{0\} \to Y$ be the identification map. Let $i : S^n \to \mathbb{E}^{n+1}$ be the natural embedding. Define $g : S^n \to Y$ by $g(x) = h \circ i(x)$. Then g is continuous with compact domain. And Y is Hausdorff because given $x, y \in Y$, $x \neq y$, $\exists\, x', y' \in \mathbb{E}^{n+1}$ such that $h(x') = x$ and $h(y') = y$. Then $x' \neq \pm y'$. Thus $\exists$ open sets $U, V \subseteq \mathbb{E}^{n+1}$ such that $x' \in U$, $y' \in V$, and $U \cap \pm V = \emptyset$. Thus $x \in h(U)$, $y \in h(V)$ and $h(U) \cap h(V) = \emptyset$. By Corollary 4.4 g is an identification map. Since g identifies x and $-x$, the identification space is exactly X.

We next show Z is homeomorphic to X. Embed $i : B^n \hookrightarrow S^n$ as the northern hemisphere $\{(x_i) \mid x_n \geq 0\}$. Let $h : S^n \to X$ be the identification map. Then $h \circ i$ is a map from a compact space B^n to a Hausdorff space $\mathbb{P}^n$. Thus by Corollary 4.4 $h \circ i$ is an identification map. Note that $h \circ i$ identifies antipodal points on the boundary of B^n and does not identify points in the interior to any other points. Thus $h \circ i$ induces a homeomorphsm from Z to X.

Problem 2. Which space do we obtain if we take a Möbius strip and identify its boundary circle to a point?

Solution: We obtain the projective plane $\mathbb{P}^2$. To see this, let R be a rectangle as in Fig. 4.1 (page 66). Instead of first identifying the right and left edges to get a Möbius and then collapsing the boundary circle, we first collapse the top and bottom borders to two points a and b, then identify the (image of) the right and left edges as they would have been identified to form a Möbius from R. This operation identifies the two points a and b. So after these two identifications we end up with the same space. So by this argument we can do the identifications in the opposite order. Proceeding in this way, we first collapse the top horizontal border to one point and the bottom border to another point. From this identification we get a space homeomorphic to a disc D. The identification map $f : R \to D$ can be seen as taking the left border of R to the left semi-circle of the boundary of D and the right border to the right semi-circle of the boundary of D. Then we follow f with the identification map from D to $\mathbb{P}^2$ given in Problem 1 above. The result is an identification map from D to $\mathbb{P}^2$ that (since it identifies antipodal points on the boundary) identifies the same two points on the two boundary semicircles that came from two points in R that would be identified if we formed the Möbius first. Thus the identification space is homeomorphic to the image $\mathbb{P}^2$.

Problem 3. Let $f : X \to Y$ be an identification map, let A be a subspace of X, and give $f(A)$ the induced topology from Y. Show that the restriction $f|_A : A \to f(A)$ need not be an identification map.

Solution: Let $X = [0, 1]$ and $Y = S^1 \subseteq \mathbb{C}$. Let $f : X \to Y$ be given by $x \mapsto e^{2\pi i x}$. Then by Corollary 4.4, f is an identification map because X is compact and Y is Hausdorff. Let $A = [0, 1)$. Then $f|_A(A) = Y$. But $f|_A$ does not identify any points. Thus if $f|_A$ were an identification map it would (by Theorem 4.2) induce a homeomorphism from A to Y. But we know f is not a homeomorphism (see Section 1.4, Example 3, page 14).

Problem 4. With the terminology of Problem 3, show that if A is open in X and if f takes open sets to open sets, or if A is closed in X and f takes closed sets to closed sets, then $f|_A : A \to f(A)$ is an identification map.

Solution: In the first case $f|_A$ is an open map and in the second case $f|_A$ is a closed map. In either case it follows from Theorem 4.3 that f is an identification map.

Problem 5. Let X denote the union of the circles $[x - (1/n)]^2 + y^2 = (1/n)^2$, $n = 1, 2, 3, \ldots$, with the subspace topology from the plane, and let Y denote the identification space obtained from the real line by identifying all the integers to a single point. Show that X and Y are not homeomorphic. (X is called the *Hawaiian earring*.

Solution: The space X is compact. To see this suppose $\{U_\alpha\}$ is an open cover. Then some U_{α_1} contains the origin $(0, 0)$. U then contains all of the circles $[x - (1/n)]^2 + y^2 = (1/n)^2$ for $n >> 0$. And U^c is homeomorphic to finitely many closed intervals which is compact. Thus $\{U_\alpha\}$ has a finite subcover. Now consider the open sets $U_n = (n + 1/10, n - 1/10)$ for each $n \in \mathbb{Z}$. And let $V = (-1/5, 1/5)$. Then the image of the open sets $\{U_\alpha\} \cup \{V\}$ in Y form an open cover and for each $n \in \mathbb{Z}$, $(2n + 1)/2$ is in one and only one U_α. Thus there cannot be a finite subcover.

Problem 6. Give an example of an identification map which is neither open or closed.

Sollution: Let $f : [0, 3\pi) \to S^1$ be given by $x \mapsto e^{ix}$. Then f wraps the interval one and a half times around the circle. Let $U \subseteq S^1$. Suppose $f^{-1}(U)$ is open. Let $y \in U$. Then either $f^{-1}(y) = \{a\}$ or $f^{-1}(y) = \{a, b\}$. In the first case $a \in [\pi, 2\pi)$. Since $f^{-1}(U)$ is open, and $[\pi, 2\pi)$ is in the interior (in $\mathbb{E}^1$) of the interval $[0, 3\pi)$, there exists an open (in $\mathbb{E}^1$) interval (c, d) such that $a \in (c, d) \subseteq f^{-1}(U)$. But then $f((c, d))$ is an open arc of S^1 such that $y \in f((c, d)) \subseteq U$. Thus y is in the interior of U. Now suppose $f^{-1}(y) = \{a, b\}$. Then wlog assume $a \in [0, \pi)$ and $b \in [2\pi, 3\pi)$. It follows that there is an interval (in $\mathbb{E}^1$) (c, d) such that $b \in (c, d) \subseteq f^{-1}(U)$ and therefore $y = f(b) \in f((c, d)) \subseteq U$. Thus, as before, y is in the interior of U. Thus in all cases if $f^{-1}(U)$ is open and $y \in U$ then y is in the interior of U. It follows that $f^{-1}(U)$ open $\Rightarrow U$ is open. Thus f is an identification map. Now $[0, \pi)$ is open in $[0, 3\pi)$ but $f([0, \pi))$ is not open in S^1 since it contains the point $z = 1$ but does not contain any points with $\mathrm{im}(z) < 0$. Thus f is not an open map. Similarly, $[2\pi, 3\pi)$ is closed in $[0, 3\pi)$ but $f([2\pi, 3\pi))$ is not closed in S^1 since it does not contain $z = -1$ but it does contain all other points with $\mathrm{im}(z) \geq 0$. Thus f is not a closed map either.

Problem 7. Describe each of the following spaces: (a) the cylinder with each of its boundary circles identified to a point; (b) the torus with the subset consisting of a meridianal and a longitudinal circle identified to a point; (c) S^2 with the equator identified to a point; (d) $\mathbb{E}^2$ with each of the circles centre the origin and of integer radius identified to a point.

Solution:

(a) This space is homeomorphic to S^2. Consider the cylinder as $X = S^1 \times [-1, 1]$ in $\mathbb{E}^3$. Define $f : X \to S^2$ by $(x, y, z) \mapsto (rx, ry, z)$ where $r = \sqrt{(1 - z^2)/(x^2 + y^2)}$. Since $x^2 + y^2$ never vanishes on X, this is a continuous function. Since X is compact and S^2 is Hausdorff, by Corollary 4.4 f is an identification map. Note that f identifies all points where $z = 1$ to the north pole of S^2 and all the points where $z = -1$ to the south pole. All other points are identified only with themselves. Thus the cylinder with its boundary circles identified to points is homeomorphic to S^2, the image of f.

(b) The identification space is S^2. Let Z be the torus with a meridianal and a longitudian identified to a point. Let $X = [0, 1] \times [0, 1]$. If for each $x \in (0, 1)$ we identify $(x, 0)$ with $(x, 1)$ and for each $y \in (0, 1)$ we identify $(0, y)$ with $(1, y)$. And we identify the four points $(0, 0), (0, 1), (1, 0), (1, 1)$. Then the identification space is the torus (see example on page 68). Identifying the image in the torus of $(x, 0)$ for all x to one point and the image of $(0, y)$ for all y to one point, then we'll obtain the space Z. If we do all of the identifying at once, then we obtain the square X with its perimeter identified to a single point. By Chapter 2, Problem 21 (page 36) X is homeomorphic to the unit disc. By the example "The identification space B^n/S^{n-1}" (pages 68-69) the unit dic with its boundary identified to a point is homeomorphic to S^2.

(c) Let X be S^2 with the equator identified to a point. Let $f : S^2 \to X$ be the identification map. For $n = 1, 2$ let C_n be the sphere in $\mathbb{E}^3$ of radius $1/2$ centered at $(n, 0, 0)$. Let $Y = C_1 \cup C_2$. Then Y is two copies of S^2 joined at the point $(3/2, 0, 0)$. We will show X is homeomorphic to Y. Let $H_1 = \{(x, y, z) \in S^2 \mid z \geq 0\}$ (the northern hemisphere), and let $H_2 = \{(x, y, z) \in S^2 \mid z \leq 0\}$ (the southern hemisphere). Then $S^2 = H_1 \cup H_2$. Let $E = H_1 \cap H_2$ the equator of S^2. Then H_1/E and H_2/E can be though of as subspaces of S^2/E. Thus $X = H_1/E \cup H_2/E$. Note that H_1 is homeomorphic to the closed unit disc B^2 (project H_1 to the x-y plane). By the example "The identification space B^n/S^{n-1}" (on pages 68-69), B^2 with its boundary identified to a point is homeomorphic to S^2. Thus there is a homeomorphism h_1 from H_1/E to C_1. By Chapter 1, Problem 13 (page 23) we can choose h_1 so that it sends the point $f(E) \in H_1/E$ to $(3/2, 0, 0)$ in C_1. Likewise there is a homeomorphism h_2 from H_2/E to C_2 that takes $f(E) \in H_2/E$ to $(3/2, 0, 0)$. Since $f^{-1}(f(E)) = E$, and E is closed in S^2, it follows (from the fact that X has the identification topology) that $f(E)$ is closed in X. By the Glueing lemma (4.6) the homeomorphisms h_1 and h_2 join to a continuous, one-to-one and onto function h from X to $C_1 \cup C_2 = Y$. Since h restricted to H_1/E is a homeomorphism, it is a closed map. Similarly h restricted to H_2/E is a closed map. Since $f^{-1}(f(H_i)) = H_i$, $i = 1, 2$, and H_i is closed in S^2, it follows that $f(H_i) = H_i/E$ are closed in X. Since both H_1/E and H_2/E are closed in X, it follows that h itself is a closed map. Thus h^{-1} is continuous. Thus h gives a homeomorphism between X and C.

(d) Let X be $\mathbb{E}^2$ with each of the circles centred at the origin and of integer radius identified to a point. For each $n \in \mathbb{N}$ let C_n be the sphere in $\mathbb{E}^3$ of radius $1/2$ centered at $(n, 0, 0)$. Let $Y = \cup_{n\in\mathbb{N}} C_n$. Then X is homeomorphic to Y.

Solution: Let X be the space in question and let $f : \mathbb{E}^2 \to X$ be the identification map. For each $n \in \mathbb{N}$ let $A_n = \{p \in \mathbb{E}^2 \mid n-1 \leq \|p\| \leq n\}$. For $n > 1$, A_n is an annulus, so homeomorphic to a cylinder. And A_0 is exactly the closed unit disc. The image of A_0 in X is A_0 with its boundary circle identified to a point. We know, by the example "The identification space B^n/S^{n-1}" (on pages 68-69), that A_0 with its boundary identified to a point is homeomorphic to S^2. Likewise, we know by part (a) of this problem that for $n > 0$, the image of A_n in X is homeomorphic to S^2. Now the image of A_n in X shares exactly one point with A_{n-1} and exactly one point with A_n, when $n > 0$ and A_0 shares exactly one point with A_1. Thus X is homeomorphic to a countable sequence of spheres each connected to the next by one point. This is exactly the space Y.

Problem 8. Let X be a compact Hausdorff space. Show that the cone on X is homeomorphic to the one-point compactification of $X \times [0, 1)$. If A is closed in X, show that X/A is homeomorphic to the one-point compactification of $X - A$.

Solution: Since $\{1\}$ is compact in $[0, 1]$ and X is compact, by Theorem 3.15 $X \times \{1\}$ is compact in $X \times [0, 1]$. Since X and $[0, 1]$ are Hausdorff, by Theorem 3.14 $X \times [0, 1]$ is Hausdorff. Thus $X \times \{1\}$ is a compact subset of a Hausdorff space. By Theorem 3.6 it follows that $X \times \{1\}$ is closed in $X \times [0, 1]$. Thus the first part follows from the second.

So suppose A is closed in X. Let $Y = X - A \cup \{\infty\}$ be the one-point compactification of $X - A$. Let P be the image of A in X/A under the identification map of $i : X \to X/A$. Define a map $h : X/A \to Y$ by $x \mapsto x$ if $x \in X - A$ and $P \mapsto \infty$. We will show h is a homeomorphism. We first show h is continuous. By Theorem 4.1 it suffices to show $h \circ i$ is continuous. Let U be an open set in Y. If $\infty \notin U$ then $U \subseteq X - A$ and $(h \circ i)^{-1}(U) = U \subseteq X$ is open. Now if $\infty \in U$ then U^c is compact in $X - A$. By Theorem 3.6 U^c is closed in X. Thus $(h \circ i)^{-1}(U^c)$ is a closed subset of X (it is just U^c itself, under the various identifications). Thus $(h \circ i)^{-1}(U)$ is an open subset of X. Thus $(h \circ i)^{-1}(U)$ is open in every case. Thus h is continuous. By Chapter 3, Problem 17 Y is Hausdorff. By Theorem 3.4 X/A is compact since it is the continuous image of X under the identification map i. Thus h is a continuous map from a compact space to a Hausdorff space. And h is clearly one-to-one and onto. Thus by Theorem 3.7 h is a homeomorphism.

Problem 9. Let $f : X \to X'$ be a continuous function and suppose we have partitions $\mathcal{P}, \mathcal{P}'$ of X and X' respectively, such that if two points of X lie in the same member of $\mathcal{P}$, their images under f lie in the same member of $\mathcal{P}'$. If Y, Y' are the identification spaces given by these partitions, show that f induces a map $\hat{f} : Y \to Y'$ and that if f is an identification map then so is $\hat{f}$.

Solution: Let i be the identification map from X to Y and i' the identification map from X' to Y'. The condition on f says that if $P \in \mathcal{P}$, then $\exists\, P' \in \mathcal{P}'$ such that $f(P) \subseteq P'$. Thus the function $\hat{f} : Y \to Y'$ defined by $\hat{f}(i(P)) = i'(f(P))$ is well-defined. We just need to show that $\hat{f}$ is continuous. By Theorem 4.1 it suffices to show $\hat{f} \circ i$ is continuous. Notice that the following diagram is commutative:

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ i \downarrow & & \downarrow i' \\ Y & \xrightarrow{\hat{f}} & Y' \end{array}$$

Let $U \subseteq Y'$ be open. Then by the commutativity of the diagram, $(\hat{f} \circ i)^{-1}(U) = (i' \circ f)^{-1}(U)$ Since f and i' are continuous, it follows that $(\hat{f} \circ i)^{-1}(U)$ is open in X.

Now suppose f is an identification map. Let $U \subseteq Y'$ be such that $\hat{f}^{-1}(U)$ is open. Then $i^{-1}(f^{-1}(U))$ is open in X. Now $f^{-1}(i'^{-1}(U)) = i^{-1}(f^{-1}(U))$. Thus $f^{-1}(i'^{-1}(U))$ is open. Thus since f is an identification map, $i'^{-1}(U)$ is open in X'. Since i' is an identification map, U is therefore open in Y'.

Problem 10. Let S^2 be the unit sphere in $\mathbb{E}^3$ and define $f : S^2 \to \mathbb{E}^4$ by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Show that f induces an embedding of the projective plane in $\mathbb{E}^4$ (embeddings were defined in Problem 14 of Chapter 3).

Solution: Let $X = \text{im}(f) \subseteq \mathbb{E}^4$. Since $\mathbb{E}^4$ is Hausdorff and a subspace of a Hausdorff space is Hausdorff, X is Hausdorff. And S^2 is compact. Thus by Corollary 4.4 $f : S^2 \to X$ is an identification map. By the example "Projective spaces (a)" on page 71, $\mathbb{P}^2$ is S^2 with antipodal points identified. Thus by Theorem 4.2 (a) we will be done if we show f identifies antipodal points (and no others). It clearly does identify antipodal points, $f(x, y, z) = f(-x, -y, -z)$. So suppose $f(x_1, y_1, z_1) = f(x_2, y_2, z_2)$. We must show $(x_1, y_1, z_1) = (x_2, y_2, z_2)$ or $(x_1, y_1, z_1) = (-x_2, -y_2, -z_2)$. We have

$$x_1^2 - y_1^2 = x_2^2 - y_2^2 \tag{1}$$

$$x_1 y_1 = x_2 y_2 \tag{2}$$

$$x_1 z_1 = x_2 z_2 \tag{3}$$

$$y_1 z_1 = y_2 z_2 \tag{4}$$

From (3) and (4) it follows that

$$(x_1^2 - y_1^2) z_1^2 = (x_2^2 - y_2^2) z_2^2 \tag{5}$$

<u>case 1</u>: $x_1^2 - y_1^2 \neq 0$.

Combining (1) and (5) we get $z_1^2 = z_2^2$. Thus $z_1 = \pm z_2$. If $z_1 = z_2$ then

case a: $z_1 \neq 0$. Then (3) and (4) imply $x_1 = x_2$ and $y_1 = y_2$. If $z_1 = -z_2$ then (3) and (4) imply $x_1 = -x_2$ and $y_1 = -y_2$. Thus either $(x_1, y_1, z_1) = (x_2, y_2, z_2)$ or $(x_1, y_1, z_1) = (-x_1, -y_1, -z_1)$.

case b: $z_1 = 0$. Then $x_1^2 + y_1^2 = 1$. So one of x_1 or y_1 must be different from zero. Assume wlog that $x_1 \neq 0$. Now $x_1^2 + y_1^2 = 1 = x_2^2 + y_2^2$ together with (1) implies $2x_1^2 = 2x_2^2$. Thus $x_1 = \pm x_2$. If $x_1 = x_2$ then (2) implies $y_1 = y_2$, thus $(x_1, y_1, z_1) = (x_2, y_2, z_2)$. If $x_1 = -x_2$ then (2) implies $y_1 = -y_2$, thus $(x_1, y_1, z_1) = (-x_2, -y_2, -z_2)$.

<u>case 2</u>: $x_1^2 - y_1^2 = 0$.

Suppose $x_1 = 0$. Then $y_1 = 0$, and combining (1) and (2) it follows that $x_2 = 0$ and $y_2 = 0$. Now if $x_1 = y_1 = 0$ then necessarily $z_1 = \pm 1$. Likewise $z_2 = \pm 1$. Thus in this case either $(x_1, y_1, z_1) = (x_2, y_2, z_2)$ or $(x_1, y_1, z_1) = (-x_1, -y_1, -z_1)$. By symmetry, the same thing happens if $y_1 = 0$, $x_2 = 0$ or $y_2 = 0$.

Therefore we have reduced to the case that none of x_1, x_2, y_1, y_2 are zero. By assumption $x_1 = \pm y_1$ and from (1) it follows that $x_2 = \pm y_2$. It then follows from (2) that $x_1 = \pm x_2$.

case a: $x_1 = x_2$. Then (2) implies $y_1 = y_2$ and (4) implies $z_1 = z_2$. Thus $(x_1, y_1, z_1) = (x_2, y_2, z_2)$.

case b: $x_1 = -x_2$. Then (2) implies $y_1 = -y_2$ and (4) implies $z_1 = -z_2$. Thus $(x_1, y_1, z_1) = (-x_2, -y_2, -z_2)$.

Problem 11. Show that the function $f : [0, 2\pi] \times [0, \pi] \to \mathbb{E}^5$ defined by $f(x, y) = (\cos x, \cos 2y, \sin 2y, \sin x \cos y, \sin x \sin y)$ induces an embedding of the Klein bottle in $\mathbb{E}^5$.

Solution: Let $Y = [0, 2\pi] \times [0, \pi]$. Let $X = \text{im}(f) \subseteq \mathbb{E}^5$. Since $\mathbb{E}^5$ is Hausdorff and a subspace of a Hausdorff space is Hausdorff, X is Hausdorff. And $[0, 2\pi] \times [0, \pi]$ is compact. Thus by Corollary 4.4 $f : Y \to X$ is an identification map. We know the Klein bottle can be obtained from Y by identifying two opposite edges in the same orientation and the other two in the opposite orientation (see Figure 1.12, page 10).

First note that f identifies all four corners of Y together. So in what follows we will examine what happens to all of the other points.

Suppose $f(x_1, y_1) = f(x_2, y_2)$. First suppose $y_1 \neq y_2$ and assume $y_1, y_2 \in (0, \pi)$. Inspection of the graph of $\sin(x)$ tells us that if $\sin 2y_1 = \sin 2y_2$ then one of two cases hold: (i) $0 < 2y_1, 2y_2 < \pi$ and $2y_1 = \pi - 2y_2$ or (ii) $\pi < 2y_1, 2y_2 < 2\pi$ and

$2y_1 = 3\pi - 2y_2$. Now $\cos(2y_1) = \cos(2y_2)$ implies $2y_1 = 2\pi - 2y_2$, or equivalently $y_1 = \pi - y_2$. But this is incompatible with both (i) and (ii). Thus if we continue to assume $y_1 \neq y_2$, it must be that at least one of y_1 or y_2 equals zero or π. Since $y_1 = \pi - y_2$, if $y_1 = 0$ then $y_2 = \pi$ and conversely. Thus we have shown that if $f(x_1, y_1) = f(x_2, y_2)$ and $y_1 \neq y_2$ then one of (x_1, y_1) or (x_2, y_2) must be on the top side and the other on the bottom side. In other words $\{y_1, y_2\}$ equals $\{0, \pi\}$. We need to examine what happens to the x-coordinate of such points.

So suppose $f(a, 0) = f(b, \pi)$. And suppose $0 < a, b < 2\pi$. We have $\cos a = \cos b$ and $\sin a = -\sin b$ (from the first and fourth coordinates of f). From $\cos a = \cos b$ we know that either $a = b$ or $a = 2\pi - b$. From $\sin a = -\sin b$ we know that the only one of these that is possible is $a = 2\pi - b$ (keep in mind we are assuming here that a and b are strictly between 0 and 2π). Thus the points where $f(x_1, y_1) = f(x_2, y_2)$ with $y_1 \neq y_2$ and $x_1, x_2 \in (0, 2\pi)$ are exactly the points of the form $((x_1, y_1), (x_2, y_2)) = ((x, 0), (2\pi - x, \pi))$ for some $x \in (0, 2\pi)$. The only other points on the top or bottom sides are the corner points which we have already handled.

Suppose now that $y_1 = y_2$, and assume $x_1 \neq x_2$. Assume $y_1, y_2 \in (0, \pi)$. We know from the first coordinate of $f(x_1, y_1) = f(x_2, y_2)$ that $\cos x_1 = \cos x_2$. Since $x_1 \neq x_2$, this holds only if $x_1 = 2\pi - x_2$. Now from the fifth coordinate of $f(x_1, y_1) = f(x_2, y_2)$ we also know $\sin x_1 = \sin x_2$ (substitute $y_2 = y_1$ and then we can cancel $\sin y_1$ from both sides since we are assuming $y_1 \in (0, \pi)$), which together with $x_1 = 2\pi - x_2$ implies $-\sin x_2 = \sin x_2$, which implies $\sin x_2 = 0$. Thus $x_2 = 0$ or $x_2 = \pi$. We do not need to further evaluate the y-coordinates of such points since in this case we have assumed $y_1 = y_2$. Thus f identifies points on the opposite vertical sides at the same vertical height to each other. The only other points on the left or right sides are the corner points which we have already handled.

In summary we have shown that any points that f identifies to each other must lie on the boundary of the square; and that f identifies the four corners of the square to one point; and f identifies points on the top and bottom sides in pairs, where the point with x-coordinate x on the top is identified with the point with x-coordinate $2\pi - x$ on the bottom; and finally f identifies points on the left and right sides in pairs, where points with the same y-coordinate are identified to each other.

It follows that the identification space is the Klein bottle.

Problem 12. With the notation of Problem 11, show that if $(2 + \cos x)\cos 2y = (2 + \cos x')\cos 2y'$ and $(2 + \cos x)\sin 2y = (2 + \cos x')\sin 2y'$, then $\cos x = \cos x'$, $\cos 2y = \cos 2y'$, and $\sin 2y = \sin 2y'$. Deduce that the function $g : [0, 2\pi] \times [0, \pi] \to \mathbb{E}^4$ given by $g(x, y) = ((2 + \cos x)\cos 2y, (2 + \cos x)\sin 2y, \sin x \cos y, \sin x \sin y)$ induces an embedding of the Klein bottle in $\mathbb{E}^4$.

Solution: Suppose that

$$(2 + \cos x)\cos 2y = (2 + \cos x')\cos 2y' \tag{1}$$

$$(2 + \cos x)\sin 2y = (2 + \cos x')\sin 2y' \tag{2}$$

We want to show

$$\cos x = \cos x' \tag{3}$$

$$\cos 2y = \cos 2y' \tag{4}$$

$$\sin 2y = \sin 2y' \tag{5}$$

case 1: $\{y, y'\} \cap \{\pi/4, 3\pi/4\} = \emptyset$. Then $\cos 2y \neq 0$ and $\cos 2y' \neq 0$. Thus we can divide (2) by (1) to get $\tan 2y = \tan 2y'$. Since $y, y' \in [0, \pi]$, the only way this is possible is if $y = y'$ or (wlog) $2y = 2y' + \pi$. If $y = y' + \pi/2$ then $\cos(2y) = -\cos(2y')$. But then (1) canont hold (remember $\{y, y'\} \cap \{\pi/4, 3\pi/4\} = \emptyset$). Thus $y = y'$, from which (4) and (5) obviously hold and it follows immediately from (1) or (2) that (3) holds.

case 2: $\{y, y'\} \cap \{\pi/4, 3\pi/4\} \neq \emptyset$. If $y = \pi/4$, then (1) implies $y' = \pi/4$ or $y' = 3\pi/4$. But if $y' = 3\pi/4$ then (2) does not hold. So it must be $y = y' = \pi/4$. Then (4) and (5) hold and (2) implies (3). If on the other hand $y = 3\pi/4$, then as before (1) implies $y' = \pi/4$ or $y' = 3\pi/4$. But if $y' = \pi/4$ then (2) does not hold. So it must be that $y = y' = 3\pi/4$. Then (4) and (5) hold and (2) implies (3).

Now let f be the function defined in Problem 11. Then g is an identification map for the same reasons f is. And it follows from what we just proved above that $g(x_1, y_1) = g(x_2, y_2) \Leftrightarrow f(x_1, y_1) = f(x_2, y_2)$. Thus the identification space for g is the same as the identification space for f. Thus the image of g is homeomorphic to the Klein bottle.

Section 4.3 - The identification topology

Notes

Page 77 At the end of the first paragraph it asks "How many components has $GL(n)$?" The answer is two. But as far as I know there's no real easy way to prove this. It is a full page proof, Theorem 3.68, p.131 in Warner's "Foundations of differentiable Manifolds and Lie Groups".

Page 77 In the last paragraph we are told to check four statements about the quaternions. The quaternions are homeomorphic to $\mathbb{E}^4$. Multiplication and inversion in the quaternions is given by polynomials and rational functions in the coordinates. Thus it is easy to see that it is a topological group. We are asked to check that conjugation induces a rotation on the subspace of pure quaternions, and that this gives a homomorphism, onto, continuous to SO(3). The fact that it induces a rotation is non-trivial, it is detailed on the wikipedia page on the quaternions. I don't know what Armstrong has in mind exactly when he asks us to check that. But it also follows from the proof that it is onto. For the rest of the details I refer to the wikipedia quaternion page. The fact that it is a homomorphism just follows from the fact that multiplication in the quaternions is associative. To see it is continuous, if $q_n \to q$ is a convergent sequence of quaternions converging to a quaternion, then conjugation by q_n must converge to conjugation by q because all of the coordinates of the transformation are given by rational functions that don't have vanishing denominators on the non-zero quaternions. The kernel of the map is $\mathbb{R} - \{0\}$ because those are exactly the quaternions that commute with all other quaternions. Thus conjugating by them induce the trivial rotation.

Problem 13. Show that the product of two topological groups is a topological group.

Solution: Let G_1 and G_2 be two topologial groups with multiplication functions m_1 and m_2 respectively and inverse functions i_1 and i_2 respectively. Let $G = G_1 \times G_2$. Then the multiplication map $m : G \times G \to G$ is given by by $m((g_1, g_2), (g_1', g_2')) = (m_1(g_1, g_2), m_2(g_1', g_2'))$ and the inverse map $i : G \to G$ is given by $i(g_1, g_2) = (i_1(g_1), i_2(g_2))$. By Theorem 3.14 G is Hausdorff. We just need to show m and i are continuous. These facts both follow from the following:

Lemma: Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be continuous functions. Then the function $(f_1 \times f_2) : X_1 \times X_2 \to Y_1 \times Y_2$ given by $(f_1 \times f_2)(x_1, x_2) = (f_1(x_1), f_2(x_2))$ is continuous.

Proof: Let $U_1 \times U_2$ be a basic open set in $Y_1 \times Y_2$. Then $(f_1 \times f_2)^{-1}(U_1 \times U_2) = f_1^{-1}(U_1) \times f_2^{-1}(U_2)$ which is open in $X_1 \times X_2$.

Problem 14. Let G be a topological group. If H is a subgroup of G, show that its closure $\bar{H}$ is also a subgroup, and that if H is normal then so is $\bar{H}$.

Solution:

Define $f : G \times G \to G$ by $f(x, y) = xy^{-1}$. Since G is a topological group f is continuous. Therefore $f^{-1}(\bar{H})$ is closed. Since H is a subgroup, $H \times H \subseteq f^{-1}(\bar{H})$. So, taking closures $\overline{H \times H} \subseteq f^{-1}(\bar{H})$. Now by Chapter 3, Problem 20, $\overline{H \times H} = \bar{H} \times \bar{H}$. Thus $f(\bar{H} \times \bar{H}) \subseteq \bar{H}$. It follows that $\bar{H}$ is a subgroup of G.

Define $f : G \times G \to G$ by $f(x, y) = xyx^{-1}$. Since G is a topological group f is continuous. Therefore $f^{-1}(\bar{H})$ is closed. Since H is normal, $G \times H \subseteq f^{-1}(\bar{H})$. So, taking closures $\overline{G \times H} \subseteq f^{-1}(\bar{H})$. Again by Chapter 3, Problem 20, $\overline{G \times H} = \bar{G} \times \bar{H} = G \times \bar{H}$. Thus $f(G \times \bar{H}) \subseteq \bar{H}$. It follows that $\bar{H}$ is a normal subgroup of G.

Problem 15. Let G be a compact Hausdorff space which has the structure of a group. Show that G is a topological group if the multiplication function $m : G \times G \to G$ is continuous.

Solution: We must show that $i : x \mapsto x^{-1}$ is continuous. Let $C \subseteq G$ be closed. We will show $i^{-1}(C)$ is closed. Let $e \in G$ be the identity element. Since m is continuous and points are closed, $m^{-1}(\{e\})$ is closed. Let π_1 be the projection map from $G \times G$ to G. We claim π_1 is a closed map. If $A \subseteq G \times G$ is closed, then by Theorem 3.15 $G \times G$ is compact, so by Theorem 3.4 $\pi_1(A)$ is compact. And since G is Hausdorff, by Theorem 3.6 $\pi_1(A)$ is closed. Thus $\pi_1((G \times C) \cap m^{-1}(\{e\})) = i^{-1}(C)$ is a closed set.

Problem 16. Prove that $\mathrm{O}(n)$ is homeomorphic to $\mathrm{SO}(n) \times \mathbb{Z}_2$. Are these two isomorphic as topological groups?

Solution: Let $M \in \mathrm{O}(n)$. We know from the proof of Theorem 3.12 that $\det : \mathrm{O}(n) \to \mathbb{Z}_2$ is continuous. And since $\det(MN) = \det(M)\det(N)$, it is a homomorphism. Since $\mathrm{SO}(n) = \det^{-1}(1)$, $\mathrm{SO}(n)$ is open in $\mathrm{O}(n)$. Define $f : \mathrm{O}(n) \to \mathrm{SO}(n) \times \mathbb{Z}_2$ by $M \to (\det(M) \cdot M, \det(M))$. Then f is a well-defined homomorphism that is clearly one-to-one and onto. Let $U \subseteq \mathrm{SO}(n)$ be open. Since $\mathrm{SO}(n)$ is open in $\mathrm{O}(n)$, U is open in $\mathrm{O}(n)$. Let $-U = \{-M \mid M \in U\}$, which is also open in $\mathrm{O}(n)$. Then $f^{-1}(U \times \{0\}) = U$ and $f^{-1}(U \times \{1\}) = -U$. Thus $f^{-1}(V)$ is open for all basic open sets $V \in \mathrm{SO}(n) \times \mathbb{Z}_2$. Thus f is continuous. By Theorem 3.7 f is a homeomorphism.

Problem 17. Let A, B be compact subsets of a topological group. Show that the product set $AB = \{ab \mid a \in A, b \in B\}$ is compact.

Solution: The multiplication map $m : G \times G \to G$ is continuous. By Theorem 3.15 $A \times B$ is compact. By Theorem 3.4 $m(A \times B) = AB$ is compact.

Problem 18. If U is a neighborhood of e in a topological group, show there is a neighborhood V of e for which $VV^{-1} \subseteq U$.

Solution: Let $f : G \times G \to G$ be given by $f(x, y) = xy^{-1}$. Then $f^{-1}(U)$ is an open neighborhood of (e, e). Thus $\exists$ a basic open set $V \times V$ such that $(e, e) \in V \times V \subseteq f^{-1}(U)$. It follows that $VV^{-1} = f(V \times V) \subseteq U$.

Problem 19. Let H be a discrete subgroup of a topological group G (i.e. H is a subgroup, and is a discrete space when given the subspace topology). Find a neighborhood N of e in G such that the translates $hN = L_h(N)$, $h \in H$, are all disjoint.

Solution: Let $U \subseteq G$ be open such that $U \cap H = \{e\}$. By Problem 18 $\exists\ N \subseteq G$ open such that $NN^{-1} \subseteq U$. Suppose $h_1, h_2 \in H$ and $h_1N \cap h_2N \neq \emptyset$. Then $\exists\ v_1, v_2 \in V$ such that $h_1v_1 = h_2v_2$. This implies $v_1v_2^{-1} = h_1^{-1}h_2$. Thus $h_1^{-1}h_2 \in U \cap H$. But $U \cap H = \{e\}$. Thus $h_1^{-1}h_2 = e$. Thus $h_1 = h_2$.

Problem 20. If C is a compact subset of a topological group G, and if H is a discrete subgroup of G, show that $H \cap C$ is finite.

Solution: We first prove the following:

Lemma: If G is a topological group and H is a discrete subgroup, then H is a closed subset of G.

Proof: Let $f : G \times G \to G$ be given by $f(a, b) = ab^{-1}$. Since G is a topological group f is continuous. Since H is discrete $\exists$ a set U open in G such that $U \cap H = \{e\}$, where e is the identity element of G. Then $f^{-1}(U)$ is open and $(x, x) \in f^{-1}(U)$. Thus $\exists$ a set V open in G such thta $(x, x) \in V \times V \subseteq f^{-1}(U)$. Suppose $a, b \in V \cap H$. Then $f((a, b)) = ab^{-1} \in U \cap H$. Thus $ab^{-1} = e$. Thus $a = b$. Thus $V \cap H$ has at most one element. Thus $V \cap H = \emptyset$ or $V \cap H = \{y\}$. Let $W = V - \{y\}$. Since $x \notin H$, $x \neq y$. Thus $x \in W$. Since G is Hausdorff, $\{y\}$ is closed (Theorem 3.6, single points are compact sets). Thus W is open. Thus we have found an open set W such that $x \in W$ and $W \cap H = \emptyset$. $\square$

Now by the lemma H is closed, thus $H \cap C$ is closed in C. By Theorem 3.5 $H \cap C$ is compact. Since H is discrete, points in H are open. Thus the sets consisting of individual points in $C \cap H$ constitute an open cover of $C \cap H$ that has no subcover. Thus $C \cap H$ must be a finite set of points.

Problem 21. Prove that every nontrivial discrete subgroup of $\mathbb{R}$ is infinite cyclic.

Solution: Let H be a non-trivial discrete subgroup of $\mathbb{R}$. Then $\exists\ h \in H$, $h \neq 0$. Thus one of $h, -h$ is positive. Let $A = \{x \in H \mid x > 0\}$. Then A is not empty. Let $a = \inf A$. By the lemma in the previous problem H is closed in G. Thus $a \in H$. Thus $< a > \subseteq H$. Since H is discrete, $\exists\ \epsilon > 0$ such that $(-\epsilon, \epsilon) \cap H = \{0\}$. Thus $a > 0$. Suppose $b \in H$ and $b \notin < a >$. Then $\exists\ n \in \mathbb{N}$ such that $na < b < (n+1)a$. But then $0 < b - na < a$ and $b - na \in H$ contradicting the definition of a. Thus $H = < a >$.

Problem 22. Prove that every non-trivial discrete subgroup of the circle is finite and cyclic.

Solution: Let $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. Let $H \subseteq S^1$ be a non-trivial discrete subgroup. Let $A = \{\theta \in (0, 2\pi) \mid e^{i\theta} \in H\}$. Since $\exists\ h \in H$, $h \neq 1$, A is not empty. Let $\theta = \inf A$. By the lemma in the Problem 20, H is closed in G. Thus $e^{i\theta} \in H$. Thus $< e^{i\theta} > \subseteq H$. Since H is discrete, $\exists\ \epsilon > 0$ such that $(-\epsilon, \epsilon) \cap A = \emptyset$. Thus $\theta > 0$. Suppose $b \in H$ and $e^{i\phi} \notin < e^{i\theta} >$. Then $\exists\ n \in \mathbb{N}$ such that $n\theta < \phi < (n+1)\theta$. But then $0 < \phi - n\theta < \theta$ and $e^{i(\phi - n\theta)} \in H$ contradicting the definition of θ. Thus $H = < e^{i\theta} >$.

Problem 23. Let $A, B \in \mathrm{O}(2)$ and suppose $\det A = +1$, $\det B = -1$. Show that $B^2 = I$ and $BAB^{-1} = A^{-1}$. Deduce that every discrete subgroup of $\mathrm{O}(2)$ is either cyclic or dihedral.

Solution: Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the conditions $BB^T = I$ and $\det B = -1$ amount to the following four equations:

$$ad - bc = -1 \tag{1}$$

$$a^2 + b^2 = 1 \tag{2}$$

$$c^2 + d^2 = 1 \tag{3}$$

$$ac + bd = 0 \tag{4}$$

Suppose $a = 0$. Then by (4) $bd = 0$ and since B is non-singular it must be that $d = 0$ (and likewise if $d = 0$ we must have $a = 0$). We then have $b^2 = c^2 = 1$. This gives two possibilities for B, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$. In both cases $B^2 = I$.

Suppose $b = 0$. Then by (4) $ac = 0$ and since B is non-singular it must be that $c = 0$ (and likewise if $c = 0$ we must have $b = 0$). Thus $a^2 = d^2 = 1$. This gives two possibilities for B, $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. In both cases $B^2 = I$.

By the above we can assume $a, b, c, d \neq 0$.

Now $B^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + dc & bc + d^2 \end{bmatrix}$. Thus we want to show

$$a^2 + bc = 1 \tag{5}$$

$$ab + bd = 0 \tag{6}$$

$$ac + dc = 0 \tag{7}$$

$$bc + d^2 = 1 \tag{8}$$

From (4) we have $c = \frac{-bd}{a}$. Substituting into (1) we get $ad + \frac{b^2 d}{a} = -1$. Thus $a^2 d + b^2 d = -a$, which implies $(a^2 + b^2)d = -a$. Using (2) we get $d = -a$. Substituting for $-a$ for d into (1) we get $a^2 + bc = 1$, which is (5). And substituting $-d$ for a into (1) we get $d^2 + bc = 1$ which is (8).

From (4) we have $d = \frac{-ac}{b}$. Substituting into (1) we get $\frac{-a^2 c}{b} - bc = -1$ which implies $c(a^2 + b^2) = b$ and thus $c = b$. Thus substiuting b for c in (4) we get $ab + bd = 0$, which is (6). And substituting c for b in (4) we get $ac + cd = 0$ which is (7).

Thus in all cases if $\det B = -1$ and $BB^T = I$, we have $B^2 = I$. We have shown in particular that $B = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ where $a^2 + b^2 = 1$.

Now suppose $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then the conditions $AA^T = I$ and $\det A = 1$ amount to the following four equations:

$$ad - bc = 1 \tag{9}$$

$$a^2 + b^2 = 1 \tag{10}$$

$$c^2 + d^2 = 1 \tag{11}$$

$$ac + bd = 0 \tag{12}$$

As before, $c = 0 \Leftrightarrow b = 0$ and $a = 0 \Leftrightarrow d = 0$. It follows that the only cases where one of a, b, c or d is zero are $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

By the above we can assume $a, b, c, d \neq 0$. Solving (12) for a and substituting into (1) we get $b = -c$. Solving (12) for b and substituting into (1) we get $a = d$. Thus A is of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ where $a^2 + b^2 = 1$. And $A^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

And we can write $B = \begin{bmatrix} x & y \\ y & -x \end{bmatrix}$ with $x^2 + y^2 = 1$.

Multiplying out we get

$$\begin{aligned}
BAB^{-1} &= \begin{bmatrix} x & y \\ y & -x \end{bmatrix} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \begin{bmatrix} x & y \\ y & -x \end{bmatrix} \\
&= \begin{bmatrix} x & y \\ y & -x \end{bmatrix} \begin{bmatrix} ax + by & ay - bx \\ -bx + ay & -by - ax \end{bmatrix} \\
&= \begin{bmatrix} ax^2 + bxy - bxy + ay^2 & axy - bx^2 - by^2 - ayx \\ axy + by^2 + bx^2 - axy & ay^2 - bxy + bxy + ax^2 \end{bmatrix} \\
&= \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = A^{-1}
\end{aligned}$$

It remains to show every discrete subgroup of O(2) is cyclic or dihedral. Let H be a discrete subgroup of O(2). By Theorem 4.13 O(2) is compact. Therefore, by Problem 20 we know H is finite. If every element of H has determinant equal to one, then H is a subgroup of SO(2). By the note after Theorem 4.13 we know that SO(2) is homeomorphic to S^1 and by Problem 22 we know every discrete subgroup of S^1 is cyclic. Thus suppose $\exists\, B \in H$ such that $\det B = -1$. We must show H is dihedral.

Let $K = H \cap \mathrm{SO}(2)$. Then K is cyclic and consists of the elements of H with determinant equal to one. Let M be a generator of K. Let N be a element of $H - K$. Then $\det N = -1$. Since $N^2 = I$ and $NMN^{-1} = M^{-1}$, the subgroup $<M, N>$ generated by N and M is dihedral. Thus we will be done if we show $H =<M, N>$. Suppose $L \in H - <M, N>$. Then $L \notin K$ thus $\det L = -1$. Then $\det LMN = 1$. Thus $LMN \in K$. Say $LMN = R \in K$. Then $L = RN^{-1}M^{-1} \in H$. This contradicts the choice of L. Thus $H - <M, N>= \emptyset$. Thus $H =<M, N>$ is dihedral.

Problem 24. If T is an automorphism of the topological group $\mathbb{R}$ (i.e., T is a homeomorphism which is also a group isomorphism) show that $T(r) = rT(1)$ for any rational number r. Deduce that $T(x) = xT(1)$ for any real number x, and hence that the automorphism group of $\mathbb{R}$ is isomorphic to $\mathbb{R} \times \mathbb{Z}_2$.

Solution:

Let $A = \operatorname{Aut}\mathbb{R}$, the automorphism group of $\mathbb{R}$. Let $n \in \mathbb{N}$. It follows immediately from the fact that T is a homomorphism that $T(nx) = nT(x)$. Also $T(0) = T(0+0) = T(0) + T(0)$. Thus $T(0) = 0$. Thus $T(0 \cdot 1) = 0T(1)$ (both sides are zero). Also $0 = T(0) = T(x - x) = T(x) + T(-x)$. Thus $T(-x) = -T(x)$. Therefore $T(-nx) = -nT(x)$. Now $T(1) = T(n\frac{1}{n}) = T(\sum_{i=1}^{n} \frac{1}{n}) = \sum_{i=1}^{n} T(\frac{1}{n}) = nT(\frac{1}{n})$. Thus $T(\frac{1}{n}) = \frac{1}{n}T(1)$. Thus $T(\frac{n}{m}) = nT(\frac{1}{m}) = \frac{n}{m}T(1)$. Thus $T(r) = rT(1)$ for all $r \in \mathbb{Q}$. Now let $x \in \mathbb{R}$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, $\exists$ a sequence of rational numbers $r_n \to x$. Since T is continuous, $T(r_n) \to T(x)$. But $T(r_n) = r_n T(1) \to xT(1)$. Thus $T(x) = xT(1)$.

It follows that every automorphism of $\mathbb{R}$ is determined completely by its value at $x = 1$. As long as $x \neq 0$, then $y \overset{T}{\mapsto} yx$ is an automorphism of $\mathbb{R}$ such that $T(1) = x$. Thus the correspondence $T \mapsto T(1)$ gives a well-defined onto function $f : A \to \mathbb{R} - \{0\}$. If we define an operation on A given by composition of functions, then A is a group. And $\mathbb{R} - \{0\}$ is a group with respect to multiplication. Clearly $f(xy) = f(x)f(y)$. Thus A is isomophic as a group to $\mathbb{R} - \{0\}$. We will be done if we show $\mathbb{R} - \{0\}$ is isomorphic to $\mathbb{R} \times \mathbb{Z}_2$. Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Then $\mathbb{R}^+$ is a group with respect to the multiplication operation and $x \mapsto e^x$ is an isomorphism from $\mathbb{R}$ (as a group with respect to the addition operation) to $\mathbb{R}^+$ (as a group with respect to the multiplication operation). Let $g : \mathbb{R} \times \mathbb{Z}_2 \to \mathbb{R} - \{0\}$ be given by $g(x, y) = e^x$ if $y = 0$ and $-e^x$ if $y = 1$. Now $\mathbb{R} \times \mathbb{Z}_2$ is a group with respect to addition and $\mathbb{R} - \{0\}$ is a group with respect to multiplication. Thus g is a homomorphism. And g is one-to-one and onto. Therefore g is an isomorphism.

Problem 25. Show that the automorphism group of the circle is isomorphic to $\mathbb{Z}_2$.

Solution: Let $f : S^1 \to S^1$ be an automorphism. Let $a = e^{i\theta}$ and $b = e^{i\phi}$ in S^1 where $0 \leq \theta \leq \phi \leq 2\pi$. Let $I_{a,b} = \{e^{ix} \mid \theta \leq x \leq \phi\}$. In words $I_{a,b}$ is the closed segment of the circle going counter-clockwise from a to b. So for example $I_{1,-1}$ is the upper half of the circle $\{z \in S^1 \mid \operatorname{im} z \geq 0\}$, and $I_{-1,1}$ is the lower half of the circle $\{z \in S^1 \mid \operatorname{im} z \leq 0\}$. By the intermediate value theorem and the fact that f is one-to-one, it must be that

$$f(I_{a,b}) = I_{f(a),f(b)} \text{ or } f(I_{a,b}) = I_{f(b),f(a)} \tag{1}$$

Let $n \in \mathbb{N}$. There are exactly n solutions to the equation $x^n - 1 = 0$ in $\mathbb{C}$, called n-th roots of unity, evenly spaced around the circle. If $z^n = 1$ then $1 = \overline{z^n} = \overline{z}^n$. Thus the conjugate of an n-th root of unity is another n-th root of unity. We will prove by induction on n that for every root of unity z, $f(z) = z$, or for every root of unity $f(z) = \overline{z}$. Precisely let the statement $S(n)$ be:

$f(z) = z$ $\forall$ n-th roots of unity z and for *consecutive* n-th roots of unity a and b, $f(I_{a,b}) = I_{a,b}$
or
$f(z) = \overline{z}$ $\forall$ n-th roots of unity z and for *consecutive* n-th roots of unity a and b, $f(I_{a,b}) = \overline{I_{a,b}}$.

Base case $n = 2$. The square roots of unity are ± 1. Since f is a homomorphism of groups $f(1) = 1$. Since $(-1)^2 = 1$, $1 = f(1) = f((-1)^2) = (f(-1))^2$ Thus $f(-1) = \pm 1$. Since $f(1) = 1$ it cannot be that also $f(-1) = 1$, thus $f(-1) = -1$. Thus f fixes 1 and -1. Then by (1) $f(I_{1,-1}) = I_{1,-1}$ or $f(I_{1,-1}) = I_{-1,1}$. The base case $n = 2$ therefore is true.

Assume $S(m)$ is true $\forall$ $m < n$. Let z be an n-th root of unity. Then $z^n = 1$, so $1 = f(z^n) = (f(z))^n$. Thus $f(z)$ is also an n-th root of unity. Let $a, b \in S^1$ be *consective* $(n-1)$-st roots of unity. Then there is one and only one n-th root of unity $z \in I_{a,b}$. By the induction hypothesis f either fixes a and b or sends them to their (respective) conjugates. If it fixes a and b then, also by the induction hypothesis, $f(I_{a,b}) = I_{a,b}$. Thus the only n-th root of unity in $f(I_{a,b})$ is z. Since $f(z) \in I_{a,b}$ must be an n-th root of unity, it must be that $f(z) = z$. If f sends a and b to their conjugates, then $f(I_{a,b}) = \overline{I_{a,b}} = I_{\overline{b},\overline{a}}$ and $\overline{b}, \overline{a}$ are consecutive $(n-1)$-st roots of unity, thus $I_{\overline{b},\overline{a}}$ also contains one and only one n-th root of unity, $\overline{z}$. Since $f(z) \in I_{\overline{b},\overline{a}}$ must be an n-th root of unity, it follows that $f(z) = \overline{z}$. Thus we have shown that f either fixes all of the n-th roots of unity or sends them all to their conjugates. The required behavior of f on the intervals between consecutive n-th roots of unity follows from (1).

Thus we have shown that f fixes *all* of the roots of unity. The roots of unity correspond to the rational numbers in $[0, 1]$ under the map $x \mapsto e^{2\pi i x}$. Thus the roots of unity are dense in S^1. Therefore if f fixes all of the roots of unity, and f is continuous, then f must be the identity function. And if f sends all roots of unity to their conjugates, then f must send all elements of S^1 to their conjugates. Thus the identity function and conjugation are the only two possibilities for f.

Section 4.4 - Orbit Spaces

Notes

Page 79 The top of the page it says without proof that $O(n) \times S^{n-1} \to S^{n-1}$ given by $(A, x) \mapsto Ax$ is continuous. This is continuous since both $O(n)$ and S^{n-1} are subsets of euclidean space and the function $(A, x) \mapsto Ax$ is given by polynomials in the coordinates.

Page 82 The first line of the page it says "one easily checks that $f^{-1}(x)$ is precisely the left coset of $AO(n-1)$, where $A \in O(n)$ satisfies $A(e_1) = x$. Clearly A is in $f^{-1}(x)$. And since everything in $O(n-1)$ fixes e_1, it is clear everything in the coset has the same effect on x. On the other hand, if $B(e_1) = x$ then $A^{-1}B$ fixes e_1 and is thus in $O(n-1)$. Thus $B \in AO(n-1)$.

Page 82 At the end of the first paragraph it says "a similar argument gives $SO(n)/SO(n-1) \cong S^{n-1}$. The main issue is that we have to show there is an element $A \in SO(n)$ such that $A(e_1) = x$. We know there is such an A in $O(n)$. If $\det A = 1$ then we're done, otherwise let M be the element of $O(n)$ that sends e_2 to $-e_2$ and fixes all other e_i's. Then $\det M = -1$ and $M \in SO(n)$ and $\det AM = (-1)(-1) = 1$. Thus $AM \in SO(n)$ and $AM(e_1) = x$. The rest of the proof is identical then to the $SO(n)$ case.

Page 83 Last line of first paragraph. It says "which is clearly impossible." I think it should say "which only happens when $r = s$.

Page 83 At the end of example 7 it says "we leave it to the reader to check for himself that the orbit is a proper dense subset of T.

We first show $\{m + n\sqrt{2} \mid n, m \in \mathbb{Z}\}$ is dense in $\mathbb{R}$. Let $x \in \mathbb{R}$ and $\epsilon > 0$. Find $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$. For each $i = 1, 2, \ldots, n+1$ find $m_i \in \mathbb{Z}$ such that $0 < m_i + i\sqrt{2} < 1$. We have strict inequalities because if $m_i + i\sqrt{2}$ were 0 or 1 then $\sqrt{2}$ would be rational. Let $A = \{m_i \mid i = 1, 2, \ldots, n+1\}$. Now by the pigeon hole principle, there must be some $j \in \{0, 1, \ldots, n-1\}$ such that the interval $(\frac{j}{n}, \frac{j+1}{n})$ contains two elements $a, b \in A$. Assume wlog that $b < a$. Then $0 < a - b < \frac{1}{n} < \epsilon$ and $a - b$ is of the form $m + n\sqrt{2}$. Thus multiples of $(a - b)$ cannot skip over an interval of length ϵ. Thus $k(a - b)$ must be in the interval $(x - \epsilon, x + \epsilon)$ for some $k \in \mathbb{Z}$.

Now let $I = [0, 1]$ and let $(x, y) \in I \times I$. We must show $\exists\ r \in \mathbb{R}$ and $k, \ell \in \mathbb{Z}$ such that $\|(r, r\sqrt{2}) - (x + k, y + \ell)\| < \epsilon$. Find $n, m \in \mathbb{Z}$ such that $|(x\sqrt{2} - y) - (m + n\sqrt{2})| < \epsilon$. Then let $r = x - n$, $k = -n$, $\ell = m$. Then

$$\|(r, r\sqrt{2}) - (x + k, y + \ell)\|$$

$$= \|(x - n, (x - n)\sqrt{2}) - (x - n, y + m)\|$$

$$= |(x\sqrt{2} - y) - (m + n\sqrt{2})| < \epsilon$$

Page 83 Example 8, bottom of page. Here is talks about "isometries" of the plane. Those are functions that preserve distances $d(a, b) = d(f(a), f(b))$.

Page 84 The second sentence of the last paragraph says "We assume as known the fact that an isometry can be written as an ordered pair (θ, v) where $\theta \in O(2)$ and $v \in \mathbb{E}^2$." I'm not sure why he didn't include the proof of this fact, I found the following elementary proof of it on `math.stackexchange.com`:

For $a \in \mathbb{E}^2$ let $T_a(y) = y + a$. Let P be an isometry of the plane. Let $P(0) = x$. Then define $U = T_{-x} \circ P$. Note that $U(0) = 0$ and U is an isometry. Now by the isometry definition, $\|U(y)\| = d(U(y), U(0)) = d(y, 0) = \|y\|$. We see that since an isometry preserves triangles, that $U(x + y) = U(x) + U(y)$, and clearly $aU(x) = U(ax)$. Thus our map U is a linear transformation. Now consider the image of $(1, 0)$. Since it lies on the unit circle (points of norm 1), its image does to, and thus $U(1, 0) = (\cos(\theta), \sin(\theta))$. Likewise $U(0, 1) = (\cos(\phi), \sin(\phi))$. Now we note that since isometries preserve angles, $|\theta - \phi| = \pi/2$. Now either we have that $\phi = \theta + \pi/2$, and we can see that U is rotation by θ, or $\phi = \theta - \pi/2$ and we can see that it is the product of a reflection and then rotation by θ. Now multiplying U by T_x to get P, we have the desired result.

Page 85 In the second to last paragraph of the section he claims a certain set is a convex polygon. I don't know why he omitted the proof but for any points p and q, $\{x \in \mathbb{R}^2 \mid \|x - p\| \leq \|x - q\|\}$ is a halfplane (or the whole plane if $p = q$), so the set is an intersection of halfplanes, hence a convex. The fact that it's bounded follows from the assumption that $\mathbb{E}^2/G$ is compact. Thus it is a polygon.

Problem 26. Give an action of $\mathbb{Z}$ on $\mathbb{E}^1 \times [0, 1]$ which has the Möbius strip as orbit space.

Solution Let $z \in \mathbb{Z}$. The define $z(x, y) = (x + z, y)$ if z is even and $z(x, y) = (x + z, 1 - y)$ is z is odd. Equivalently $z(x, y) = (x + z, \frac{1}{2}((-1)^z(2y - 1) + 1))$. Then the set $[0, 1] \times [0, 1]$ has representatives from every orbit, and $(0, y)$ is identified with $(1, 1 - y)$. The resulting orbit space is therefore the mobius strip.

Problem 27. Find an action of $\mathbb{Z}_2$ on the torus with orbit space the cylinder.

Solution: Represent the torus in $\mathbb{E}^3$ as in Figure 4.3 (page 80). Using the notation of example 3 on page 81, define $g(x, y, z) = (x, -y, z)$. Reflection through the x-z-plane. Let $A = \{(x, y, z) \in T \mid y \geq 0\}$. Then A is one-to-one with the orbits. Unlike the cases in example 3, no point on the boundary of A is identified with any other point on the boundary of A (or any other point in A for that matter). Thus A is homeomorphic to orbit space and it is obviously homeomorphic to the cylinder.

Problem 28. Describe the orbits of the natural action of $\mathrm{SO}(n)$ on $\mathbb{E}^n$ as a group of linear transformations, and identify the orbit space.

Solution: Let $r \in \mathbb{R}$, $r \geq 0$. Let $S_r = \{p \in \mathbb{E}^n \mid \|p\| = r\}$. Then since $\mathrm{SO}(n)$ preserves distances, $\mathrm{SO}(n)$ must take S_r to itself. Furthermore, the action on S_r is transitive, because it is transitive on $S^{n-1} \subseteq \mathbb{E}^n$ and $S_r = r \cdot S^{n-1}$. To see the action on S^{n-1} is transitive, for any vector $\mathbf{v} \in S^{n-1}$, it can be put into an orthnormal basis $\mathbf{B}$. Then there is a change of coordinates matrix M from the standard basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ to $\mathbf{B}$ that takes $\mathbf{e}_1$ to v. Since both bases are orthnormal, $M \in \mathrm{O}(n)$. Clearly M can be chosen to be in $\mathrm{SO}(n)$ such that $M(\mathbf{e}_1) = v$ (if it's not already in $\mathrm{SO}(n)$, just multiply one of the other $\mathbf{e}_i$'s by -1). Since any element of S^{n-1} can be taken to $\mathbf{e}_1$, the action must be transitive. Now if $\mathbf{v} \in \mathbb{E}^n$ is arbitrary ($v \neq 0$) then just scale v to be in S^{n-1}, transform within S^{n-1} and scale back. The two scaling operations assure that the resulting transformation has determinant equal to one, and therefore is in $\mathrm{SO}(n)$.

Now let $r, r' \in \mathbb{R}$, $r, r' > 0$, $r \neq r'$. Since things in $\mathrm{SO}(n)$ are length preserving, an element of $\mathrm{SO}(n)$ cannot take an element of S_r to an element of $S_{r'}$. Thus each S_r is exactly one orbit (true also if $r = 0$ since S_0 consists of one point).

Let $f : \mathbb{E}^n \to [0, \infty)$ be given by $f(\mathbf{v}) = \|\mathbf{v}\|$. Then f is a continuous function that identifies each orbit of the action to a single point. Let B be an open ball in $\mathbb{E}^n$. Then clearly $f(B)$ is an interval, open in $[0, \infty)$. Since functions respect unions, it follows that f is an open map. By Corollary 4.4 f is an identification map. Thus the identification space is homeomorphic to the image of f, which is $[0, \infty)$. Thus the orbit space of $\mathrm{SO}(n)$ on $\mathbb{E}^n$ is homeomorphic to $[0, \infty)$.

Problem 29. If $\pi : X \to X/G$ is the natural identification map, and if O is open in X, show that $\pi^{-1}(\pi(O))$ is the union of the sets $g(O)$ where $g \in G$. Deduce that π takes open sets to open sets. Does π always take closed sets to closed sets?

Solution: For the first part we don't need to use the fact that O is open, so for the first part assume O is an arbitrary subset of X.

Let $a \in O$ and $g \in G$. Since ga is in the same orbit as a, $\pi(ga) = \pi(a)$. Thus $\pi(g(O)) = \pi(O)\ \forall\ g \in G$. Thus $g(O) \subseteq \pi^{-1}(\pi(O))$ $\forall\ g \in G$. Thus $\cup_{g \in G} g(O) \subseteq \pi^{-1}(\pi(O))$. Now suppose $x \in \pi^{-1}(\pi(O))$. Then $\pi(x) \in \pi(O)$. Thus x is in the same orbit as some element of O. Thus $x = ga$ for some $g \in G$ and $a \in O$. Thus $x \in g(O)$. Thus $\pi^{-1}(\pi(O)) \subseteq \cup_{g \in G} g(O)$. Since we have containment in both directions, we can conclude that $\pi^{-1}(\pi(O)) = \cup_{g \in G} g(O)$.

Now suppose O is open in X. Recall a set $U \subseteq X/G$ is open in $X/G \Leftrightarrow \pi^{-1}(U)$ is open in X. Now $\pi^{-1}(\pi(O))$ is a union of sets of the form $g(O)$ and (since each g induces a homeomorphism of X) $g(O)$ is open in X $\forall\, g \in G$. Thus $\pi^{-1}(\pi(O))$ is open. Thus $\pi(O)$ is open in X/G. Thus π is an open map.

We will show by counter-example that π is not a closed map. Let $\mathbb{Z}$ act on $\mathbb{R}$ by translation $x \mapsto x + z$. The orbit space is S^1. For each $n = 0, 1, 2, \ldots$, let $A_n = [n + \frac{1}{n+3}, n + \frac{1}{n+2}]$. So

$$A_0 = [1/3, 1/2]$$

$$A_1 = [1 + \frac{1}{4}, 1 + \frac{1}{3}]$$

$$A_2 = [2 + \frac{1}{5}, 2 + \frac{1}{4}]$$

$$\text{etc } \ldots$$

Let $A = \cup_n A_n$. Then A is closed in $\mathbb{E}^1$, but $\pi(A) = \pi((0, 1/2])$ which is not a closed subset of S^1, since (considering $S^1 \subseteq \mathbb{C}$) it does not contain $z = 1$, but it contains all points with $\operatorname{im} z < 0$.

Problem 30. Show that X may be Hausdorff yet X/G non-Hausdorff. If X is a compact topological group and G is a closed subgroup acting on X by left translation, show that X/G is Hausdorff.

Solution: Let $X = \mathbb{R}$ and $G = \mathbb{Q}$. Let G act on X by translations (see example 4 on page 81). Let π be the identification map. For $A \subseteq X$ let $\overline{A} = \pi(A)$. Then $\mathbb{Q}$ represents exactly one subset of the partition, so $\overline{\mathbb{Q}}$ is a single point in X/G. Now suppose $U \subseteq X/G$ is an open set. Then $\pi^{-1}(U)$ is open in X. So there is an open interval $(a, b) \subseteq X$ such that $(a, b) \subseteq \pi^{-1}(U)$. Now $\mathbb{Q} \cap (a, b) \neq \emptyset$. Thus $\overline{\mathbb{Q}} \in U$. Thus every open set in X/G contains the point $\overline{\mathbb{Q}}$. It follows that every pair of open sets in X/G have non-empty intersection. Thus X/G cannot be Hausdorff.

Now suppose G is a closed subgroup acting by translations. We will show X/G is Hausdorff.

Note: I don't think we need X to be compact for this. I have the following proof that does not require compactness. Maybe there's a mistake but I can't find one.

Let $C = \{(x, y) \in X \times X \mid x^{-1}y \in G\}$. Let $h : X \times X \to X$ be the map $h(x, y) = x^{-1}y$. Then $h^{-1}(G) = C$. Since G is closed and h is continuous, it follows that C is closed. Let $f : X \to X/G$ be the identification map. Let $g : X \times X \to X/G \times X/G$ be the map $g(x, y) = (f(x), f(y))$. By Problem 29 f is an open map. It follows that g is an open map. Thus by Theorem 4.3 g is an identification map. Let Δ be the diagonal in $X/G \times X/G$. Then $g^{-1}(\Delta) = C$. Since g is an identification map and C is closed in $X\times X$, it follows that Δ is closed in $X/G\times X/G$. By Chapter 3, Problem 25 (page 55) it follows that X/G is Hausdorff.

Problem 31. The stabilizer of a point $x \in X$ consists of those elements $g \in G$ for which $g(x) = x$. Show that the stabilizer of any point is a closed subgroup of G when X is Hausdorff, and that points in the same orbit have conjugate stabilizers for any X.

Note: In the 1987 edition X was not required to be Hausdorff. But this was a mistake, one can construct a counter-example. For example $\mathbb{R}$ acting on $\mathbb{R}/\mathbb{Q}$ by translation. The stabilizer of $\overline{\mathbb{Q}}$ is $\mathbb{Q}$ which is not closed in $\mathbb{R}$. Note that $\mathbb{R}/\mathbb{Q}$ is not Hausdorff (see Problem 30).

Solution: If g and g' are in the stabilizer of x, then $gg'x = gx = x$ so gg' is in the stabilizer of x. And $g^{-1}gx = 1 \cdot x = x$, but also $g^{-1}gx = g^{-1}x$. Thus $g^{-1}x = x$ so g^{-1} is in the stabilizer of x. It follows that the stabilizer of x is a subgroup of G. Now, let $f : G \to X$ be given by $f(g) = gx$. Then f is continuous. Since X is Hausdorff, by Theorem 3.6 points are closed (finite sets are always compact). Thus $f^{-1}(x)$ is closed in X. But $f^{-1}(x)$ is exactly the stabilizer of x. Thus the stabilizer of x is closed in X.

It remains to show points in the same orbit have conjugate stabilizers. Let x, y be in the same orbit, so $x = gy$ for some $g \in G$. Let $a \in \operatorname{stab}(x)$. Then $g^{-1}agy = g^{-1}ax = g^{-1}x = y$. Thus $g^{-1}ag \in \operatorname{stab}(y)$. Thus $g^{-1}\operatorname{stab}(x)g \subseteq \operatorname{stab}(y)$. Now let

$a \in \text{stab}(y)$. Then $gag^{-1}x = gay = gy = x$ so $gag^{-1} \in \text{stab}(x)$. Thus $\text{stab}(y) \subseteq g^{-1}\text{stab}(x)g$. Since we have set containment in both directions it follows that $g^{-1}\text{stab}(x)g = \text{stab}(y)$.

Problem 32. If G is compact, X is Hausdorff, and G acts transitively on X, show that X is homeomorphic to the orbit space $G/(\text{stabilizer of } x)$ for any $x \in X$.

Solution: Let $x \in X$. Let $f : G \to X$ be given by $f(g) = gx$. Since G acts transitively, f is onto. Since G is compact and X is Hausdorff, f is an identification map (Corollary 4.4). So G^*, the identification space associated to f, is homeomorphic to X. Suppose $f(g_1) = f(g_2)$. Then $g_2^{-1}g_1$ is in the stabilizer of x. Thus g_1 is in the same coset as g_2 with respect to the subgroup (stabilizer of x). Thus G^* is exactly $G/(\text{stabilizer of } x)$.

Problem 33. Let p, q be integers which have highest common factor 1. Let P be a regular polygonal region in the plane with center of gravity at the origin and vertices $a_0, a_1, \ldots, a_{p-1}$, and let X be the solid double pyramid formed from P by joining each of its points by straight lines to the points $b_0 = (0, 0, 1)$ and $b_q = (0, 0, -1)$ of $\mathbb{E}^3$ (see Fig. 4.6). Identify the triangles with vertices a_i, a_{i+1}, b_0, and a_{i+q}, a_{i+q+1}, b_q for each $i = 1, \ldots, p-1$, in such a way that a_i is identified to a_{i+q}, a_{i+1} to a_{i+q+1}, and b_0 to b_q. (The subscripts $i+1$, $i+q$, $i+q+1$ are course read mod p.) Prove that the resulting space is homeomorphic to the Lens space $L(p, q)$.

Solution: First off, I think there's a problem with this description when $p = 2$. But regardless, I'm stumped. I found the following proof online (credit to Mariano Surez-Alvarez), however I don't understand it too well. I've ordered a book where apparently the proof can be found, I will update this if I find a proof with more details. For now I'll call this a sketch.

Sketch of Proof: The sphere can be identified with

$$S^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\},$$

and the action of a generator of $\mathbb{Z}_p$ is then given by

$$(z, w) \mapsto (\lambda z, \lambda^q w)$$

with λ a primitive pth root of unity.

The orbit of each point of S^3 has a point (z, w) such that the argument of z is in $[0, 2\pi/p]$. Consider the subset L of S^3 of such points. If you look at it correctly, you will see it is a (curved) bipiramid, whose central vertical axis is the curve of points of the form $(z, 0)$ with z of modulus 1. Moreover, it is easy to see that the lens space can be obtained from L by doing identifications along its boundary, which is the set of points of the form (z, w) in S^3 with z of argument either 0 or $2\pi/p$.

If you work out exactly *what* identifications are induced by the action of the group, you will find the alternate.

Problem 34. Show that $L(2, 1)$ is homeomorphic to $\mathbb{P}^3$. If p divides $q - q'$, prove that $L(p, q)$ is homeomorphic to $L(p, q')$.

Solution: Using the definition of $L(2, 1)$ on page 82, $L(2, 1)$ is an identification space of S^3, the orbit space under the action of $\mathbb{Z}_2$. The action of the generator of $\mathbb{Z}_2$ is the homeomorphism $(z_0, z_1) \mapsto (e^{\pi i}z_0, e^{\pi i}z_1) = (-z_0, -z_1)$. And $(-z_0, -z_1)$ is the antipodal point to (z_0, z_1). Thus by example 2 on page 80, the identification space is exactly $\mathbb{P}^3$.

Now suppose $p|(q - q')$. Then $q = q' + np$ for some $n \in \mathbb{N}$. Thus $e^{2\pi qi/p} = e^{2\pi(q'+np)i/p} = e^{2\pi q'i/p + 2\pi npi/p} = e^{2\pi q'i/p}e^{2\pi ni} = e^{2\pi q'i/p}$. Thus the action of $\mathbb{Z}_p$ is identical in both cases. Thus the resulting orbit spaces are identical (and therefore homeomorphic).

Chapter 5 - The Fundamental Group

Section 5.1 - Homotopic maps

Problem 1. Let C denote the unit circle in the plane. Suppose $f : C \to C$ is a map which is not homotopoic to the identity. Prove that $f(x) = -x$ for some point x of C.

Solution: C is just S^1. Suppose that $f(x) \neq -x \ \forall\ x$. We can apply the homotopy of example 2, page 89 to f and $g(x) = x$. The assumption imples $f(x)$ and $g(x) = x$ are never antipodal. It follows that f and g are homotopic. But by assumption f is not homotopic to the identity. Thus there must be at least one x such that $f(x) = -x$.

Problem 2. With C as above, show that the map which takes each point of C to the point diametrically opposite is homotopic to the identity.

Solution: Let $f(x) = -x$. Then f is the same as rotating the circle 180°. So we just need to parameterize rotation by $t \in [0, 1]$ so that $t = 0$ is a rotation of 180° and $t = 1$ is a rotation of 0°. Rotation of θ radians is given by multiplying by the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$. Let $F : C \times I \to C$ be given by $F((x, y), t) = (x\cos(\pi(1-t)) - y\sin(\pi(1-t)), x\sin(\pi(1-t)) + y\cos(\pi(1-t)))$. Then F is the desired homotopy.

Problem 3. Let D be the disc bounded by C, parametrize D using polar coordinates, and let $h : D \to D$ be the homeomorphism defined by $h(0) = 0$, $h(r, \theta) = (r, \theta + 2\pi r)$. Find a homotopy F from h to the identity map such that the functions $F|_{D\times\{t\}} D \times \{t\} \to D$, $0 \le t \le 1$, are all homeomorphisms.

Solution: Let $i(r, \theta) = (r, \theta)$ the identity map. Define $F : D \times I \to D$ by $F((r, \theta), t) = (r, \theta + 2\pi r(1 - t))$. F is given by polynomials in r, θ and t, so F is continuous. And $F((r, \theta), 0) = h(r, \theta)$ and $F((r, \theta), 1) = (r, \theta) = i(r, \theta)$. Thus F is a homotopy between h and i. Since $F|_{D\times\{t\}} \to D$ is a one-to-one continuous map from a compact space to a Hausdorff space, Theorem 3.7 implies that $F|_{D\times\{t\}} \to D$ is a homeomorphism.

Problem 4. With the terminology of Problem 3, show that h is homotopic to the identity map relative to C.

Solution: The map h rotates the circle of radius r by $2\pi r$ radians. So $h(1, \theta)$ acts as the identity on C. The homotopy we gave in Problem 3 does not fix C for all t, we will have to find a different one. Define $F : D \times I \to D$ by $F((r, \theta), t) = (r, \theta + 2\pi r^{1-t})$. Since $t \in [0, 1]$ this is a continuous function of r, θ and t. Then $F((r, \theta), 0) = (r, \theta + 2\pi r) = h(r, \theta)$ and $F((r, \theta), 1) = (r, \theta + 2\pi) = (r, \theta)$. So $F((r, \theta), 1)$ is the identity function as a function of r and θ. Finally, note that $F((1, \theta), t) = (1, \theta + 2\pi) = (1, \theta)$. So F fixes C for all values of t.

Problem 5. Let $f : X \to S^n$ be a map which is not onto. Prove that f is *null homotopoic*, that is to say f is homotopic to a map which takes all of X to a single point of S^n.

Solution: Let p be a point in S^n such that its antipodal point $-p$ is not in the image of f. Now let $g : X \to S^n$ be the constant function $g(x) = p$. Then $g(x)$ and $f(x)$ never give a pair of antipodal points for any $x \in X$. By example 2 on page 89, f and g are homotopic.

Problem 6. As usual, CY denotes the cone on Y. Show that any two maps $f, g : X \to CY$ are homotopic.

Solution: