Math 4431, Exam 3: Ch. 4 (practice) Name and section: ________________

1. (20 points) Let $Q = [0, 2\pi] \times [0, 2\pi] \subset \mathbb{R}^2$, and let $Q_0 = \{\{(x, y)\} : 0 < x < 2\pi, 0 < y < 2\pi\}$ be the singleton partition of the interior of Q. Furthermore, set

$$\begin{aligned} X_0 &= \{(x, 0) : 0 < x < 2\pi\}, \\ X_1 &= \{(x, 2\pi) : 0 < x < 2\pi\}, \\ Y_0 &= \{(0, y) : 0 < y < 2\pi\}, \quad \text{and} \\ Y_0 &= \{(2\pi, y) : 0 < y < 2\pi\}. \end{aligned}$$

These are the (open) sides of Q. Finally, let

$$Q_1 = \{\{(0, 0), (0, 2\pi), (2\pi, 0), (2\pi, 2\pi)\}\}$$

be the set containing the set of corners.

Note that the map $(x, 0) \mapsto (x, 2\pi)$ identifies X_0 and X_1 **in the same direction** while the map $(x, 0) \mapsto (2\pi - x, 2\pi)$ identifes X_0 and X_1 **in the opposite direction**. Similarly, the pair of sides Y_0 and Y_1 may be identified either in the same direction or the opposite direction. Thus, we have two possible identifications of two pairs of sides of Q; this makes four possible combinations of identifications. List the associated identification spaces for each possible choice (give the associated partition) and identify the space. For example, here is the first partition corresponding to identifying both pairs of sides in the same direction:

$$\mathcal{P}_1 = Q_0 \cup \{\{(x, 0), (x, 2\pi)\} : 0 < x < 2\pi\} \cup \{\{(0, y), (2\pi, y)\} : 0 < y < 2\pi\} \cup Q_1.$$

You need to identify this space and then repeat the same procedure for the other three possibilities. Hint: Draw some pictures of squares to represent the identification of sides.

Solution: The first space $\mathcal{P}_1$ is the (one holed) torus, $\mathbb{T}^2$.

The partition for identifying X_0 with X_1 in the same direction, but Y_0 with Y_1 in the opposite direction is

$$\mathcal{P}_2 = Q_0 \cup \{\{(x, 0), (x, 2\pi)\} : 0 < x < 2\pi\} \cup \{\{(0, y), (2\pi, 2\pi - y)\} : 0 < y < 2\pi\} \cup Q_1.$$

This is the Klein bottle.

Identifying X_0 and X_1 in the opposite direction and Y_0 and Y_1 in the same direction also gives the Klein bottle in the form

$$\mathcal{P}_3 = Q_0 \cup \{\{(x, 0), (2\pi - x, 2\pi)\} : 0 < x < 2\pi\} \cup \{\{(0, y), (2\pi, y)\} : 0 < y < 2\pi\} \cup Q_1.$$

Identifying both sides in the opposite direction gives

$$\begin{aligned} \mathcal{P}_4 = Q_0 &\cup \{\{(x, 0), (2\pi - x, 2\pi)\} : 0 < x < 2\pi\} \cup \{\{(0, y), (2\pi, y)\} : 0 < y < 2\pi\} \\ &\cup \{\{(0, 0), (2\pi, 2\pi)\}\} \cup \{\{(0, 2\pi), (2\pi, 0)\}\}. \end{aligned}$$

This is the projective plane.

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2. (20 points) Assume $f : X \twoheadrightarrow Y$ is continuous and surjective. Let $p : X \to \mathcal{P}$ be the generalized projection onto the partition

$$\mathcal{P} = \{f^{-1}(\{y\}) : y \in Y\}.$$

Show that if Y is Hausdorff, then the quotient space $\mathcal{P}$ is Hausdorff.

Solution: Consider the bijection $\phi : \mathcal{P} \to Y$ by $\phi(P) = y$ where $P = f^{-1}(\{y\})$. Since $f = \phi \circ p$ is continuous, we know ϕ is continuous. Now, if P_1 and P_2 are distinct partition sets in $\mathcal{P}$, then there are distinct points $y_j \in Y$ for $j = 1, 2$ with $f^{-1}(\{y_j\}) = P_j$. Since Y is Hausdorff, there are disjoint open sets V_1 and V_2 with $y_j \in V_j$ for $j = 1, 2$. The sets $U_1 = \phi^{-1}(V_1)$ and $U_2 = \phi^{-1}(V_2)$ are open sets in $\mathcal{P}$ with $P_j \in U_j$ for $j = 1, 2$. These sets are also disjoint since if $P \in U_1 \cap U_2$, we know $P = f^{-1}(\{y\})$ for some (unique) y and $y = \phi(P) \in V_j$. Since this is true for both $j = 1$ and $j = 2$, we have $y \in V_1 \cap V_2$ which is a contradiction. We have shown that $\mathcal{P}$ is Hausdorff.

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3. (20 points) Is it necessarily true that the spaces $\mathcal{P}$ and Y in the previous problem are homeomorphic? Justify your answer.

Solution: No, it is not true that the spaces $\mathcal{P}$ and Y in the previous problem are always homeomorphic. If f is an identification map, then it is true, so what we need (for a counterexample) is a continuous surjective map $f : X \to Y$ which is **not** an identification map. In order to fail to be an identification map, there must be a set A in Y which is **not** open but whose inverse image is open in X.

We know such a map. Let $X = [0, 2\pi)$ and $Y = \mathbb{S}^1 \subset \mathbb{R}^2$. The map $f : X \to Y$ by $f(x) = (\cos x, \cos y)$ is a continuous bijection. However, the image of the interval $[0, \pi) \subset X$ is not open in $\mathbb{S}^1$, but $[0, \pi)$ is open in the interval $X = [0, 2\pi)$. In fact, the partition in this case consists entirely of singletons: $\mathcal{P} = \{\{x\} : x \in X\}$, and $\mathcal{P}$ is homeomorphic to the interval X which is not homeomorphic to the circle $Y = \mathbb{S}^1$.

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4. Let X denote the rectangle $[\pi/4, 3\pi/4] \times [0, 2\pi] \subset \mathbb{R}^2$, and consider $q_1 : X \to \mathbb{R}^3$ by $q_1(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$.
   (a) (5 points) Show that the antipodal map $\mathrm{a}n : \mathbb{R}^3 \to \mathbb{R}^3$ induces a bijection on $X_1 = q_1(X)$.
   (b) (5 points) Determine the partition $\mathcal{P}$ of X induced by the antipodal map on X_1. Let $p_1 : X \to \mathcal{P}$ be the generalized projection into the quotient $\mathcal{P}$
   (c) (10 points) Let $\tilde{X}$ denote the rectangle $[0, 2\pi] \times [-1/2, 1/2]$, and consider the identification space $\mathcal{M}$ obtained by identifying the two vertical sides in the reverse direction. Let $p : \tilde{X} \to \mathcal{M}$ be the associated identification map. Show that $\mathcal{P}$ in the identification topology is homeomorphic to $\mathcal{M}$ by defining an explicit map h from $\tilde{X}$ to X so that $q = p_1 \circ h : \tilde{X} \to \mathcal{P}$ is an identification map inducing the partition $\mathcal{M}$.

Solution:

(a) The spherical coordinates map q_1 is one-to-one on $[\pi/4, 3\pi/4] \times [0, 2\pi)$ and has image a band symmetric with the equator on the sphere. We see, however, that

$$\begin{aligned}\mathrm{a}n(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) &= (-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi)\\ &= (\sin(\pi-\phi)\cos(\pi+\theta), \sin(\pi-\phi)\sin(\pi+\theta), \cos(\pi-\phi))\\ &= q_1(\pi-\phi, \pi+\theta).\end{aligned}$$

Note that the map $\phi \mapsto \pi - \phi$ is a bijection between $[\pi/4, 3\pi/4]$ and itself. Also, $\theta \mapsto \pi + \theta$ maps the interval $[0, \pi)$ bijectively onto $[\pi, 2\pi)$.

At least for $0 \le \theta < \pi$, this shows the corresponding half of $q_1(X)$ is mapped bijectively onto the other half. For the other half corresponding to $\pi \le \theta < 2\pi$, we note that the expression above can also be written as

$$\begin{aligned}\mathrm{a}n(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) &= (-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi)\\ &= (\sin(\pi-\phi)\cos(\theta-\pi), \sin(\pi-\phi)\sin(\theta-\pi), \cos(\pi-\phi))\\ &= q_1(\pi-\phi, \pi-\theta).\end{aligned}$$

Since $\theta \mapsto \theta - \pi$ maps $[\pi, 2\pi)$ bijectively onto $[0, \pi)$, we see the antipodal maps also maps the half of $q_1(X)$ corresponding to $\pi \le \theta < 2\pi$ onto the first half, and $\mathrm{a}n$ gives a global bijection.

(b) As mentioned, q_1 already identifies $(\phi, 0)$ with $(\phi, 2\pi)$, so the remaining points are paired into the partition

$$\begin{aligned}\mathcal{P} = \{\{(\phi, 0), (\phi, 2\pi), (\pi - \phi, \pi)\} &: \pi/4 \le \phi \le 3\pi/4\}\\ \cup \{\{(\phi, \theta), (\pi - \phi, \pi + \theta)\} &: 0 < \theta < \pi, \pi/4 \le \phi \le 3\pi/4\}.\end{aligned}$$

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(c) The whole Möbius strip is given by half of X (under the antipodal identification), so we need to map half $\tilde{X}$ onto one of the halves of X with the vertical edges going to horizontal lines and the horizontal boundary of $\tilde{X}$ mapping to vertical boundary lines in X. If we use coordinates (θ, t) in $\tilde{X}$, the map is

$$h(\theta, t) = \left(\frac{\pi}{2}(t+1), \frac{\theta}{2}\right).$$

The composition $p_1 \circ h$ is clearly continuous and onto $\mathcal{P}$. Furthermore, the preimages of the sets in $\mathcal{P}$ are as follows:

For $\pi/4 \leq \phi \leq 3\pi/4\}$,

$$\begin{aligned}(p_1 \circ h)^{-1}(\{(\phi, 0), (\phi, 2\pi), (\pi - \phi, \pi)\}) &= h^{-1}(\{(\phi, 0), (\pi - \phi, \pi)\}) \\ &= \{(0, 2\phi/\pi - 1), (2\pi, 1 - 2\phi/\pi)\},\end{aligned}$$

and for $0 < \theta < \pi$ and $\pi/4 \leq \phi \leq 3\pi/4$,

$$\begin{aligned}(p_1 \circ h)^{-1}(\{(\phi, \theta), (\pi - \phi, \pi + \theta)\}) &= h^{-1}(\{(\phi, \theta)\}) \\ &= \{(2\theta, 2\phi/\pi - 1)\}.\end{aligned}$$

These are precisely the partition sets in $\tilde{X}$ giving the Möbius strip. Therefore, $\mathcal{M}$ and $\mathcal{P}$ are homeomorphic.

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5. (20 points) Consider the following partition of $\mathbb{R}^2$:

$$\mathcal{P} = \{\{(0, y)\} : y \in \mathbb{R}\} \cup \{\{(x, y) : y \in \mathbb{R}\} : x \in \mathbb{R}\backslash\{0\}\}.$$

Show that the identification space of $\mathbb{R}^2$ determined by $\mathcal{P}$ is not a Hausdorff space.

Solution: Let $p : \mathbb{R}^2 \to \mathcal{P}$ be the usual generalized projection with $p(x, y) = P$ where $(x, y) \in P$.

Let V be and open set in $\mathcal{P}$ containing $\{(0, 0)\}$. Since $p(0, 0) = \{(0, 0)\}$, we know $p^{-1}(V)$ is an open set in $\mathbb{R}^2$ with $(0, 0) \in p^{-1}(V)$. In particular, there is an open ball $B_\epsilon(0, 0) \subset p^{-1}(V)$. Thus, for every x with $0 < |x| < \epsilon$, we know $p(x, 0) \in V$. That is the set $\{(x, y) : y \in \mathbb{R}\} \in V$.

On the other hand, if W is an open set in $\mathcal{P}$ containing $\{(0, 1)\}$, then $p(0, 1) = \{(0, 1)\}$, we know $p^{-1}(W)$ is an open set in $\mathbb{R}^2$ with $(0, 1) \in p^{-1}(W)$. There is some open ball $B_\delta(0, 1) \subset p^{-1}(W)$. Again, for every x with $0 < |x| < \delta$, we have $p(x, 1) \in W$. That is, the set $\{(x, y) : y \in \mathbb{R}\} \in W$.

Evidently, taking x with $0 < |x| < \min\{\epsilon, \delta\}$, we obtain a partition set $\{(x, y) : y \in \mathbb{R}\} \in V \cap W$. This shows that every open set V in $\mathcal{P}$ containing $\{(0, 0)\}$ and every open set W in $\mathcal{P}$ containing $\{(0, 1)\}$ have a nontrivial intersection. Therefore, $\mathcal{P}$ is not Hausdorff.